22. Properties of exponential and logarithm. The empirical basis of allometry

In this lecture we establish important properties of exp and ln and apply them to the study of allometry, i.e., "growth of a part at a different rate from that of a body as a whole or of a standard" (Huxley and Teissier, 1936). The part that grows at a different rate is often a horn or other appendage, in which case it is called the allometric organ. The part exhibits positive or negative allometry according to whether it grows faster or slower than the standard. For example, analysis of small male fiddler crabs *(Uca pugnax)* predicts that a 20% increase in body weight would be accompanied by a 34% increase in weight of the great claw (Huxley, 1932, pp. 10-12); whereas a similar analysis of stick insects *(Dixippus morosus)* predicts that a 20% increase in total length would be accompanied by only a 9% increase in eye diameter (Huxley, 1932, p. 27).[1] So the great claw of a fiddler crab is a positively allometric organ, whereas the eye of a stick insect is a negatively allometric organ.

We begin with properties of ln. In Lecture 7 we defined the logarithm as the inverse of the exponential function. Later, in Lecture 20, we showed that

$$\frac{d}{dx}\{\ln(x)\} \quad = \quad \frac{1}{x} \tag{22.1}$$

on [1, ∞). So it follows from the fundamental theorem that

$$\ln(x) \quad - \quad \ln(1) \quad = \quad \int_1^x \frac{1}{t}\,dt \tag{22.2}$$

on [1, ∞). But ln(1) = 0, from (7.14). So an alternative definition of ln on [1, ∞) is

$$\ln(x) \quad = \quad \int_1^x \frac{1}{t}\,dt. \tag{22.3}$$

This formula provides an explicit definition of ln, which is very handy for theoretical purposes (although the implicit definition, that ln is the inverse of exp, is generally more useful for practical purposes).

An advantage of (3) is that it extends the domain of definition of ln from [1, ∞) to (0, ∞), provided we first agree on what we mean by Int(f, [b, a]) for b > a. Any sensible definition of Int(f, [b, a]) for b > a must preserve the truth of the fundamental theorem, i.e., it must ensure Int(f, [b, a]) = F(a) – F(b), where F is an antiderivative of f. But the fundamental theorem already yields F(b) – F(a) = Int(f, [a, b]), and F(a) – F(b) = –{F(b) – F(a)}. So we define Int(f, [b, a]) = – Int(f, [a, b]) or, in Leibniz notation,

$$\int_b^a f(x)\,dx \quad = \quad -\int_a^b f(x)\,dx. \tag{22.4}$$

Now ln(x) has meaning not only when $1 \le x < \infty$ but also when $0 < x < 1$. Specifically,

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[1] Small means less than 1 gm, and body weight excludes the weight of the large chela. For *U. pugnax*, Huxley estimated $\beta = 1.62$ in (32); thus increasing body weight by a factor of 1.20 increases chela weight by a factor of $1.20^{1.62} = 1.34$. For *D. morosus*, Huxley quotes $\beta = 0.48$ (apparently from Teissier); thus increasing body length by a factor of 1.20 increases eye diameter by a factor of only $1.20^{0.48} = 1.09$. Note that Huxley used "heterogony" in place of allometry in his earlier work.

$$\ln(x) = \int_1^x \frac{1}{t}\,dt = \begin{cases} -\int_x^1 \frac{1}{t}\,dt & 0 < x < 1 \\ \int_1^x \frac{1}{t}\,dt & 1 \le x < \infty. \end{cases} \qquad (22.5)$$

The extended graph of ln is shown in Figure 1. Its domain of ln is now $(0, \infty)$; and its range is now $(-\infty, \infty)$, by which we mean that every positive or negative number is a label for ln (precisely once). But ln remains strictly increasing and therefore invertible, and its inverse is still called exp. So extending the domain and range of ln from $[1, \infty)$ and $[0, \infty)$ to $(0, \infty)$ and $(-\infty, \infty)$, respectively, automatically extends the domain and range of exp from $[0, \infty)$ and $[1, \infty)$ to $(-\infty, \infty)$ and $(0, \infty)$, respectively; see Table 1. The extended graph of exp is shown in Figure 2. Note that exp is always strictly positive.

ln	OLD	NEW	exp	OLD	NEW
DOMAIN	$(1,\infty)$	$(0,\infty)$	DOMAIN	$(0,\infty)$	$(-\infty,\infty)$
RANGE	$(0,\infty)$	$(-\infty,\infty)$	RANGE	$(1,\infty)$	$(0,\infty)$

Table 22.1 Extending the domains and ranges of exp and ln

The function ln has two very nice properties. The first is that

$$\ln(1/x) = -\ln(x) \qquad (22.6)$$

for any positive x. To establish this result, we fix x and substitute

$$u = \phi(t) = \frac{1}{t} \qquad (22.7)$$

in the definition

$$\ln(1/x) = \int_1^{1/x} \frac{1}{t}\,dt. \qquad (22.8)$$

Then $t = 1/u$. So from Lecture 21 the inverse substitution is $\zeta(u) = 1/u = u^{-1}$, implying $\zeta'(u) = -u^{-2}$; and, from (21.21) with $f(t) = 1/t$, $a = 1$ and $b = 1/x$,

$$\begin{aligned} \int_1^{1/x} \frac{1}{t}\,dt &= \int_a^b f(t)\,dt = \int_{\phi(a)}^{\phi(b)} f(\zeta(u))\,\zeta'(u)\,du \\ &= \int_{\phi(1)}^{\phi(1/x)} \frac{1}{\zeta(u)}\,\zeta'(u)\,du \\ &= \int_1^x \frac{1}{u^{-1}}\{-u^{-2}\}\,du \\ &= -\int_1^x \frac{1}{u}\,du = -\int_1^x \frac{1}{t}\,dt \\ &= -\ln(x), \end{aligned} \qquad (22.9)$$

as required. Note, incidentally, that ϕ is a decreasing function. Although we assumed in Lecture 21 that ϕ is an increasing function, it turns out that (21.21) holds also when ϕ is decreasing (and therefore whenever ϕ is invertible). Details are in Appendix 22.

The second nice property is that

$$\ln(wx) \quad = \quad \ln(w) \quad + \quad \ln(x) \tag{22.10}$$

for any positive w or x. To establish this result, we fix w, x and substitute

$$u \quad = \quad \phi(t) \quad = \quad \frac{t}{x} \tag{22.11}$$

in the definition

$$\ln(wx) \quad = \quad \int_1^{wx} \frac{1}{t}\,dt. \tag{22.12}$$

Then t = ux. So from Lecture 21 the inverse substitution is $\zeta(u) = xu$, implying $\zeta'(u) = x$; and, from (21.21) with $f(t) = 1/t$, $a = 1$ and $b = wx$,

$$\begin{aligned}
\int_1^{wx} \frac{1}{t}\,dt \quad = \quad \int_a^b f(t)\,dt \quad &= \quad \int_{\phi(a)}^{\phi(b)} f(\zeta(u))\,\zeta'(u)\,du \\
&= \quad \int_{\phi(1)}^{\phi(wx)} \frac{1}{\zeta(u)}\,\zeta'(u)\,du \\
&= \quad \int_{1/x}^{w} \frac{1}{xu}\,x\,du \\
&= \quad \int_{1/x}^{w} \frac{1}{u}\,du \\
&= \quad \ln(u)\Big|_{1/x}^{w} \\
&= \quad \ln(w) \quad - \quad \ln(1/x) \\
&= \quad \ln(w) \quad + \quad \ln(x),
\end{aligned} \tag{22.13}$$

from (6). In particular, on setting x = w, we have $\ln(x^2) = \ln(x) + \ln(x) = 2\ln(x)$, so that $w = x^2$ yields $\ln(x^3) = \ln(x^2) + \ln(x) = 2\ln(x) + \ln(x) = 3\ln(x)$, and so on. We conjecture the more general result that

$$\ln(x^m) \quad = \quad m\ln(x) \tag{22.14}$$

for any integer m, which is easy to prove by mathematical induction; see Exercise 2. In fact, (14) holds even if m is not an integer; see Exercise 3.

These two nice properties of ln are paralleled by two nice properties of exp, both of which follow directly from the definition of inverse; more precisely, from

$$\exp(\ln(\bullet)) \quad = \quad \bullet \tag{22.15}$$

for any $\bullet$ in the domain of ln (i.e., any positive $\bullet$) and

$$\ln(\exp(\oplus)) \quad = \quad \oplus \tag{22.16}$$

for any $\oplus$ in the domain of exp (i.e., any real number). Because $\ln(1/x) = -\ln(x)$ for any positive x and $\exp(x) > 0$, we have

$$\ln(1/\exp(x)) \quad = \quad -\ln(\exp(x)) \quad = \quad -x, \tag{22.17}$$

from (16). Directly from (17),

$$\exp\big(\ln(1/\exp(x))\big) \quad = \quad \exp(-x), \tag{22.18}$$

But from (15) with $\bullet = 1/\exp(x)$ we have $\exp(\ln(1/\exp(x))) \; = \; 1/\exp(x)$. So (18) implies

$$\exp(-x) = \frac{1}{\exp(x)}. \tag{22.19}$$

This property is illustrated in Figure 3.

Again, because ln(🍎) = ln(🍎 + ln(•) for any positive 🍎 and •, and because exp is strictly positive, we have

$$\begin{aligned}\ln\big(\exp(w)\exp(x)\big) &= \ln\big(\exp(w)\big) + \ln\big(\exp(x)\big) \\ &= w + x\end{aligned} \tag{22.20}$$

by (16), implying

$$\exp\Big(\ln\big(\exp(w)\exp(x)\big)\Big) = \exp(w + x). \tag{22.21}$$

But from (15) with • = exp(w)exp(x) we have exp(ln(exp(w)exp(x))) = exp(w)exp(x). So (21) implies

$$\exp(w + x) = \exp(w)\exp(x) \tag{22.22}$$

for any w or x. In particular, on setting w = x, we have $\exp(2x) = \{\exp(x)\}^2$, so that w = 2x yields $\exp(3x) = \exp(2x + x) = \exp(2x)\exp(x) = \{\exp(x)\}^2\exp(x) = \{\exp(x)\}^3$, and so on. In this way we obtain the more general result that

$$\exp(mx) = \{\exp(x)\}^m \tag{22.23}$$

for any integer m; see Exercise 4.

Now, if w and x are integers or rational numbers and c > 0, then

$$c^{-x} = \frac{1}{c^x}, \quad c^0 = 1, \quad c^{w+x} = c^w c^x. \tag{22.24}$$

But we have established that

$$\exp(-x) = \frac{1}{\exp(x)}, \quad \exp(0) = 1, \quad \exp(w + x) = \exp(w)\exp(x). \tag{22.25}$$

A comparison shows that the effect of exp, acting on x, is to raise some number, say e, to the power of x. Whatever number e is, if we raise e to the power of 1 then we must obtain e itself. Hence

$$e = \exp(1) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \approx 2.718281828 \tag{22.26}$$

and

$$\exp(x) = e^x. \tag{22.27}$$

Now we know why exp is called the exponential function.

Given these properties, for any x and c > 0 we *define* c to the power of x by

$$c^x = e^{x\ln(c)}. \tag{22.28}$$

With this definition, properties of exponents are extended at once from rational numbers to all real numbers. For example, we have $(c^x)^w = (c^w)^x$ because

$$\begin{aligned}(c^x)^w &= (e^{x\ln(c)})^w = e^{w\ln(e^{x\ln(c)})} = e^{wx\ln(c)} \\ &= e^{xw\ln(c)} = e^{x\ln(e^{w\ln(c)})} = (e^{w\ln(c)})^x \\ &= (c^w)^x.\end{aligned} \tag{22.29}$$

Moreover, the derivatives of (28) and of any power function follow immediately from the chain rule because

$$\begin{aligned}\frac{d}{dx}\left(c^x\right) &= \frac{d}{dx}\left(e^{x\ln(c)}\right) = e^{x\ln(c)}\frac{d}{dx}(x\ln(c)) \\ &= e^{x\ln(c)}\ln(c) = c^x\ln(c)\end{aligned} \tag{22.30}$$

and

$$\frac{d}{dx}\{x^\beta\} = \frac{d}{dx}\{e^{\beta\ln(x)}\} = e^{\beta\ln(x)}\frac{d}{dx}\{\beta\ln(x)\} = x^\beta\frac{d}{dx}\{\beta\ln(x)\} = x^\beta\beta\frac{1}{x} = \beta x^{\beta-1} \qquad (22.31)$$

for arbitrary exponent β. Properties of exp and ln are summarized in Table 2.

$e^{-x} = \frac{1}{e^x}, \quad -\infty < x < \infty$	$\ln(\frac{1}{x}) = -\ln(x), \quad 0 < x < \infty$
$e^{w+x} = e^w e^x, \quad -\infty < w < \infty, -\infty < x < \infty$	$\ln(wx) = \ln(w) + \ln(x), \quad 0 < w, x < \infty$
$(e^x)^\beta = e^{\beta x}, \quad -\infty < x < \infty$	$\ln(x^\beta) = \beta\ln(x), \quad 0 < x < \infty$
$e^{\ln(x)} = x, \quad 0 < x < \infty$	$\ln(e^x) = x, \quad -\infty < x < \infty$
$\frac{d}{dx}(e^x) = e^x, \quad -\infty < x < \infty$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad 0 < x < \infty$

Table 22.2 Properties of exponential and logarithm

The properties of exp and ln are the basis of allometry, which in essence is the study of how relative sizes differ according to the formula

$$y = \alpha x^\beta, \qquad (22.32)$$

where x is the standard (e.g., an animal's body weight), y is the magnitude of the part that grows at a different rate (e.g., an antler) and α, β are parameters. According to (32), if a standard increases in magnitude from x to kx, then an allometric organ increases in magnitude from $y = \alpha x^\beta$ to $\alpha(kx)^\beta = \alpha k^\beta x^\beta = k^\beta(\alpha x^\beta) = k^\beta y$. So increasing the standard by factor k increases the magnitude of the allometric organ by factor k^β, and relative growth is positively or negatively allometric according to whether $\beta > 1$ or $\beta < 1$.

Allometry depends on exp and ln in at least two ways. First, by the properties of these two functions, (32) is well defined for any β as

$$y = \alpha e^{\beta\ln(x)}. \qquad (22.33)$$

Second, by the same properties, (33) is equivalent to

$$\ln(y) = \ln\left(\alpha e^{\beta\ln(x)}\right) = \ln(\alpha) + \ln\left(e^{\beta\ln(x)}\right) = \ln(\alpha) + \beta\ln(x). \qquad (22.34)$$

That is, if $y = \alpha x^\beta$, then ln(y) is a linear function of ln(x). So we can test the hypothesis that x and y are related allometrically by plotting (ln(x), ln(y)) data pairs and drawing the straight line that fits them best. The closer the fit, the more confidence we have in the hypothesis.

Total length	# specimens	Mean total length, x	Mean head length, y	ln(x)	ln(y)
17-18 m	13	17.5 m	2.77 m	2.862	1.019
18-19 m	14	18.5 m	2.97 m	2.918	1.089
19-20 m	18	19.5 m	3.24 m	2.97	1.176
20-21 m	14	20.5 m	3.52 m	3.02	1.258
21-22 m	16	21.5 m	3.76 m	3.068	1.324
22-23 m	21	22.5 m	3.99 m	3.114	1.384
23-24 m	38	23.5 m	4.33 m	3.157	1.466
24-25 m	56	24.5 m	4.67 m	3.199	1.541
25-26 m	25	25.5 m	4.78 m	3.239	1.564
26-27 m	3	26.5 m	5.04 m	3.277	1.617

Table 22.3 Head and body lengths of 218 male blue whales

Taxon or taxa	Whole or standard	Organ or other part	β	Source
Wax moth	Dry weight	Total fat	1.32	Huxley (1932, p. 30)
Mouse	Total length	Tail length	1.41	Huxley (1932, p. 22)
Sheep dog	Cranial length	Facial length	1.5	Huxley (1932, p. 18)
Various plants	Stem diameter	Leaf area	1.84	Niklas (1994, p. xv)
Stagbeetle	Length of elytron	Head breadth	2.0	Huxley (1932, p. 25)
Mature rabbits	Adrenal weight	Testis weight	2.3	Huxley (1932, p. 257)

Table 22.4 Some examples of positive allometry

Taxon or taxa	Whole or standard	Organ or other part	β	Source
Crayfish	Carapace length	Ommatidium diameter	0.4	Huxley (1932, p. 27)
Cancer pagurus	Carapace breadth	Interocular breadth	0.7	Huxley (1932, p. 27)
Immature rabbits	Adrenal weight	Testis weight	0.74	Huxley (1932, p. 257)
Carcinus maenas	Carapace breadth	Interocular breadth	0.85	Huxley (1932, p. 27)
Various plants	Stem diameter	Overall plant height	0.896	Niklas (1994, p. xv)
Various plants	Fruit mass	Average seed mass	0.93	Niklas (1994, p. xvi)

Table 22.5 Some examples of negative allometry

For example, Table 3, which is adapted from Huxley (1932, p. 136), shows some of N.A. Mackintosh and J.F.G. Wheeler's on 218 male blue whales *(Balaenoptera musculus)* . Figure 4, where ln(y) is plotted against ln(x), shows how closely the data points fall to a straight line, especially for whales up to 25 m in total length. The line, fitted by the method of Appendix 2A (for whales up to 25 m), has equation

$$\ln(y) = 1.545\{\ln(x) - 2.209\}. \qquad (22.35)$$

Comparing with (34), $\beta = 1.54$ and $\ln(\alpha) = -3.41$, implying $\alpha = \exp(-3.41) = 0.033$. So

$$y = 0.033x^{1.54} \qquad (22.36)$$

appears to provide an excellent description of relative head size, except perhaps for the largest whales. The model predicts, for example, that 20% more body would imply 33% more head, because $1.2^\beta = 1.33$. Further examples of both positive and negative allometry appear in Tables 4-5. Note in particular that the sign of allometry may change during an organism's life history.

Restrictions	DERIVATIVE on $[a,b]$, $b>a$	ANTIDERIVATIVE on $[a,b]$, $b>a$	SOURCE
	$\frac{d}{dx}\{\exp(cx)\} = c\exp(cx)$	$\int_a^x \exp(ct)\,dt = \frac{1}{c}\exp(cx) + \text{const}$	Exercise 5
$c>0$	$\frac{d}{dx}\{c^x\} = c^x\ln(c)$	$\int_a^x c^t\,dt = \frac{c^x}{\ln(c)} + \text{const}$	(30)
$a>0$ if $\beta<1$ or not integer	$\frac{d}{dx}\{x^\beta\} = \beta x^{\beta-1}$	$\int_a^x t^{\beta-1}\,dt = \frac{x^\beta}{\beta} + \text{const},\ \beta\neq 0$	(31)
$a>0$	$\frac{d}{dx}\{\ln(x)\} = \frac{1}{x}$	$\int_a^x t^{-1}\,dt = \ln(x) + \text{const}$	Table 2
$a>c$	$\frac{d}{dx}\{\ln(x-c)\} = \frac{1}{x-c}$	$\int_a^x (t-c)^{-1}\,dt = \ln(x-c) + \text{const}$	Exercise 20.1
$b<c$	$\frac{d}{dx}\{\ln(c-x)\} = -\frac{1}{c-x}$	$\int_a^x (c-t)^{-1}\,dt = -\ln(c-x) + \text{const}$	Exercise 6
$a>0, b<c$	$\frac{d}{dx}\left\{\ln\left(\frac{x}{c-x}\right)\right\} = \frac{c}{x(c-x)}$	$\int_a^x \frac{1}{t(c-t)}\,dt = \frac{1}{c}\ln\left(\frac{x}{c-x}\right) + \text{const}$	Exercise 7
$\beta\neq 0$; $a>c$ if $\beta<1$ or not integer	$\frac{d}{dx}\{(x-c)^\beta\} = \beta(x-c)^{\beta-1}$	$\int_a^x (t-c)^{\beta-1}\,dt = \frac{(x-c)^\beta}{\beta} + \text{const}$	Exercise 8

Table 22.6 Some derivatives and integrals considered known by the end of this lecture

We will postpone a discussion of allometry's conceptual basis until Lecture 29. Meanwhile, note that properties of exp and ln have two other consequences. First, it is no longer necessary for the Weibull distribution's shape parameter, say c, to be an integer; with s as scale parameter, we can now write its c.d.f. and p.d.f. as

$$F(x) = 1 - \exp(-\{x/s\}^c) = 1 - e^{-(x/s)^c} \tag{22.37}$$

$$f(x) = \frac{c}{s}(x/s)^{c-1}\exp(-\{x/s\}^c) = \frac{c}{s}(x/s)^{c-1}e^{-(x/s)^c}, \tag{22.38}$$

less cumbersome than (20.33)-(20.34). Second, the properties enable us considerably to expand our list of known integrals and derivatives. We pursue this matter largely in the exercises, but the main results are recorded for convenience in Table 6.

References

Huxley, Julian S. (1932). Problems of Relative Growth. The Dial Press, New York
Huxley, J. S. & G. Teissier (1936). Nature **137**, 780-781.
Niklas, Karl J. (1994). Plant Allometry. University of Chicago Press

Exercises 22

22.1 Use (6) and (10) to find a simpler expression for $\ln(w) - \ln(z)$, where w and z are both positive.

22.2 Use mathematical induction (Appendix 17B) to prove that $\ln(x^m) = m\ln(x)$ for any nonnegative integer m.

22.3 Show that $\ln(x^\beta) = \beta\ln(x)$ on $(0, \infty)$ even if β is not an integer. Hint: Use (28).

22.4 Use mathematical induction (Appendix 17B) to prove that $\exp(mx) = \{\exp(x)\}^m$ for any nonnegative integer m.

22.5 For f defined on $(-\infty, \infty)$ by $f(x) = e^{\lambda x}$ with λ a constant, what are f' and f''?

22.6 For f defined on $(-\infty, c)$ by $f(x) = \ln(c-x)$ with c a constant, what are f' and f''?

22.7 For f defined on (0, c) by

$$f(x) = \ln\left(\frac{x}{c-x}\right)$$

with c (> 0) a constant, what are f' and f''?

22.8 For f defined on (c, ∞) by $f(x) = (x-c)^\beta$ with c, β constant, what are f' and f''?

22.9 $F(x) = 1 - e^{-x}\left(1 + x + \frac{1}{2}x^2\right)$ defines a probability distribution on $[0, \infty)$. What is its probability density function?

22.10 $F(x) = 1 - e^{-x}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)$ defines a probability distribution on $[0, \infty)$. What is its probability density function?

22.11 $F(x) = 1 - e^{-x}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4\right)$ defines a probability distribution on $[0, \infty)$. What is its probability density function?

22.12 Using the chain rule in conjunction with the product rule and properties of exponentials and logarithms, find $F'(x)$ in each of the following cases:

(i) $F(x) = \left(e^{(x^3+4x^2+2x+1)}\right)^2$ (ii) $F(x) = x^5 e^{(x^3+4x^2+2x+1)}$

(iii) $F(x) = \ln\left(x^3 + 4x^2 + 2x + 1\right)$ (iv) $F(x) = \ln\left(x^5 e^{(x^3+4x^2+2x+1)}\right)$

22.13 (i) For $c \geq 0$ and $t > 0$, use the chain rule to calculate $\frac{d}{dt}\{\ln(c+t)\}$.

(ii) What is $\frac{d}{dt}\{2\ln(4+t) - \ln(1+t)\}$? Simplify your answer.

(iii) A function G is defined on $(0, \infty)$ by

$$G(t) = \int_1^t \frac{x-2}{(x+1)(x+4)}\,dx.$$

Using properties of logarithms, deduce from (ii) that $G(t) = \ln\left\{\frac{2(4+t)^2}{25(1+t)}\right\}$.

(iv) On which subdomains is G positive? On which subdomains is G negative?

22.14 (i) For $c \geq 0$ and $t > 0$, use the chain rule to calculate $\frac{d}{dt}\{\ln(c+t)\}$.

(ii) What is $\frac{d}{dt}\{4\ln(1+t) - 3\ln(t)\}$? Simplify your answer.

(iii) A function F is defined on $(0, \infty)$ by

$$F(t) = \int_1^t \frac{x-3}{x(x+1)}\,dx.$$

Using properties of logarithms, deduce from (ii) that $F(t) = \ln\left\{\frac{(1+t)^4}{16t^3}\right\}$.

(iv) On which subdomains is F positive? On which subdomains is F negative?

22.15 For f defined on $(0, c^2)$ by

$$f(x) = \ln\left(\frac{x^2}{c^2 - x^2}\right)$$

with c a constant, what are f' and f''?

22.16 The function G defined on $[0, \infty)$ by $G(t) = \ln\left(1+t^4\right)$ is known to satisfy

$$\frac{G(t+h) - G(t)}{h} = \frac{4t^3}{1+t^4} + O[h]$$

as $h \to 0+$. What must be the value of

$$\int_1^3 \frac{4x^3}{1+x^4}\,dx?$$

Write your answer as simply as possible.

22.17 A function f is defined on [2, 4] by $f(x) = 2 + \ln\left(\frac{x}{x^2+9}\right)$.

(i) Find the derivative, f'.
(ii) Hence find all global extrema and extremizers of f.

22.18 A function f is defined on [1, 4] by $f(x) = \ln\left(\frac{x^2+4}{x}\right)$.

(i) Find the derivative, f'.
(ii) Hence find all global extrema and extremizers of f.

22.19 For f defined on (0, c) by

$$f(x) = \ln\left(\frac{x^5}{c^6 - x^6}\right)$$

where c is a constant, what are f' and f''? Hence show that f is strictly increasing, with an inflection point where $x \approx 0.707c$.

Appendix 22: On integration by substitution

In the previous lecture we estabilished that

$$\int_a^b f(x)\,dx \;=\; \int_{\phi(a)}^{\phi(b)} f(\zeta(u))\,\zeta'(u)\,du \qquad (22.A1)$$

on the assumption that ϕ is increasing. If instead ϕ is decreasing, then (21.15) becomes

$$\mathrm{Prob}(\phi(b) \le A \le \phi(a)) \;=\; G(\phi(a)) - G(\phi(b)) \;=\; \int_{\phi(b)}^{\phi(a)} g(u)\,du \qquad (22.A2)$$

and $\mathrm{Prob}(R \le x) = \mathrm{Prob}(\phi(R) \ge \phi(x)) = \mathrm{Prob}(A \ge \phi(x)) = 1 - \mathrm{Prob}(A \le \phi(x))$, so that (21.17) becomes

$$F(x) \;=\; 1 - G(\phi(x)), \qquad (22.A3)$$

yielding

$$F(\zeta(u)) \;=\; 1 - G(u). \qquad (22.A4)$$

in place of (21.18). On differentiating with respect to u, we find that (21.20) is replaced by

$$f(\zeta(u))\cdot\zeta'(u) \;=\; -g(u). \qquad (22.A5)$$

Now, in place of (21.16), (A2) yields

$$\begin{aligned}\int_a^b f(x)\,dx \;&=\; \int_{\phi(b)}^{\phi(a)} g(u)\,du \\ &=\; -\int_{\phi(a)}^{\phi(b)} g(u)\,du \qquad\qquad . \qquad (22.A6)\\ &=\; -\int_{\phi(a)}^{\phi(b)} \{-f(\zeta(u))\cdot\zeta'(u)\}\,du,\end{aligned}$$

on using (4) and then (A5). So (A1) holds if ϕ is decreasing. But (A1) also holds if ϕ is decreasing, by Lecture 21. Therefore (A1) holds whenever ϕ is invertible.

Answers and Hints for Selected Exercises

22.1 By (6), we have $-\ln(z) = \ln(1/z)$. So

$$\ln(w) - \ln(z) = \ln(w) + \ln(1/z) = \ln(w \cdot 1/z) = \ln(w/z),$$

on using (10) with $x = 1/z$.

22.5 From Exercise 20.2 with $P(x)$ and hence $P'(x) = \lambda$, we have

$$\frac{d}{dx}\{e^{\lambda x}\} = \frac{d}{dx}\{e^{P(x)}\} = P'(x)e^{P(x)} = \lambda e^{\lambda x},$$

implying

$$f''(x) = \frac{d}{dx}\{f'(x)\} = \frac{d}{dx}\{\lambda e^{\lambda x}\} = \lambda\frac{d}{dx}\{e^{\lambda x}\} = \lambda \cdot \lambda e^{\lambda x} = \lambda^2 e^{\lambda x}.$$

22.6 From Exercise 20.3 with $P(x) = c - x$ we have

$$f'(x) = \frac{d}{dx}\{\ln(P(x))\} = = \frac{P'(x)}{P(x)} = \frac{-1}{c-x} = \frac{1}{x-c}.$$

So, on using the chain rule with $Q(y) = y^{-1} \Rightarrow Q'(y) = -y^{-2}$ and with P redefined by $P(x) = x - c \Rightarrow P'(x) = 1$, we have $f''(x) =$

$$\frac{d}{dx}\left\{\frac{1}{x-c}\right\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)Q'(P(x)) = 1\cdot\left(-\{P(x)\}^{-2}\right) = -\frac{1}{(x-c)^2}.$$

22.7 From Exercise 1 with $w = x$ and $z = c - x$, $f(x) = \ln(x) - \ln(c-x) = \ln(x) - Q(P(x))$, where $P(x) = c - x \Rightarrow P'(x) = -1$ and $Q(y) = \ln(y) \Rightarrow Q'(y) = 1/y$. Thus $Q'(P(x)) = 1/P(x)$ and, on using the chain rule,

$$f'(x) = \frac{d}{dx}\{\ln(x)\} - \frac{d}{dx}\{Q(P(x))\}$$
$$= \frac{1}{x} - P'(x)\cdot Q'(P(x)) = \frac{1}{x} + \frac{1}{c-x} = \frac{c}{x(c-x)}.$$

and, on using the product rule,

$$f''(x) = \frac{d}{dx}\{x^{-1}\} + \frac{d}{dx}\{(c-x)^{-1}\}$$

Using the chain rule again, still with $P(x) = c - x$ but now with $Q(y) = 1/y = y^{-1}$ $\Rightarrow Q'(y) = -y^{-2} \Rightarrow Q'(P(x)) = -\{P(x)\}^{-2}$, we have

$$\frac{d}{dx}\{(c-x)^{-1}\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)\cdot Q'(P(x))$$
$$= -1\cdot\{-P(x)^{-2}\} = \frac{1}{(c-x)^2}.$$

So, from above,

$$f''(x) = \frac{d}{dx}\{x^{-1}\} + \frac{d}{dx}\{(c-x)^{-1}\} = -x^{-2} + (c-x)^{-2} = \frac{c(2x-c)}{x^2(c-x)^2}.$$

22.9 $f(x) = F'(x) = \frac{1}{2}x^2e^{-x}$. See the solution to Exercise 10.

22.10 By the product rule, $F(x) = 1 - e^{-x}\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)$ implies

$$\begin{aligned} f(x) = F'(x) &= 0 - \frac{d}{dx}\left\{e^{-x}\left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\right)\right\} \\ &= -\left\{\frac{d}{dx}\left\{e^{-x}\right\}\cdot\left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\right) + e^{-x}\cdot\frac{d}{dx}\left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\right)\right\} \\ &= -\left\{-e^{-x}\left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3\right) + e^{-x}\cdot\left(0 + 1 + x + \tfrac{1}{2}x^2\right)\right\} \\ &= e^{-x}\left\{1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \left(1 + x + \tfrac{1}{2}x^2\right)\right\} = \frac{1}{6}x^3e^{-x} \end{aligned}$$

on making use of Exercise 5 with $\lambda = -1$.

22.11 $f(x) = F'(x) = \frac{1}{24}x^4e^{-x}$. See the solution to Exercise 10.

22.12 Set $\Omega(x) = x^3 + 4x^2 + 2x + 1$.

(i) By properties of the exponential function, we have

$$F(x) = \left(e^{\Omega(x)}\right)^2 = e^{\Omega(x)}\cdot e^{\Omega(x)} = e^{\{\Omega(x)+\Omega(x)\}} = e^{2\Omega(x)}.$$

From Exercise 20.2, we have

$$\frac{d}{dx}\left\{e^{P(x)}\right\} = P'(x)e^{P(x)}.$$

So, with $P(x) = 2\Omega(x)$,

$$F'(x) = 2\Omega'(x)e^{2\Omega(x)} = 2(3x^2 + 8x + 2)e^{2(x^3+4x^2+2x+1)}.$$

(ii) Here $F(x) = x^5e^{\Omega(x)}$. By the product rule and chain rule,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left\{x^5e^{\Omega(x)}\right\} = \frac{d}{dx}\left\{x^5\right\}\cdot e^{\Omega(x)} + x^5\cdot\frac{d}{dx}\left\{e^{\Omega(x)}\right\} \\ &= 5x^4\cdot e^{\Omega(x)} + x^5\cdot\Omega'(x)e^{\Omega(x)} \\ &= x^4e^{\Omega(x)}\{5 + x\Omega'(x)\} \\ &= x^4e^{\Omega(x)}\{5 + x(3x^2 + 8x + 2)\} \\ &= (3x^3 + 8x^2 + 2x + 5)x^4e^{(x^3+4x^2+2x+1)} \end{aligned}$$

Alternatively, because $x^5 = e^{5\ln(x)}$, we have $F(x) = e^{5\ln(x)}\, e^{5\ln(x)}e^{\Omega(x)} = e^{P(x)}$ with $P(x) = 5\ln(x) + \Omega(x)$. So

$$\begin{aligned} F'(x) &= P'(x)e^{P(x)} = \frac{d}{dx}\{5\ln(x) + \Omega(x)\}e^{P(x)} \\ &= \left(\frac{5}{x} + \Omega'(x)\right)e^{P(x)} = \left(\frac{5}{x} + \Omega'(x)\right)x^5e^{\Omega(x)} \end{aligned}$$

which yields the same result as above.

(iii) From Exercise 20.3,

$$\frac{d}{dx}\{\ln(\Omega(x))\} = \frac{\Omega'(x)}{\Omega(x)} = \frac{3x^2+8x+2}{x^3+4x^2+2x+1}.$$

(iv) By properties of the exponential and logarithm, we have,

$$F(x) = \ln\left(x^5 e^{\Omega(x)}\right) = \ln\left(x^5\right) + \ln\left(e^{\Omega(x)}\right) = 5\ln(x) + \Omega(x).$$

So

$$F'(x) = 5\frac{d}{dx}\{\ln(x)\} + \Omega'(x) = \frac{5}{x}+2+8x+3x^2$$

22.13 (i) Define P, Q and S by $P(t) = c + t$, $Q(y) = \ln(y)$ and $S(t) = \ln(c + t)$, so that $P'(t) = 0 + 1 = 1$, $Q'(y) = 1/y$ and $S(t) = Q(c + t) = Q(P(t))$. Then

$$\frac{d}{dt}\{\ln(c+t)\} = S'(t) = P'(t)Q'(P(t)) = 1\cdot\frac{1}{P(t)} = \frac{1}{c+t}.$$

(ii) On using the above result, first with $c = 4$, then with $c = 0$, we have

$$\frac{d}{dt}\{2\ln(4+t)-\ln(1+t)\} = 2\frac{d}{dt}\{\ln(4+t)\}-\frac{d}{dt}\{\ln(1+t)\} = 2\frac{1}{4+t} - \frac{1}{1+t} = \frac{t-2}{(t+1)(t+4)}$$

(iii) On using the above result together with the fundamental theorem and properties of logarithms,

$$\begin{aligned}G(t) &= \int_1^t \frac{x-2}{(x+1)(x+4)}\,dx = \int_1^t \frac{d}{dx}\{2\ln(4+x)-\ln(1+x)\}\,dx \\ &= 2\ln(4+x)-\ln(1+x)\Big|_1^t \\ &= 2\ln(4+t)-\ln(1+t) - \{2\ln(4+1)-\ln(1+1)\}. \\ &= \ln\left(\{4+t\}^2\right)-\ln(1+t) - 2\ln(5)+\ln(2) \\ &= \ln\left(\frac{\{4+t\}^2}{(1+t)}\right) - \ln(25)+\ln(2) = \ln\left\{\frac{2(4+t)^2}{25(1+t)}\right\}\end{aligned}$$

(iv) If $0 < x \le 1$, then $(x-2)/\{(x+1)(x+4)\} < 0$. So, for $0 < t < 1$, we must have

$$G(t) = -\int_t^1 \frac{x-2}{(x+1)(x+4)}\,dx < 0.$$

For $t \ge 1$, put $P(t) = 2(4+t)^2/\{25(1+t)\}$, so that $G(t) = \ln(P(t))$. Then $G(t)$ is positive or negative according to whether $P(t) > 1$ or $P(t) < 1$. But $P(t) > 1$ if and only if $2(4+t)^2 > 25(1+t)$ or $2t^2-9t+7 > 0$, i.e., $(t-1)(2t-7) > 0$. So $G(t) > 0$ if $t > 7/2$, but $G(t) < 0$ if $1 < t < 7/2$. In sum, G is positive on subdomains $(0, 1)$ and $(7/2, \infty)$ but negative on subdomain $(1, 7/2)$.

22.14 (iv) G is positive on subdomains $(0, 1)$ and (c, ∞) but negative on subdomain $(1, c)$, where c is the only positive root of the equation $c^3 - 11c^2 - 5c - 1 = 0$, i.e., $c = 11.44$.

22.15 By properties of logarithms, $f(x) = \ln(x^2) - \ln(c^2 - x^2) = 2\ln(x) - Q(P(x))$, where $P(x) = c^2 - x^2 \Rightarrow P'(x) = -2x$ and $Q(y) = \ln(y) \Rightarrow Q'(y) = 1/y \Rightarrow Q'(P(x)) = 1/P(x)$. So, on using the chain rule

$$f'(x) = 2\frac{d}{dx}\{\ln(x)\} - \frac{d}{dx}\{Q(P(x))\}$$

$$= 2\cdot\frac{1}{x} - P'(x)\cdot Q'(P(x)) = \frac{2}{x} + \frac{2x}{c^2 - x^2}$$

and, on using the product rule,

$$f''(x) = 2\frac{d}{dx}\left\{x^{-1}\right\} + 2\frac{d}{dx}\left\{x\cdot\frac{1}{c^2 - x^2}\right\}$$

$$= 2\cdot\left\{-x^{-2}\right\} + 2\left\{\frac{d}{dx}\{x\}\cdot\frac{1}{c^2 - x^2} + x\cdot\frac{d}{dx}\left\{\frac{1}{c^2 - x^2}\right\}\right\}$$

Using the chain rule again, but this time with $Q(y) = 1/y = y^{-1} \Rightarrow Q'(y) = -y^{-2}$ $\Rightarrow Q'(P(x)) = -\{P(x)\}^{-2}$, we have

$$\frac{d}{dx}\left\{\frac{1}{c^2 - x^2}\right\} = \frac{d}{dx}\{Q(P(x))\} = P'(x)\cdot Q'(P(x))$$

$$= -2x\cdot\{-P(x)^{-2}\} = \frac{2x}{(c^2 - x^2)^2}$$

So, from above,

$$f''(x) = -2x^{-2} + 2\left\{1\cdot\frac{1}{c^2 - x^2} + x\cdot\frac{2x}{(c^2 - x^2)^2}\right\}$$

$$= -\frac{2}{x^2} + \frac{2}{c^2 - x^2} + \frac{4x^2}{(c^2 - x^2)^2}$$

$$= \frac{2c^2(3x^2 - c^2)}{x^2(c^2 - x^2)^2}$$

after simplification.

22.16 Extracting the leading term of the difference quotient, we have

$$G'(t) = \frac{4t^3}{1 + t^4}.$$

Hence, by the fundamental theorem,

$$\int_1^3 \frac{4x^3}{1 + x^4}\,dx = \int_1^3 G'(x)\,dx = G(3) - G(1)$$

$$= \ln\left(1 + 3^4\right) - \ln\left(1 + 1^4\right)$$

$$= \ln(82) - \ln(2)$$

$$= \ln(82/2) = \ln(41) = 3.71.$$

22.17 By properties of logarithms, $f(x) = \ln(x^2+4) - \ln(x) = Q(P(x)) - \ln(x)$, where $P(x) = x^2+4 \Rightarrow P'(x) = 2x$ and $Q(y) = \ln(y) \Rightarrow Q'(y) = 1/y \Rightarrow Q'(P(x)) = 1/P(x)$. So, on using the chain rule

$$\begin{aligned} f'(x) &= \frac{d}{dx}\{Q(P(x))\} - \frac{d}{dx}\{\ln(x)\} \\ &= P'(x)\cdot Q'(P(x)) - \frac{1}{x} = 2x\cdot\frac{1}{x^2+4} - \frac{1}{x} \\ &= \frac{2x^2-x^2-4}{x(x^2+4)} = \frac{(x-2)(x+2)}{x(x^2+4)} \end{aligned}$$

after simplication. Because f' is negative on $[1, 2)$ but positive on $(2, 4]$, f has global minimum $f(2) = \ln(4) = 2\ln(2) \approx 1.386$. The global minimizer 2 is unique. Because $f(1) = \ln(5) = f(4)$, however, there are two global maximizers, namely, 1 and 4. But the global maximum itself, namely, $\ln(5) \approx 1.609$, must be unique.

22.18 By properties of logarithms, $f(x) = 2 + \ln(x) - \ln(x^2+9) = 2 + \ln(x) - Q(P(x))$, where $P(x) = x^2+9 \Rightarrow P'(x) = 2x$ and $Q(y) = \ln(y) \Rightarrow Q'(y) = 1/y \Rightarrow Q'(P(x)) = 1/P(x)$. So, on using the chain rule

$$\begin{aligned} f'(x) &= \frac{d}{dx}\{2\} + \frac{d}{dx}\{\ln(x)\} - \frac{d}{dx}\{Q(P(x))\} \\ &= 0 + \frac{1}{x} - P'(x)\cdot Q'(P(x)) \\ &= \frac{1}{x} - 2x\cdot\frac{1}{x^2+9} = \frac{x^2+9-2x^2}{x(x^2+9)} = \frac{(3-x)(3+x)}{x(x^2+9)} \end{aligned}$$

after simplication. Because f' is positive on $[2, 3)$ but negative on $(3, 4]$, f has global maximum $f(3) = 2 - \ln(6) \approx 0.208$, and the global minimizer is either 2 or 4. But $f(2) = 2 + \ln(2/13)$ is less than $f(4) = 2 + \ln(4/25)$ because $2/13 = 4/26$ is less than $4/25$ and ln is an increasing function. So the global minimizer is 2, and the global minimum is $f(2) = 2 + \ln(2/13) \approx 0.128$.

22.19 By properties of logarithms, $f(x) = \ln(x^5) - \ln(c^6 - x^6) = 5\ln(x) - Q(P(x))$, where $P(x) = c^6 - x^6 \Rightarrow P'(x) = -6x^5$ and $Q(y) = \ln(y) \Rightarrow Q'(y) = 1/y \Rightarrow Q'(P(x)) = 1/P(x)$. So, on using the chain rule

$$\begin{aligned} f'(x) &= 5\frac{d}{dx}\{\ln(x)\} - \frac{d}{dx}\{Q(P(x))\} \\ &= 5\cdot\frac{1}{x} - P'(x)\cdot Q'(P(x)) = \frac{5}{x} + \frac{6x^5}{c^6-x^6} \\ &= \frac{x^6+5c^6}{x(c^6-x^6)} \end{aligned}$$

after simplication. Also, on using the product rule,

$$\begin{aligned} f''(x) &= 5\frac{d}{dx}\left\{x^{-1}\right\} + 6\frac{d}{dx}\left\{x^5 \cdot \frac{1}{c^6 - x^6}\right\} \\ &= 5\cdot\left\{-x^{-2}\right\} + 6\left\{\frac{d}{dx}\left\{x^5\right\}\cdot\frac{1}{c^6 - x^6} + x^5\cdot\frac{d}{dx}\left\{\frac{1}{c^6 - x^6}\right\}\right\} \end{aligned}$$

Using the chain rule again, but this time with $Q(y) = 1/y = y^{-1} \Rightarrow Q'(y) = -y^{-2}$ $\Rightarrow Q'(P(x)) = -\{P(x)\}^{-2}$, we have

$$\begin{aligned} \frac{d}{dx}\left\{\frac{1}{c^6 - x^6}\right\} &= \frac{d}{dx}\{Q(P(x))\} = P'(x)\cdot Q'(P(x)) \\ &= -6x^5\cdot\{-P(x)^{-2}\} = \frac{6x^5}{(c^6 - x^6)^2} \end{aligned}$$

So, from above,

$$\begin{aligned} f''(x) &= -5x^{-2} + 6\left\{5x^4\cdot\frac{1}{c^6 - x^6} + x^5\cdot\frac{6x^5}{(c^6 - x^6)^2}\right\} \\ &= -\frac{5}{x^2} + \frac{30x^4}{c^6 - x^6} + \frac{36x^{10}}{(c^6 - x^6)^2} \\ &= \frac{x^{12} + 40x^6c^6 - 5c^{12}}{x^2(c^6 - x^6)^2} \end{aligned}$$

after simplification. Clearly, f' is positive on $(0, c)$; but f'' changes sign from positive to negative where $x^6 = (9\sqrt{5} - 20)c^6 \approx 0.125\,c^6$, or $x \approx 0.707c$.