26. The mean and median of a distribution.

What does it mean to be above average in some respect? A possible answer is above the middle, or **median**, of a relevant distribution, defined as the value exceeded with probability 0.5. That is, if the relevant random variable, say X, is distributed on $[0, \infty)$ with p.d.f. f and c.d.f. F and M denotes the median, then

$$\mathrm{Prob}(X \le M) \;=\; F(M) \;=\; \int_0^M f(x)\,dx \;=\; \frac{1}{2} \tag{26.1a}$$

or, equivalently,

$$\mathrm{Prob}(X \ge M) \;=\; 1 - F(M) \;=\; \int_M^\infty f(x)\,dx \;=\; \frac{1}{2}. \tag{26.1b}$$

For example, from (22.37)-(22.38), the c.d.f. and p.d.f. of a Weibull distribution with shape parameter c (≥ 1) and scale parameter s (> 0) are defined by[1]

$$F(x) \;=\; 1 \;-\; e^{-(x/s)^c} \tag{26.2a}$$

$$f(x) \;=\; F'(x) \;=\; \frac{c}{s}(x/s)^{c-1}e^{-(x/s)^c} \tag{26.2b}$$

where, in general, s and c may be any positive numbers, although in this lecture we assume that $c \ge 1$. If c = 1 then

$$f(x) \;=\; \frac{1}{s}e^{-x/s} \tag{26.3}$$

and the distribution is more commonly known as the **exponential**: its p.d.f. is strictly decreasing, i.e., m = 0. If, on the other hand, c > 1, then the distribution is unimodal with $m = (1-1/c)^{1/c}s > 0$, by (20.35). Either way, (1)-(2) imply $\exp(-\{M/s\}^c) = 1/2$ or

$$M \;=\; s\{\ln(2)\}^{\frac{1}{c}} \tag{26.4}$$

(Exercise 1). The median is illustrated for c = 2 and c = 5 by Figure 1, where shaded area equals 0.5. Note that M lies above the mode for c = 2 but below the mode for c = 5, in accordance with a more general result obtained in Exercise 1.

The median has the disadvantage of giving too little weight to the "tail" of a distribution, which may extend far to the right. If comparing an observation to the average of its distribution is meant to suggest not only whether the observed value is above or below the middle of that distribution, but also whether it is large or small (relative to the distribution), then a better definition of average is the balance point, or **mean**. The mean is defined as the value μ at which a cardboard lamina of area 1, cut to the shape of the region between the horizontal axis and the graph of the p.d.f., would (in principle) balance on a knife-edge.[2] Because the lamina has uniform thickness, area (=probability) is equivalent to weight. Weight to the right of the balance point tends to turn the lamina clockwise, weight to the left tends to turn it anticlockwise, and the balance point is precisely where these turning effects, or **moments**, are equal.

Now, if a weight f(x) were concentrated at distance $x - \mu$ from the balance point, then its turning effect about the balance point would be $(x-\mu)f(x)$. It would be positive,

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1 The Weibull is defined for c > 0, but we require only the unimodal (c > 1) and exponential (c = 1) cases.
2 As biology majors, you may be interested to know that lamina is animal backwards.

or clockwise, for $x > \mu$ and negative, or anticlockwise, for $x < \mu$. But f(x) is not a weight; rather, it is a weight (= probability) *density*, i.e., a weight per unit length. Therefore

$$T(x) = (x-\mu)f(x) \tag{26.5}$$

is not a turning effect; rather, it is a turning effect density, or turning effect per unit length. Accordingly, just as Int(f, [a, b]) is the weight (= probability) associated with the interval [a, b], so Int(T, [a, b]) is the turning effect associated with [a, b]. Hence the total positive or clockwise moment about the balance point is

$$\mathrm{Int}(T,[\mu,\infty)) = \int_{\mu}^{\infty}(x-\mu)f(x)\,dx; \tag{26.6}$$

see Figure 2, where Int(T, [μ,∞)) is the positive shaded area. Correspondingly, the total negative or anticlockwise moment about the balance point is

$$\mathrm{Int}(T,[0,\mu]) = \int_{0}^{\mu}(x-\mu)f(x)\,dx, \tag{26.7}$$

i.e., the negative shaded area in Figure 2. If the lamina is to balance, however, then net turning effect about $x = \mu$ must be precisely zero. That is, Int(T, [0, μ]) + Int(T, [μ, ∞)) = 0. Hence (8.25) implies Int(T, [0, ∞)) = 0 or, from (6)-(7),

$$\int_{0}^{\infty}(x-\mu)f(x)\,dx = 0. \tag{26.8}$$

Using elementary properties of integrals, we can rewrite (8) as

$$\int_{0}^{\infty}xf(x)\,dx - \mu\int_{0}^{\infty}f(x)\,dx = 0. \tag{26.9}$$

But Int(f, [0, ∞)) = 1. So (9) implies

$$\mu = \int_{0}^{\infty}xf(x)\,dx, \tag{26.10}$$

which defines the mean. For all of the distributions we commonly use, the mean is a well defined average. Nevertheless, we will discover in Lecture 27 that there exist well defined (and potentially useful) distributions for which μ is not a finite quantity. So an advantage of the median is that it is guaranteed to exist.

Typically, we calculate means by invoking the fundamental theorem of calculus. To illustrate, consider mean survival time for Lecture 15's melanoma patients. From (19.2), the p.d.f. is defined by

$$f(t) = \begin{cases} A + \frac{1}{4}\{1-3A\}t & \text{if } 0 \le t < 2 \\ \frac{4(1-A)}{t^3} & \text{if } 2 \le t < \infty \end{cases} \tag{26.11}$$

with A = 0.768. Clearly,

$$t\,f(t) = \begin{cases} At + \frac{1}{4}\{1-3A\}t^2 & \text{if } 0 \le t < 2 \\ \frac{4(1-A)}{t^2} & \text{if } 2 \le t < \infty \end{cases} \tag{26.12}$$

So, from (10), and on using (16.20) in conjunction with Table 18.1, we have

$$\begin{aligned}\mu &= \int_0^\infty t f(t)\,dt = \int_0^2 t f(t)\,dt + \int_2^\infty t f(t)\,dt \\ &= \int_0^2 \left\{At + \frac{1}{4}\{1-3A\}t^2\right\}dt + 4(1-A)\int_2^\infty t^{-2}\,dt \\ &= \int_0^2 \frac{d}{dt}\left\{\frac{1}{2}At^2 + \frac{1}{12}\{1-3A\}t^3\right\}dt + 4(1-A)\int_2^\infty \frac{d}{dt}\left\{-t^{-1}\right\}dt \\ &= \int_0^2 U'(t)\,dt + 4(1-A)\int_2^\infty V'(t)\,dt, \end{aligned} \tag{26.13}$$

where U is defined on [0, 2] by

$$U(t) = \frac{1}{2}At^2 + \frac{1}{12}\{1-3A\}t^3 \tag{26.14}$$

and V is defined on [2, ∞) by

$$V(t) = -\frac{1}{t}. \tag{26.15}$$

By the fundamental theorem, we easily find that

$$\int_0^2 U'(t)\,dt = U(2) - U(0) = \frac{2}{3} - 0 = \frac{2}{3}, \tag{26.16}$$

but the second integral in (13) requires a little more care. We first observe that, again by the fundamental theorem,

$$\int_2^K V'(t)\,dt = V(K) - V(2) = -\frac{1}{K} - \left(-\frac{1}{2}\right) = \frac{1}{2} - \frac{1}{K}. \tag{26.17}$$

We now allow K to be come infinitely large. Then K^{-1} approaches zero, implying that $1/2 - K^{-1}$ approaches 1/2, and so (17) yields

$$\int_2^\infty V'(t)\,dt = \frac{1}{2}. \tag{26.18}$$

Now, on substituting from (16) and (18) into (13), we find that

$$\mu = \int_0^2 U'(t)\,dt + 4(1-A)\int_2^\infty V'(t)\,dt = \frac{2}{3} + 4(1-A)\cdot\frac{1}{2} = \frac{8}{3} - 2A. \tag{26.19}$$

That is, with A = 0.768, the mean survival time is 1.13 years. For further practice with calculating means by invoking the fundamental theorem, see Exercises 5-10.

We don't invariably invoke the fundamental theorem to calculate a mean, however, and the mean of the Weibull is a case in point. From (3) and (10), the mean of a Weibull with arbitrary shape parameter c is

$$\mu = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{cx}{s}(x/s)^{c-1}e^{-(x/s)^c}\,dx = c\int_0^\infty (x/s)^c e^{-(x/s)^c}\,dx. \tag{26.20}$$

We can simplify this integral by using the substitution

$$u = \phi(x) = \frac{x}{s}. \tag{26.21}$$

Because u = x/s implies x = su, the inverse substitution is defined by

$$x = \zeta(u) = su, \tag{26.22}$$

implying

$$\zeta'(u) \quad = \quad s. \tag{26.23}$$

From (21), $\phi(0) = 0$ and, because $s > 0$, $\phi(\infty) = \infty$. (Note, however, that $s < 0$ would imply $\phi(\infty) = -\infty$.) From (21.21), we have

$$\int_a^b g(x)\,dx \quad = \quad \int_{\phi(a)}^{\phi(b)} g(\zeta(u))\,\zeta'(u)\,du, \tag{26.24}$$

for arbitrary g. With g defined by

$$g(x) \quad = \quad c(x/s)^c e^{-(x/s)^c}, \tag{26.25}$$

(20) reduces to

$$\begin{aligned}\mu \quad = \quad \int_0^\infty g(x)\,dx \quad &= \quad \int_{\phi(0)}^{\phi(\infty)} g(\zeta(u))\,\zeta'(u)\,du \\ &= \quad \int_{\phi(0)}^{\phi(\infty)} c\big(\zeta(u)/s\big)^c e^{-(\zeta(u)/s)^c}\zeta'(u)\,du \\ &= \quad c\int_0^\infty u^c e^{-u^c} s\,du,\end{aligned} \tag{26.26}$$

on using (21)-(23). Because the right-hand side of (26) depends only on s and c, we can replace x by u:

$$\mu \quad = \quad sc\int_0^\infty x^c e^{-x^c}\,dx. \tag{26.27}$$

We now make a fresh substitution,

$$u \quad = \quad \phi(x) \quad = \quad x^c, \tag{26.28}$$

whose inverse is

$$x \quad = \quad u^{1/c} \quad = \quad \zeta(u), \tag{26.29}$$

implying

$$\zeta'(u) \quad = \quad \frac{1}{c}u^{1/c\,-\,1}. \tag{26.30}$$

Because c is positive, if $x \to \infty$ then $\phi(x) \to \infty$ also. So (24) with $g(x) = sc\,x^c e^{-x^c}$ and (27) imply

$$\begin{aligned}\mu \quad = \quad sc\int_0^\infty x^c e^{-x^c}\,dx \quad &= \quad sc\int_{\phi(0)}^{\phi(\infty)} \big(\zeta(u)\big)^c e^{-\{\zeta(u)\}^c}\zeta'(u)\,du \\ &= \quad sc\int_0^\infty u e^{-u}\frac{1}{c}u^{1/c\,-\,1}\,du \qquad (26.31) \\ &= \quad s\int_0^\infty u^{(1+1/c)\,-\,1}e^{-u}\,du.\end{aligned}$$

Now, $1/c + 1 - 1 = 1/c$, and in mathematics one always prefers the simplest form of an expression. So why do we not write

$$\mu \quad = \quad s\int_0^\infty u^{1/c}e^{-u}\,du \tag{26.32}$$

in place of (31)? It turns out that (31) yields greater simplicity in the long run, because the function Γ defined by

$$\Gamma(x) \quad = \quad \int_0^\infty u^{x-1} e^{-u}\, du, \qquad (26.33)$$

called the **Gamma function**, is a "known" function of mathematics, just like exp or ln. In terms of Γ, the mean of the Weibull is simply

$$\mu \quad = \quad s\Gamma(1+1/c), \qquad (26.34)$$

on using (31).

The domain of the Gamma function is the largest interval on which the integral in (26) corresponds to a finite area, or "converges," which turns out to be $(0, \infty)$. On this domain, Γ is concave up with global minimum 0.8856 and range $[0.8856, \infty)$; see Figure 3. Because $1 \le c < \infty$, however, we have $1 < 1 + 1/c \le 2$. Thus, as far as the mean of the Weibull is concerned, it suffices to know Γ only on [1, 2]. The restriction of Γ to this subdomain is graphed in Figure 4. Note that $1 \le x \le 2$ implies $0.8856 \le \Gamma(x) \le 1$, because

$$\Gamma(1) \quad = \quad 1 \quad = \quad \Gamma(2) \qquad (26.35)$$

(Exercise 3). So the mean of a Weibull always lies between 0.8856s and s. See Figure 5, where μ, M and m are plotted versus c.[3]

For example, rat pupil radius in Lecture 22 has a Weibull distribution with shape parameter $c = 2$ (as in Figure 2) and scale parameter $s = 0.713$. So, by (34) and Figure 4, mean rat pupil radius is $s\Gamma(3/2) = 0.886s = 0.886 \times 0.713 = 0.63$ mm. Similarly, $c = 7$ and $s = 0.152$ for the Weibull in Figure 19.3, implying that mean leaf thickness in *Dicerandra linearifolia* is $s\Gamma(8/7) = 0.152 \times 0.935 = 0.14$ mm. Again, $c = 5$ and $s = 17.84$ for the Weibull in Figure 19.5, so that mean size (above base length) in D'Arcy Thompson's minnows is $s\Gamma(6/5) = 17.8 \times 0.9182 = 16.4$ mm. Finally, $c = 1$ and $s = 1.286$ in Figure 19.1, so mean life expectancy among prairie dogs is $s\Gamma(2) = s = 1.286$ years.

Although we restricted Γ to subdomain [1, 2], which suffices for the mean of a Weibull, it turns out that [1, 2] is the only subdomain on which Γ need ever be known (so that Figure 4 is an extremely useful diagram). Why? The answer is that the Gamma function has a recursive property, namely,

$$\Gamma(r+1) \quad = \quad r\Gamma(r) \qquad (26.36)$$

for any $r > 0$ (see Exercise 4 and Appendix 26). If, for example, we require both $\Gamma(0.5)$ and $\Gamma(3.7)$, we can use (36) to obtain reasonably accurate answers from Figure 4, even though neither 0.5 nor 3.7 belongs to [1, 2]. In the first case, setting $r = 0.5$ in (36) yields $\Gamma(1.5) = 0.5\Gamma(0.5)$, so that $\Gamma(0.5) = 2\Gamma(1.5) = 2 \times 0.886 = 1.772$. In the second case, setting $r = 2.7$ in (36) yields $\Gamma(3.7) = 2.7\Gamma(2.7)$, and setting $r = 1.7$ yields $\Gamma(2.7) = 1.7\Gamma(1.7)$, so that $\Gamma(3.7) = 2.7 \times 1.7 \times \Gamma(1.7) = 2.7 \times 1.7 \times 0.909 = 4.17$.

The quantity $\Gamma(0.5)$ will surface again in Lecture 28, in an important context. So we conclude by noting for later reference that $\Gamma(0.5) = 1.772$ is merely a numerical approximation to a precise relationship, namely,

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[3] Note that the curves in Figure 5 do not all intersect at the same point: The mode rises above the median at $c = 3.26$ and above the mean at $c = 3.31$, whereas the mean does not fall below the median until $c = 3.44$.

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}, \qquad (26.37)$$

where π is the ratio between circumference and diameter of a circle.

Exercises 26

26.1 (i) Verify that $M = s\{\ln(2)\}^{\frac{1}{c}}$ for the Weibull distribution defined by (2).
(ii) Verify that M lies above or below the mode according to whether $c < c^*$ or $c > c^*$, where $c^* = \{1 - \ln(2)\}^{-1} \approx 3.26$.

26.2 Find the median of the distribution defined on $[0, \infty)$ by

$$f(x) = \begin{cases} \frac{2}{3}x & \text{if } 0 \le x < 1 \\ \frac{1}{3}(3-x) & \text{if } 1 \le x < 3 \\ 0 & \text{if } 3 \le x < \infty \end{cases}$$

26.3 Show that $\frac{d}{du}\{-e^{-u}\} = e^{-u}$ and $\frac{d}{du}\{-(u+1)e^{-u}\} = u\,e^{-u}$. Hence establish (35).

26.4 Use mathematical induction (Appendix 17B) to show that if r is an integer, then $\Gamma(r+1) = r!$, where r! (or r factorial) is the product of the first r positive integers, i.e., $r! = 1 \cdot 2 \cdot 3 \cdot \ \ldots \ \cdot (r-1)r$.

26.5 The p.d.f. of a distribution on $[0, \infty)$ is defined by

$$f(x) = \begin{cases} x(2-x)/L & \text{if } 0 \le x < 2 \\ 0 & \text{if } 2 \le x < \infty \end{cases}$$

where L is a constant. Find (i) L (ii) m (iii) μ (iv) the c.d.f.

26.6 The p.d.f. of a distribution on $[0, \infty)$ is defined by

$$f(x) = \begin{cases} x^2(2-x)/L & \text{if } 0 \le x < 2 \\ 0 & \text{if } 2 \le x < \infty \end{cases}$$

where L is a constant. Find (i) L (ii) m (iii) μ (iv) the c.d.f.

26.7* The p.d.f. of a distribution on $[0, \infty)$ is defined by

$$f(x) = \begin{cases} x(2-x)^2/L & \text{if } 0 \le x < 2 \\ 0 & \text{if } 2 \le x < \infty \end{cases}$$

where L is a constant. Find (i) L (ii) m (iii) μ (iv) the c.d.f.

26.8 The p.d.f. of a distribution on $[0, \infty)$ is defined by

$$f(x) = \begin{cases} x^2(2-x)^2/L & \text{if } 0 \le x < 2 \\ 0 & \text{if } 2 \le x < \infty \end{cases}$$

where L is a constant. Find (i) L (ii) m (iii) μ (iv) the c.d.f.

26.9 The exponential distribution defined by (36) and the distributions defined in Exercises 22.9-22.11 are all special cases of the "Gamma" distribution. The p.d.f. of the Gamma with shape parameter c and scale parameter s is f defined by

$$f(x) = \frac{x^{c-1}e^{-x/s}}{L},$$

where L is a constant chosen to ensure Int(f, [0, ∞)) = 1. Find (i) L (ii) μ (iii) m.
<u>Hint</u>: For (i)-(ii), use (21)-(24) and (33)

26.10* A truncated exponential distribution is defined on [0, ∞) by

$$f(x) = \begin{cases} \frac{1}{L}e^{-\lambda x} & \text{if } 0 \le x \le b \\ 0 & \text{if } b \le x \le \infty \end{cases}$$

where λ, b are parameters and L is a constant to ensure that Int(f, [0, ∞)) = 1. Find (i) L (ii) μ (iii) M (iv) the c.d.f. Verify that the c.d.f. is continuous.

Appendix 26: The recursive property of the Gamma function

The purpose of this appendix is to establish that $\Gamma(r + 1) = r\,\Gamma(r)$ for any $r > 0$. The key observation is that $u^r e^{-u}$ approaches zero as $u \to \infty$, no matter how large the value of r. To be sure, for any $r > 0$, u^r increases with u; and the larger the value of r, the more rapidly u^r increases. At the same time, however, e^{-u} decreases with u. So $u^r e^{-u}$ is the product of something that gets bigger and bigger as $u \to \infty$ and something that gets smaller and smaller. It is as though an arms race exists between u^r, which is on its way to infinity, and e^{-u}, which is on its way to zero; but the winner of this arms race is e^{-u}, no matter how large the value of r.

The easiest way to see this result is to compare the graph of $z = u/r$ with that of $z = \ln(u)$. The first is a straight line with with positive slope through the origin of coordinates; the second is a concave down curve (Figure 22.1). Because $z = \ln(u)$ keeps turning down as u increases, there must come a point beyond which $z = \ln(u)$ stays below $z = u/r$ forever; and the further u increases beyond this point, the greater the divergence between $z = u/r$ and $z = \ln(u)$. In other words, $u/r - \ln(u)$ must approach infinity as $u \to \infty$. Hence $u - r\ln(u)$ must also approach infinity as $u \to \infty$, implying that $\exp(-\{u - r\ln(u)\})$ must approach zero as $u \to \infty$. But $\exp(-\{u - r\ln(u)\}) = u^r e^{-u}$, from (22.22) and (22.28). Therefore $u^r e^{-u} \to 0$ as $u \to \infty$.

We now apply the product rule to $u^r e^{-u}$. From (17.21), (22.31) and Exercise 3, we have

$$\begin{aligned}\frac{d}{du}\left\{u^r e^{-u}\right\} &= \frac{d}{du}\left\{u^r\right\} e^{-u} + u^r \frac{d}{du}\left\{e^{-u}\right\} \\ &= r u^{r-1} e^{-u} + u^r \left\{-e^{-u}\right\} \\ &= r u^{r-1} e^{-u} - e^{-u} u^r,\end{aligned} \tag{26.A1}$$

implying

$$\int_0^\infty \left\{r u^{r-1} e^{-u} - e^{-u} u^r\right\} du = \int_0^\infty \frac{d}{du}\left\{u^r e^{-u}\right\} du. \tag{26.A2}$$

So (12.25) and (18.20) imply

$$r\int_0^\infty u^{r-1} e^{-u}\, du - \int_0^\infty u^{(r+1)-1} e^{-u}\, du = u^r e^{-u}\Big|_0^\infty. \tag{26.A3}$$

Now (33) implies

$$r\Gamma(r) - \Gamma(r+1) = \lim_{u\to\infty} u^r e^{-u} - 0 = 0 - 0 = 0, \tag{26.A4}$$

from which $\Gamma(r + 1) = r\,\Gamma(r)$, as required.

Answers and Hints for Selected Exercises

26.2 $\int_0^1 f(x)dx = \int_0^1 \frac{2}{3}x\,dx = \int_0^1 \frac{d}{dx}\left(\frac{1}{3}x^2\right)dx = \frac{1}{3}x^2\Big|_0^1 = \frac{1}{3}1^2 - \frac{1}{3}0^2 = \frac{1}{3}$, which is less than 1/2, implying $M > 1$. So use $\int_M^\infty f(x)dx = \frac{1}{2} \Rightarrow \int_M^3 \frac{1}{3}(3-x)dx = \frac{1}{2}$. But $\int_M^3 \frac{1}{3}(3-x)dx = \int_M^3 \frac{d}{dx}\left(-\frac{1}{6}\{3-x\}^2\right)dx = -\frac{1}{6}\{3-x\}^2\Big|_M^3 = -\frac{1}{6}0^2 + \frac{1}{6}\{3-M\}^2 = \frac{1}{6}\{M-3\}^2$. So $\frac{1}{6}\{M-3\}^2 = \frac{1}{2}$, implying $\{M-3\}^2 = 3 \Rightarrow M-3 = -\sqrt{3}$ (because, clearly, $M < \sqrt{3}$). So $M = 3 - \sqrt{3} = 1.268$.

26.3 $\frac{d}{du}\{e^{-u}\} = -e^{-u}$ follows from Exercise 22.5 with $\lambda = -1$. The product rule yields

$$\begin{aligned}\frac{d}{du}\left\{-(u+1)e^{-u}\right\} &= \frac{d}{du}\left\{(u+1)(-e^{-u})\right\}\\ &= \left\{\frac{d}{du}(u+1)\right\}\cdot(-e^{-u}) + (u+1)\frac{d}{du}(-e^{-u})\\ &= \{1+0\}\cdot(-e^{-u}) + (u+1)e^{-u} = ue^{-u}.\end{aligned}$$

Now, from (26) with $x = 1$ and the fundamental theorem,

$$\begin{aligned}\Gamma(1) &= \int_0^\infty u^{1-1}e^{-u}\,du = \int_0^\infty \frac{d}{du}\left\{-e^{-u}\right\}du\\ &= -e^{-u}\Big|_0^\infty = \lim_{x\to\infty} -e^{-u}\Big|_0^x = \lim_{x\to\infty}\left\{-e^{-x}-(-e^{-0})\right\}\\ &= \lim_{x\to\infty}\left\{1-e^{-x}\right\} = \lim_{x\to\infty}\left\{1-\frac{1}{e^x}\right\} = 1\end{aligned}$$

because e^x becomes arbitrarily large as $x \to \infty$. Similarly, from (26) with $x = 2$ and the fundamental theorem,

$$\begin{aligned}\Gamma(2) &= \int_0^\infty u^{2-1}e^{-u}\,du = \int_0^\infty ue^{-u}\,du = \int_0^\infty \frac{d}{du}\left\{-(u+1)e^{-u}\right\}du\\ &= -(u+1)e^{-u}\Big|_0^\infty = \lim_{x\to\infty} -(u+1)e^{-u}\Big|_0^x = \lim_{x\to\infty}\left\{-(x+1)e^{-x} + (0+1)e^0\right\}\\ &= \lim_{x\to\infty}\left\{1-(x+1)e^{-x}\right\} = \lim_{x\to\infty}\left\{1 - \frac{x+1}{e^x}\right\} = 1\end{aligned}$$

because e^x approaches infinity much more rapidly than $x + 1$ as $x \to \infty$ from Figure 7.2 (or Appendix 26).

26.5 (i) $L = 4/3$ (ii) $m = 1$ (iii) $\mu = 1$ (iv) $F(t) = \frac{1}{4}t^2(3-t)$

26.6 (i) $L = 4/3$ (ii) $m = 4/3$ (iii) $\mu = 6/5$ (iv) $F(t) = \frac{1}{16}t^3(8-3t)$

26.7 (i) Define g by

$$g(x) = \begin{cases} x(2-x)^2 & \text{if } 0 \le x < 2 \\ 0 & \text{if } 2 \le x < \infty \end{cases}$$

Then $f(x) = g(x)/L$. So

$$\int_0^\infty f(x)\,dx = 1 \Rightarrow \int_0^\infty \frac{g(x)}{L}\,dx = 1 \Rightarrow \frac{1}{L}\int_0^\infty \frac{g(x)}{L}\,dx = 1,$$

implying

$$\begin{aligned}
L &= \int_0^\infty g(x)\,dx = \int_0^2 x(2-x)^2\,dx + \int_2^\infty 0\,dx \\
&= \int_0^2 x\{4-4x+x^2\}dx + 0 \\
&= \int_0^2 \{4x-4x^2+x^3\}dx \\
&= \int_0^2 \frac{d}{dx}\{2x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{4}x^4\}\,dx \\
&= 2x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{4}x^4\Big|_0^2 \\
&= 2\cdot 2^2 - \tfrac{4}{3}2^3 + \tfrac{1}{4}2^4 - \{2\cdot 0^2 - \tfrac{4}{3}0^3 + \tfrac{1}{4}0^4\} = \frac{4}{3}.
\end{aligned}$$

(ii) Clearly, $0 < m < 2$. For $x < 2$, we have $f(x) = \frac{3x}{4}(2-x)^2 \Rightarrow$

$$\begin{aligned}
f'(x) &= \frac{3}{4}\frac{d}{dx}\left\{x(2-x)^2\right\} \\
&= \frac{3}{4}\left\{\frac{d}{dx}\{x\}\cdot(2-x)^2 + x\frac{d}{dx}\left\{2-x)^2\right\}\right\} \\
&= \frac{3}{4}\left\{1\cdot(2-x)^2 + x\left\{2(2-x)\frac{d}{dx}(2-x)\right\}\right\} \\
&= \frac{3}{4}\left\{1\cdot(2-x)^2 + x\{2(2-x)(-1)\}\right\} \\
&= \frac{3}{4}(2-x)\{2-x+x(-2)\} = \frac{3}{4}(2-x)\{2-3x\},
\end{aligned}$$

on using the product rule. So $f'(x) > 0$ if $0 < x < 2/3$ but $f'(x) < 0$ if $2/3 < x < 2$, implying that f has a maximum $8/9$ where $x = 2/3$. This maximizer is the mode. That is, $m = 2/3$.

(iii) From (12),

$$\mu = \int_0^\infty x f(x)\,dx = \int_0^2 x\left\{\frac{3}{4}x(2-x)^2\right\}dx + \int_2^\infty x\,0\,dx$$
$$= \frac{3}{4}\int_0^2 x^2(2-x)^2\,dx + 0$$
$$= \frac{3}{4}\int_0^2 x^2(4-4x+x^2)\,dx$$
$$= \frac{3}{4}\int_0^2 \{4x^2-4x^3+x^4\}\,dx$$
$$= \frac{3}{4}\int_0^2 \frac{d}{dx}\{\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5\}\,dx$$
$$= \frac{3}{4}\left(\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5\Big|_0^2\right) = \frac{3}{4}\left(\tfrac{4}{3}\cdot 2^3 - 2^4 + \tfrac{1}{5}2^5 - 0\right)$$
$$= \frac{3}{4}\left(2^3\left(\tfrac{4}{3} - 2 + \tfrac{1}{5}2^2\right)\right) = 2^3\left(\tfrac{4}{3} - 2 + \tfrac{1}{5}2^2\right) = \frac{3}{4}\cdot 8\cdot\frac{2}{15} = \frac{4}{5}.$$

(iv) If $t > 2$ then $F(t) = 1$. If $0 \le t \le 2$, then

$$F(t) = \int_0^t f(x)\,dx = \int_0^t \tfrac{3}{4}x(2-x)^2\,dx$$
$$= \frac{3}{4}\int_0^t \frac{d}{dx}\{2x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{4}x^4\}\,dx$$
$$= \frac{3}{4}\left(2x^2 - \tfrac{4}{3}x^3 + \tfrac{1}{4}x^4\Big|_0^t\right) = \frac{3}{4}\left(2t^2 - \tfrac{4}{3}t^3 + \tfrac{1}{4}t^4 - 0\right)$$
$$= \tfrac{3}{2}t^2 - t^3 + \tfrac{3}{16}t^4.$$

Note that $F(2) = 1$.

26.8 (i) Define g by

$$g(x) = \begin{cases} x^2(2-x)^2 & \text{if } 0 \le x < 2 \\ 0 & \text{if } 2 \le x < \infty \end{cases}$$

Then $f(x) = g(x)/L$. So, as in the previous exercise, $L = \text{Int}(g, [0, \infty))$, implying

$$L = \int_0^2 x^2(2-x)^2\,dx + \int_2^\infty 0\,dx$$
$$= \int_0^2 x^2\{4-4x+x^2\}\,dx + 0$$
$$= \int_0^2 \{4x^2-4x^3+x^4\}\,dx$$

$$= \int_0^2 \frac{d}{dx}\{\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5\}\,dx$$

$$= \tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5\Big|_0^2$$

$$= \tfrac{4}{3}\cdot 2^3 - 2^4 + \tfrac{1}{5}2^5 \; - \; \{\tfrac{4}{3}\cdot 0^3 - 0^4 + \tfrac{1}{5}0^5\} \; = \; \frac{16}{15}.$$

(ii) Clearly, $0 < m < 2$. For $x < 2$, we have $f(x) \; = \; \dfrac{15x^2}{16}(2-x)^2 \; \Rightarrow$

$$f'(x) \; = \; \frac{15}{16}\frac{d}{dx}\left\{x^2(2-x)^2\right\}$$

$$= \; \frac{15}{16}\left\{\frac{d}{dx}\left\{x^2\right\}\cdot(2-x)^2 + x^2\frac{d}{dx}\left\{2-x)^2\right\}\right\}$$

$$= \; \frac{15}{16}\left\{2x\cdot(2-x)^2 + x^2\{2(2-x)(-1)\}\right\}$$

$$= \; \frac{15}{16}x(2-x)\{2(2-x)+2x(-1)\} \; = \; \frac{15}{4}x(2-x)\{1-x\}.$$

So $f'(x) > 0$ when $0 < x < 1$ but $f'(x) < 0$ when $1 < x < 2$, implying that f has a maximum 15/16 where $x = 1$. This maximizer is the mode. That is, $m = 1$.

(iii) From (12),

$$\mu \; = \; \int_0^\infty x f(x)dx \; = \; \int_0^2 x\left\{\frac{15}{16}x^2(2-x)^2\right\}dx$$

$$= \; \frac{15}{16}\int_0^2 x^3(2-x)^2\,dx$$

$$= \; \frac{15}{16}\int_0^2 x^3(4-4x+x^2)dx$$

$$= \; \frac{15}{16}\int_0^2 \{4x^3 - 4x^4 + x^5\}dx$$

$$= \; \frac{15}{16}\int_0^2 \frac{d}{dx}\{x^4 - \tfrac{4}{5}x^5 + \tfrac{1}{6}x^6\}\,dx$$

$$= \; \frac{15}{16}\left(x^4 - \tfrac{4}{5}x^5 + \tfrac{1}{6}x^6\Big|_0^2\right) \; = \; \frac{15}{16}\left(2^4 - \tfrac{4}{5}2^5 + \tfrac{1}{6}2^6 \; - \; 0\right) \; = \; 1.$$

A simpler way to show that $m = 1 = \mu$ in this case will emerge in Lecture 27.

(iv) If $t > 2$ then $F(t) = 1$. If $0 \le t \le 2$, then

$$\begin{aligned} F(t) &= \int_0^t f(x)\,dx = \int_0^t \tfrac{15}{16} x^2 (2-x)^2\,dx \\ &= \frac{15}{16}\int_0^t \frac{d}{dx}\{\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5\}\,dx \\ &= \frac{15}{16}\left(\tfrac{4}{3}x^3 - x^4 + \tfrac{1}{5}x^5\Big|_0^t\right) = \frac{15}{16}\left(\tfrac{4}{3}t^3 - t^4 + \tfrac{1}{5}t^5 - 0\right) \\ &= \frac{1}{16}t^3(20 - 15t + 3t^2). \end{aligned}$$

Note that $F(2) = 1$.

26.9 (i) $L = s^c\Gamma(c)$ (ii) See (27.28)

(iii) On using the product rule,

$$\begin{aligned} f'(x) &= \frac{1}{L}\frac{d}{dx}\left\{x^{c-1}e^{-x/s}\right\} \\ &= \frac{1}{L}\left\{\frac{d}{dx}\left\{x^{c-1}\right\}e^{-x/s} + x^{c-1}\frac{d}{dx}\left\{e^{-x/s}\right\}\right\} \\ &= \frac{1}{L}\left\{(c-1)x^{c-2}e^{-x/s} + x^{c-1}\left\{-\frac{1}{s}e^{-x/s}\right\}\right\} \\ &= \frac{x^{c-2}e^{-x/s}}{L}\left\{c - 1 - \frac{x}{s}\right\}. \end{aligned}$$

So $f'(x) > 0$ when $0 < x < (c-1)s$ but $f'(x) < 0$ when $(c-1)s < x < \infty$, implying that f has a maximum where $x = (c-1)s$. That is, $m = (c-1)s$.