4 Epstein's Adaptive Model of War

Lanchester's equations[1] are symmetric with regard to role: neither side is specifically regarded as the attacker or the defender. By contrast, in constructing an adaptive model to address his criticisms of Lanchester's models, Epstein (1997) introduces an asymmetry by distinguishing between those roles at the very outset. We introduce his model below.

Before proceeding, however, we digress to state the obvious: there is more than one way to generalize Lanchester's original equations. We have already used two such different generalizations in Lecture 3, since it is clear that (3.1) is not a special case of (3.9), and easily checked that (3.9) is a special case of (3.1) only if $g(n/m) = (n/m)^{\theta} g(m/n)$, which is true if $g(r) = r^{\theta/2}$ but not in general—for example, it fails to hold for (3.17). In this regard, Epstein generalizes Lanchester's equations from (2.1) and (2.11) to

$$\frac{dm}{dt} = -\alpha_n \, m^a n^{b+\theta_2} \tag{4.1a}$$

$$\frac{dn}{dt} = -\alpha_m \, m^{a+\theta_1} n^b \tag{4.1b}$$

with instantaneous casualty-exchange ratio

$$\frac{dm}{dn} = \frac{\alpha_n}{\alpha_m} \frac{n^{\theta_2}}{m^{\theta_1}} \tag{4.2}$$

whereas

$$\frac{dm}{dn} = \left(\frac{\alpha_n}{\alpha_m}\right)^{\lambda} \left(\frac{m}{n}\right)^{1-\theta} \tag{4.3}$$

is the instantaneous casualty-exchange ratio for (3.5) in Lecture 3. Clearly, neither is—in general—a special case of the other.

Epstein's model is formulated as a discrete dynamical system. Two key variables are $A(t)$ and $D(t)$, the number of surviving units of force on the attacking side and on the defending side, respectively, at the *start* of day t. Thus, as a matter of definition, the attrition rates per day on day t are

$$\alpha_a(t) = \frac{A(t) - A(t+1)}{A(t)} \tag{4.4}$$

for the attacking side and

$$\alpha_d(t) = \frac{D(t) - D(t+1)}{D(t)} \tag{4.5}$$

for the defender, and the casualty-exchange ratio on day t—that is, the ratio of attackers lost on day t to defenders lost on day t—is

$$\rho(t) = \frac{A(t) - A(t+1)}{D(t) - D(t+1)} = \frac{\alpha_a(t) A(t)}{\alpha_d(t) D(t)}. \tag{4.6}$$

[1] Excluding (2.25), which are Deitchman's (1962) equations.

Epstein regards the casualty-exchange ratio as something to be specified. Moreover, by analogy with (4.2), he chooses

$$\rho(t) \;=\; \rho_0 \, \frac{D(t)^{\lambda_d}}{A(t)^{\lambda_a}} \tag{4.7}$$

as "a plausible and relatively general functional form" for it. In effect, he replaces m, n, θ_1, θ_2 and α_n/α_m in (4.2) by $A(t)$, $D(t)$, λ_a, λ_d and ρ_0, respectively. Note that, because Epstein allows for the possibility that $\lambda_a = 0 = \lambda_d$, he allows for the possibility that

$$\rho(t) \;=\; \rho_0 \tag{4.8}$$

is constant.

Two further key variables are $W(t)$, the defender's rate of withdrawal on day t, and $P(t)$, the attacker's "prosecution" rate on day t. In turn, two key parameters—on which W and P depend—are threshold attrition rates, $\alpha_{dT} \in (0,1)$ for the defender and $\alpha_{aT} \in (0,1)$ for the attacker. The first of these rates, α_{dT}, is the maximum daily attrition rate that the defender is willing to suffer to hold the territory it currently occupies. Thus unacceptable previous-day attrition rates for the defender all fall in the interval $(\alpha_{dT}, 1)$, with values of $\alpha_d(t-1)$ near 1 being least acceptable and values of $\alpha_d(t-1)$ near α_{dT} being most acceptable (though all are unacceptable). The defending side will remain in place on day t if its attrition rate on day $t-1$ does not exceed the critical threshold α_{dT}, or $\alpha_d(t-1) \le \alpha_{dT}$. Otherwise, the defending side will increase its rate of withdrawal in such a way that the relative reduction of its capacity to withdraw corresponds to the relative decrease in acceptability of the attrition rate or

$$\frac{W_{\max} - W(t)}{W_{\max} - W(t-1)} \;=\; \frac{1 - \alpha_d(t-1)}{1 - \alpha_{dT}} \tag{4.9}$$

where $W_{\max}$ is the maximum rate of withdrawal. That is,

$$W(t) \;=\; \begin{cases} 0 & \text{if } \alpha_d(t-1) \le \alpha_{dT} \\ \left(\frac{\alpha_d(t-1)-\alpha_{dT}}{1-\alpha_{dT}}\right) W_{\max} + \left(\frac{1-\alpha_d(t-1)}{1-\alpha_{dT}}\right) W(t-1) & \text{if } \alpha_d(t-1) > \alpha_{dT} \end{cases} \tag{4.10}$$

Likewise, α_{aT} is the maximum daily attrition rate that the attacker is willing to suffer to take the territory. Acceptable previous-day attrition rates for the attacker all fall in the interval $(0, \alpha_{aT})$, with values of $\alpha_a(t-1)$ near α_{aT} being least acceptable and values of $\alpha_a(t-1)$ near 0 being most acceptable; and unacceptable previous-day attrition rates for the attacker all fall in the interval $(\alpha_{aT}, 1)$. If the actual attrition rate on day $t-1$ equals α_{aT}, then the prosecution rate does not change, that is, $P(t) = P(t-1)$. Otherwise, the attacking side will increase or decrease its prosecution rate according to whether its rate of attrition is below or above the critical threshold, in such a way that the relative change corresponds to the relative change in acceptability of the attrition rate. That is,

$$\frac{P(t)}{P(t-1)} \;=\; \frac{1 - \alpha_a(t-1)}{1 - \alpha_{aT}}. \tag{4.11}$$

Finally, the attacking side's attrition rate $\alpha_a(t)$ will increase with its prosecution rate $P(t)$ but decrease with the defender's rate of withdrawal $W(t)$, in such a way that $\alpha_a(t) \to 0$

as $W(t) \to W_{\max}$. It is therefore reasonable to posit that

$$\alpha_a(t) \;=\; \left(1 - \frac{W(t)}{W_{\max}}\right) P(t). \tag{4.12}$$

The values of $A(t)$, $D(t)$, $P(t)$, $W(t)$, $\alpha_a(t)$ and $\alpha_d(t)$ can now be generated for all $t \geq 1$ from given values of the parameters λ_a, λ_d, ρ_0, α_{aT}, α_{dT}, $W_{\max}$ and the recursion

$$A(t) \;=\; \big(1 - \alpha_a(t-1)\big) A(t-1) \tag{4.13a}$$

$$D(t) \;=\; D(t-1) - \frac{\alpha_a(t-1) A(t-1)^{\lambda_a + 1}}{\rho_0 D(t-1)^{\lambda_d}} \tag{4.13b}$$

$$P(t) \;=\; \frac{1 - \alpha_a(t-1)}{1 - \alpha_{aT}} P(t-1) \tag{4.13c}$$

$$W(t) \;=\; \begin{cases} 0 & \text{if } \alpha_d(t-1) \leq \alpha_{dT} \\ \left(\frac{\alpha_d(t-1) - \alpha_{dT}}{1 - \alpha_{dT}}\right) W_{\max} + \left(\frac{1 - \alpha_d(t-1)}{1 - \alpha_{dT}}\right) W(t-1) & \text{if } \alpha_d(t-1) > \alpha_{dT} \end{cases} \tag{4.13d}$$

$$\alpha_a(t) \;=\; \left(1 - \frac{W(t)}{W_{\max}}\right) P(t) \tag{4.13e}$$

$$\alpha_d(t) \;=\; \frac{\alpha_a(t) A(t)^{\lambda_a + 1}}{\rho_0 D(t)^{\lambda_d + 1}} \tag{4.13f}$$

together with initial values for $A(t)$, $D(t)$, $P(t)$ and $W(t)$. Note that (4.13a) is just a rearrangement of (4.4) with t replaced by $t-1$; (4.13b) is likewise a rearrangement of (4.5) with t replaced by $t-1$, and using (4.6) to substitute for $\alpha_d(t)D(t)$ and (4.7) to substitute for $\rho(t)$; (4.13c) is a rearrangement of (4.11); (4.13d) and (4.13e) are just restatements of (4.10) and (4.12), respectively; and (4.13f) is obtained by substituting in (4.6) from (4.7) for $\rho(t)$. Also note that we do not need initial values for $\alpha_a(t)$ and $\alpha_d(t)$ because

$$\alpha_a(1) \;=\; \left(1 - \frac{W(1)}{W_{\max}}\right) P(1), \qquad \alpha_d(1) \;=\; \frac{\alpha_a(1) A(1)^{\lambda_a + 1}}{\rho_0 D(1)^{\lambda_d + 1}} \tag{4.14}$$

from (4.13e)–(4.13f). Results from this recursion are exemplified by Figures 4.1–4.4.

According to Epstein (1997, p. 40), the parameters α_{dT} and α_{aT} "allow one to reflect the different ways in which given forces can behave." But one could also, for example, allow λ_a and λ_d to differ from zero (as in Figures 4.3–4.4) or consider plausible functional forms for ρ that differ from (4.7). Epstein unleashes the "full apparatus" (Epstein, 1997, p. 30) of his adaptive dynamic model only in an earlier book (Epstein, 1990, pp. 85-89), where he distinguishes between ground forces, ground reinforcements and air power to construct a more elaborate discrete dynamical system than the one discussed above. If any of you is sufficiently interested, perhaps this work could form the basis of a more in-depth investigation leading to an end-of-term presentation.

For the rest of us, however, this is as far as we go for now!

Figure 4.1: Solution of discrete-time dynamical system (4.13) for $\lambda_a = 0 = \lambda_d$, $\rho_0 = 1.1$, $\alpha_{aT} = 0.6$, $\alpha_{dT} = 0.3$ and $W_{\max} = 20$ with $P(1) = 0.1$, $W(1) = 0$ and $A(1) = 0.5 \cdot 10^6 = D(1)$; attacker in green, defender in red. (a) The solid curve is for $P(t)$, the thick dashed curves for $\alpha_a(t)$ and $\alpha_d(t)$ and the thin dashed curves for the thresholds, α_{aT} and α_{dT}. (b) The solid curve is for $W(t)$, the dashed curves for $A(t)$ and $D(t)$. The curves for P and W agree with Figure 2.4 of Epstein (1997, p. 38). Note that, by (4.12), $\alpha_a(t) = P(t)$ for $t \geq 7$ because $W(t) = 0$.

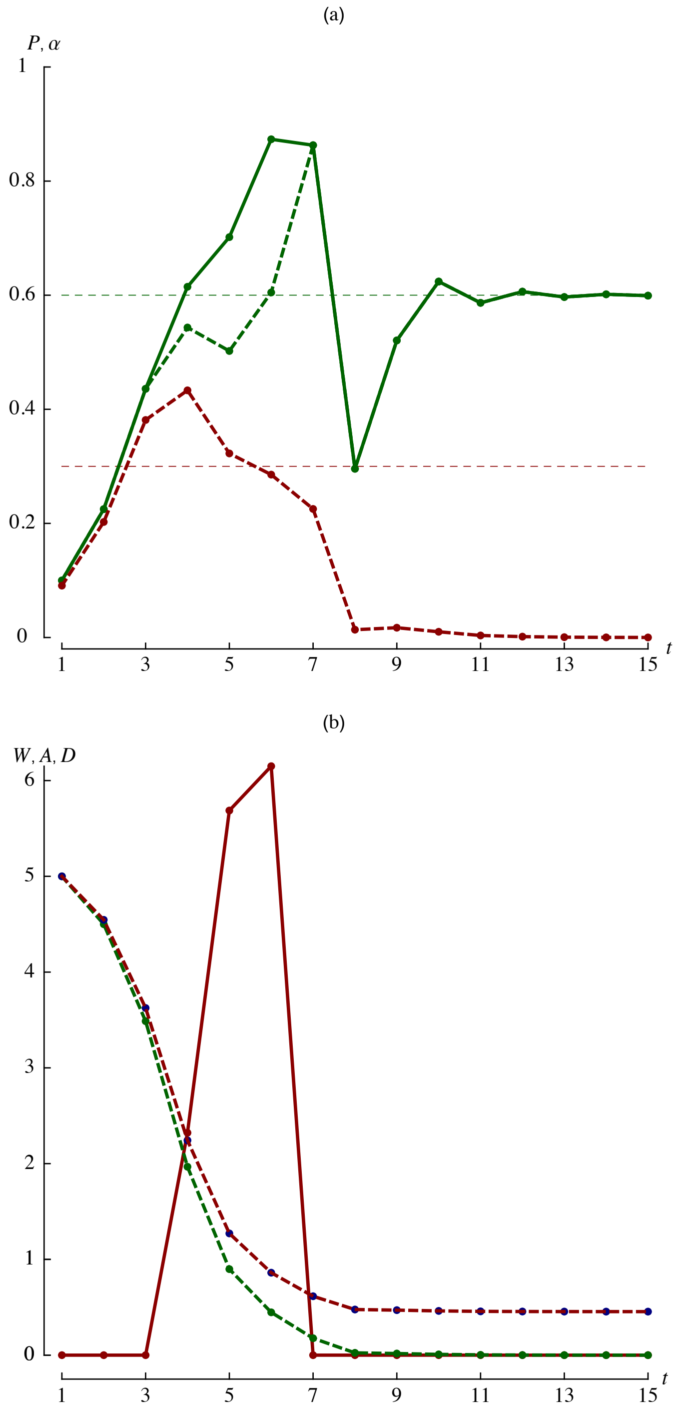


Figure 4.2: Solution of discrete-time dynamical system (4.13) for $\lambda_a = 0 = \lambda_d$, $\rho_0 = 1.1$, $\alpha_{aT} = 0.1$, $\alpha_{dT} = 0.1$ and $W_{\max} = 20$ with $P(1) = 0.2$, $W(1) = 0$ and $A(1) = 0.5 \cdot 10^6 = D(1)$; attacker in green, defender in red. (a) The solid curve is for $P(t)$, the thick dashed curves for $\alpha_a(t)$ and $\alpha_d(t)$ and the thin dashed curves for the thresholds, α_{aT} and α_{dT}. (b) The solid curve is for $W(t)$, the dashed curves for $A(t)$ and $D(t)$. The curves for P and W agree with Figure 2.5 of Epstein (1997, p. 39). Note that, by (4.12), $\alpha_a(t) = P(t)$ for $t \geq 10$ because $W(t) = 0$.

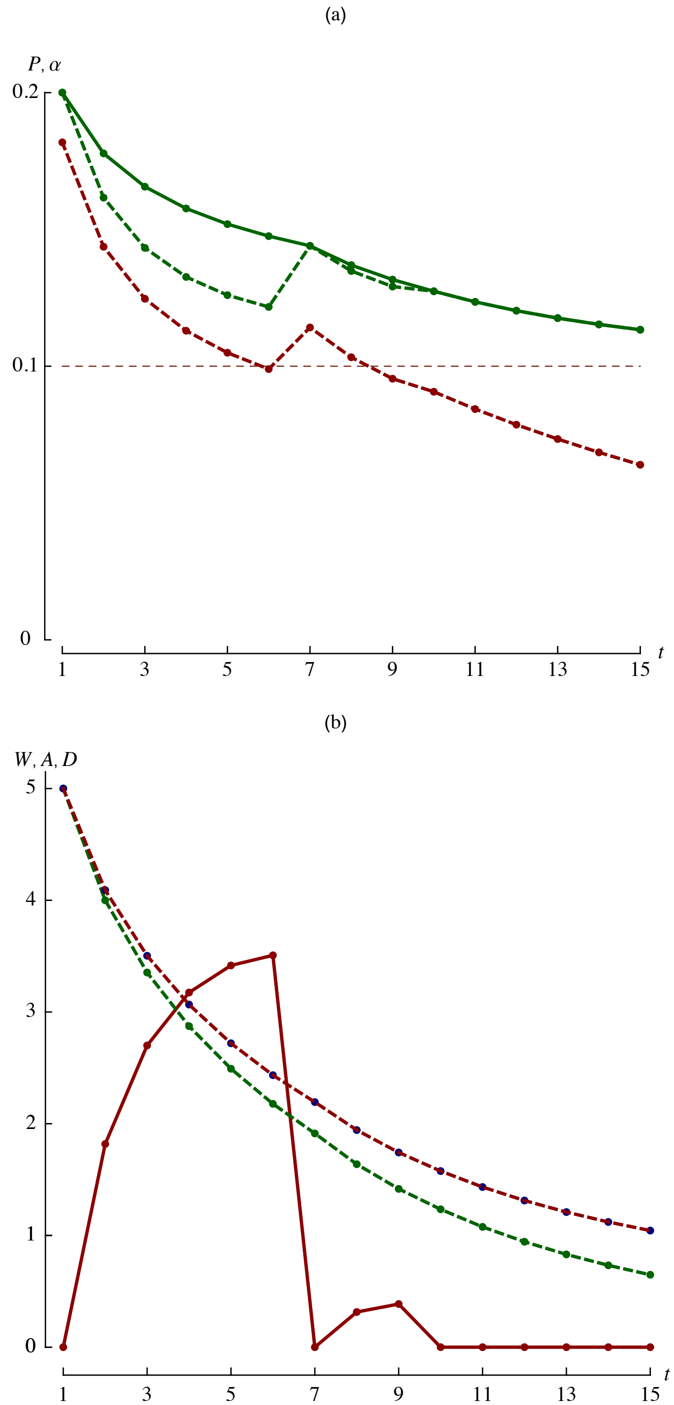
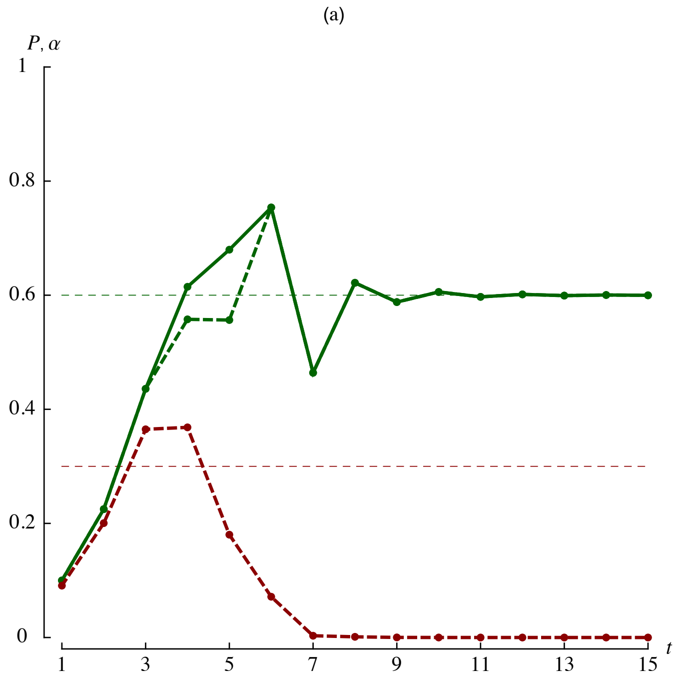


Figure 4.3: Solution of discrete-time dynamical system (4.13) for $\lambda_a = 1 = \lambda_d$, $\rho_0 = 1.1$, $\alpha_{aT} = 0.6$, $\alpha_{dT} = 0.3$ and $W_{\max} = 20$ with $P(1) = 0.1$, $W(1) = 0$ and $A(1) = 0.5 \cdot 10^6 = D(1)$; attacker in green, defender in red. (a) The solid curve is for $P(t)$, the thick dashed curves for $\alpha_a(t)$ and $\alpha_d(t)$ and the thin dashed curves for the thresholds, α_{aT} and α_{dT}. (b) The solid curve is for $W(t)$, the dashed curves for $A(t)$ and $D(t)$. The curves for P and W agree with Figure 2.4 of Epstein (1997, p. 38). Note that, by (4.12), $\alpha_a(t) = P(t)$ for $t \geq 6$ because $W(t) = 0$.



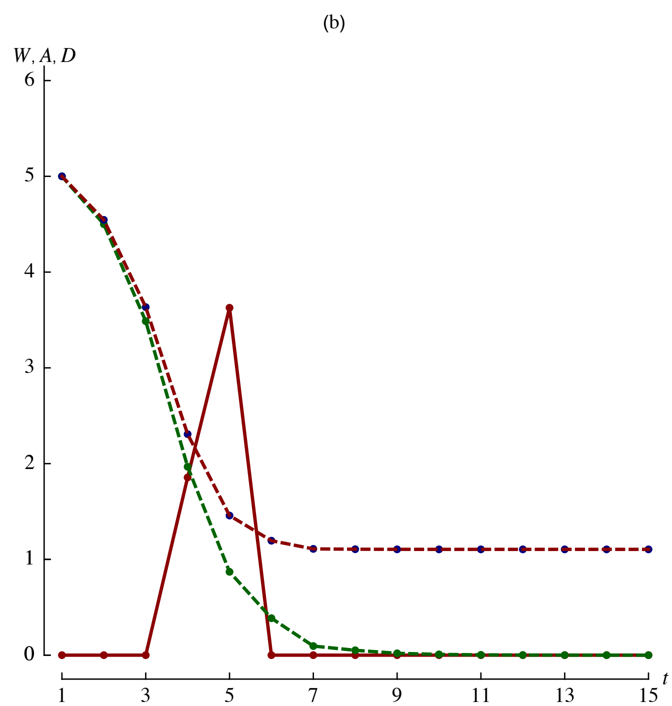
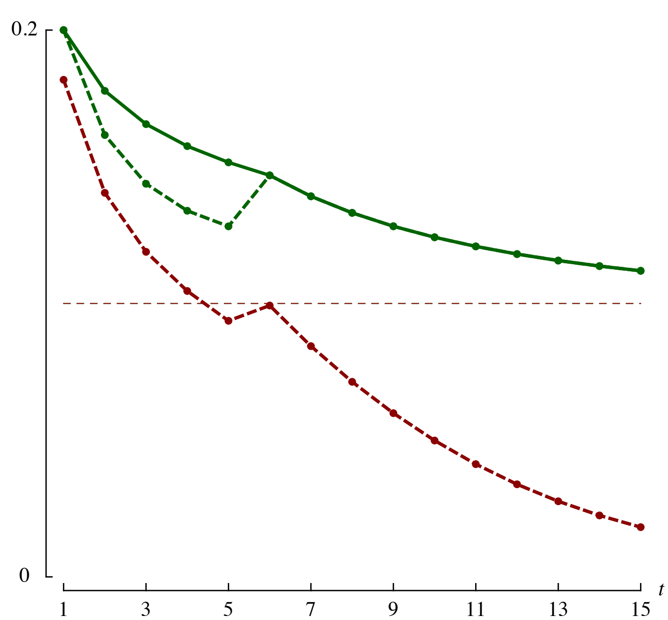


Figure 4.4: Solution of discrete-time dynamical system (4.13) for $\lambda_a = 1 = \lambda_d$, $\rho_0 = 1.1$, $\alpha_{aT} = 0.1$, $\alpha_{dT} = 0.1$ and $W_{\max} = 20$ with $P(1) = 0.2$, $W(1) = 0$ and $A(1) = 0.5 \cdot 10^6 = D(1)$; attacker in green, defender in red. (a) The solid curve is for $P(t)$, the thick dashed curves for $\alpha_a(t)$ and $\alpha_d(t)$ and the thin dashed curves for the thresholds, α_{aT} and α_{dT}. (b) The solid curve is for $W(t)$, the dashed curves for $A(t)$ and $D(t)$. The curves for P and W agree with Figure 2.5 of Epstein (1997, p. 39). Note that, by (4.12), $\alpha_a(t) = P(t)$ for $t \geq 6$ because $W(t) = 0$.

(a)



(b)