18. How flat must a flatworm be, not to have a heart? The fundamental theorem.

Let $V(t)$ ml be the volume of blood at time t in a human left ventricle, and let $v(t)$ ml/s be the corresponding net inflow. Then, as we saw in Lecture 13,

$$v(t) \quad = \quad \underset{h \to 0}{\mathrm{Lim}}\, DQ(V,[t,t+h]) \quad = \quad V'(t). \tag{18.1}$$

This equation holds regardless of whether volume is increasing ($V'(t) > 0$) and inflow is positive ($v(t) > 0$) or volume is decreasing ($V'(t) < 0$) and inflow is negative ($v(t) < 0$). From Lecture 12, however, ventricular volume at time t equals volume at any earlier time, say a, plus subsequent net recharge. That is,

$$V(t) \quad = \quad V(a) \quad + \quad \mathrm{Int}\big(v,[a,t]\big) \tag{18.2a}$$

in standard notation or

$$V(t) \quad = \quad V(a) \quad + \quad \int_a^t v(x)\,dx \tag{18.2b}$$

in Leibniz notation (where $t \geq a$). We have thus identified a pair of relationships between volume and inflow, namely, (1) and (2).

These relationships are symmetric to the extent that they enable either V or v to be obtained from the other: We use (1) to find inflow if we know volume, and we use (2) to find volume if we know inflow. But the symmetry is imperfect: Although (1) determines v uniquely if V is known on $[a, b]$, (2) fails to determine V uniquely if v is known on $[a, b]$ because the right-hand side of (2) depends on $V(a)$. We can emphasize that $V(a)$ need not be known by rewriting (2) as

$$V(t) \quad = \quad C \quad + \quad \mathrm{Int}\big(v,[a,t]\big), \quad a \leq t \leq b \tag{18.3a}$$

$$\quad = \quad C \quad + \quad \int_a^t v(x)\,dx, \quad a \leq t \leq b. \tag{18.3b}$$

Nevertheless, it is always still true that

$$C \quad = \quad V(a). \tag{18.4}$$

In other words, (2) and (3) are equivalent.

Physiologically speaking, (2)-(3) say that volume implies inflow and that inflow implies recharge, but that inflow does not imply final volume because initial volume is arbitrary. Suppose, for simplicity, that $a = 0$. Then, for the particular human subject whose V and v are traced in Figures 1.3 and 1.4, $V(a) = V(0) = 120$ ml. It is possible, however, for another human subject to have a different ventricular capacity, say C, but to have *exactly* the same inflow trace throughout a cardiac cycle, and hence to have *exactly* the same stroke volume. The only difference between the first subject and the second is that the former's entire volume trace must be shifted upwards by $C - 120$ to yield the latter's. In Figure 1, the solid curves are identical to those in Figures 1.3 and 1.4 for a subject with $C = 120$ ml. The dashed curves show how volume differs, by a fixed amount everywhere, for a subject with $C = 112$ ml or $C = 124$ ml instead. The inflow, however, is the same in all cases, and (2)-(3) always hold.

The above relationship between inflow and volume is a special case of a more general relationship between integrals and derivatives (of ordinary, piecewise-smooth functions), which is known as the **fundamental theorem of calculus**. Details appear in the appendix. According to this theorem, if v has domain $[a, b]$ and V is defined in terms of v on the same domain by $V(t) = C + \mathrm{Int}(v, [a, b])$, then $v(t) = V'(t)$ on $[a, b]$.[1] Conversely, if V has domain $[a, b]$, and if v is defined in terms of V on the same domain by $v(t) = V'(t)$, then $V(t) = C + \mathrm{Int}(v, [a, b])$. In other words, the fundamental theorem is a simple two-way implication:

$$V'(t) = v(t) \quad \Leftrightarrow \quad V(t) = C + \mathrm{Int}(v,[a,t]). \tag{18.5}$$

The equivalent statement in differential notation is

$$\frac{dy}{dt} = v(t) \quad \Leftrightarrow \quad y = C + \int_a^t v(x)\,dx, \tag{18.6}$$

where it is understood that

$$y = V(t). \tag{18.7}$$

Needless to say, a theorem's validity in no way depends on the symbol we choose for an arbitrary element of a function's domain. So, in terms of information content,

$$V'(x) = v(x) \quad \Leftrightarrow \quad V(x) = C + \mathrm{Int}(v,[a,x]) \tag{18.8}$$

is *identical* to (5) and

$$\frac{dy}{dx} = v(x) \quad \Leftrightarrow \quad y = C + \int_a^x v(t)\,dt \tag{18.9}$$

is *identical* to (6), because it is now understood that $y = V(x)$. Be sure you appreciate that (6) and (9) really are identical, despite – or, if you prefer, precisely because of – the juxtaposition of x and t.

These expressions of the fundamental theorem are most appropriate when C is unknown. If C is considered known, then (4)-(6) yield

$$V'(t) = v(t) \quad \Leftrightarrow \quad \mathrm{Int}(v,[a,t]) = V(t) - V(a) \tag{18.10}$$

or

$$\frac{dy}{dt} = v(t) \quad \Leftrightarrow \quad \int_a^t v(x)\,dx = y\big|_a^t, \tag{18.11}$$

on using the customary Leibniz notation $y\big|_a^t$ for $V(t) - V(a)$. Equivalent statements are

$$V'(x) = v(x) \quad \Leftrightarrow \quad \mathrm{Int}(v,[a,x]) = V(x) - V(a) \tag{18.12}$$

or

$$\frac{dy}{dx} = v(x) \quad \Leftrightarrow \quad \int_a^x v(t)\,dt = y\big|_a^x. \tag{18.13}$$

Any function whose derivative is v is called an **antiderivative** of v. It is not in general unique, because C need not be known. Hence the use of an indefinite article: *The* antiderivative would imply that C is always known. In these terms, the practical consequence of the fundamental theorem is that every result about derivatives is also a result about antiderivatives, and vice versa. For example, because

$$\frac{d}{dx}\{x^3\} = 3x^2 \tag{18.14}$$

from Table 16.2, (8) and (12) immediately imply both

$$x^3 = C + \int_a^x 3t^2\,dt \tag{18.15}$$

and

---

[1] Extension (13.26) is assumed.

$$\int_a^x 3t^2\,dt \;=\; t^3\Big|_a^x. \tag{18.16}$$

From (16) and (12.25) we readily deduce that

$$\int_a^x t^2\,dt \;=\; \tfrac{1}{3}t^3\Big|_a^x \;=\; \tfrac{1}{3}x^3 - \tfrac{1}{3}a^3, \tag{18.17}$$

which agrees with (12.26). From (15), on the other hand, we have

$$\int_a^x t^2\,dt \;=\; \frac{1}{3}x^3 \;-\; \frac{C}{3}. \tag{18.18}$$

Because C is just an arbitrary constant, however, –C/3 is also just an arbitrary constant, so it is neater to write

$$\int_a^x t^2\,dt \;=\; \frac{1}{3}x^3 \;+\; \text{const} \tag{18.19}$$

instead: that $C = a^3$ is quite irrelevant if neither a nor C is known. Similar reasoning yields the more general result in Table 1; see Exercise 10.

Restrictions	DERIVATIVE on [a, b]	ANTIDERIVATIVE on [a, b]	SOURCE
	$\frac{d}{dx}\{C\} = 0$	$\int_a^x 0\,dt = \text{const}$	Table 16.2
n integer, $n \ge 1$	$\frac{d}{dx}\{x^n\} = nx^{n-1}$	$\int_a^x t^{n-1}\,dt = \frac{x^n}{n} + \text{const}$	(17.B1)
n integer, $n \ge 1,\ a > 0$	$\frac{d}{dx}\{x^{-n}\} = -nx^{-n-1}$	$\int_a^x t^{-n-1}\,dt = -\frac{x^{-n}}{n} + \text{const}$	(17.B2)
$a \ge 0$	$\frac{d}{dx}\{x\sqrt{x}\} = \frac{3}{2}\sqrt{x}$	$\int_a^x \sqrt{t}\,dt = \frac{2}{3}x\sqrt{x} + \text{const}$	Exercise 17.4
$a > 0$	$\frac{d}{dx}\{\sqrt{x}\} = \frac{1}{2\sqrt{x}}$	$\int_a^x t^{-1/2}\,dt = 2\sqrt{x} + \text{const}$	Exercise 17.4

Table 18.1 Some results that are equivalent by Lectures 16-17 and the fundamental theorem

For calculating integrals by reinterpreting results about derivatives as results about antiderivatives, we can most easily apply the fundamental theorem in the form

$$\int_a^t V'(x)\,dx \;=\; V(x)\Big|_a^t \;=\; V(t) - V(a), \tag{18.20a}$$

which follows directly from (11)-(12). An equivalent statement is

$$\int_a^x V'(t)\,dt \;=\; V(t)\Big|_a^x \;=\; V(x) - V(a). \tag{18.20b}$$

For example, because (16.17) and Exercise 17.4 yield

$$\sqrt{x} \;=\; \frac{d}{dx}\left\{\frac{2}{3}x\sqrt{x}\right\}, \tag{18.21}$$

(20a) holds with $V(x) = 2x\sqrt{x}/3$ and $V'(x) = \sqrt{x}$, and so

$$\int_a^t \sqrt{x}\,dx \;=\; \int_a^t \frac{d}{dx}\left\{\frac{2}{3}x\sqrt{x}\right\}dx \;=\; \frac{2}{3}x\sqrt{x}\Big|_a^t$$
$$= \frac{2}{3}t\sqrt{t} - \frac{2}{3}a\sqrt{a} \;=\; \frac{2}{3}t\sqrt{t} + \text{const.} \qquad (18.22)$$

Many other useful results can be obtained in a similar way; see Exercises 1-9.

To see how useful the fundamental theorem can be in practice, we turn our attention to flatworms.[2] Small animals such as platyhelminths are able to "breathe" without the help of a vascular system. They obtain all the oxygen they need by diffusion across the surface of their bodies from the surrounding respiratory medium. Why can't large animals do this? The answer, of course, is that they are too large: they have too little surface area for oxygen to diffuse across compared to the volume of cells that must be supplied with oxygen. But how large is too large? Or to put the question another way: How big can a flatworm be (and still survive without a vascular system)?

To simplify this question let us assume at the outset that flatworms are flat (in the sense that most of their surface area is on two opposite sides), so that as much of their cell tissue is as near to the surface as possible. Then instead of asking how big a flatworm can be, we ask more specifically: How wide can a flatworm be? For the sake of definiteness, let our flatworm have thickness 2s millimeters, and let x measure depth from its dorsal (upper) surface; see Figure 2. Then we seek an upper bound on s.

Let F(x) be the rate per unit area (mm^2) at which oxygen at depth x diffuses downward, perpendicular to the flatworm's dorsal surface (Figure 2). That is, F(x) is the volume of oxygen transported downward per unit time (second) per unit area (mm^2) at depth x, and so –F(x) is the rate per unit area at which oxygen at depth x diffuses upward. Now, diffusion of oxygen is simply flow of oxygen from regions of higher oxygen concentration to regions of lower oxygen concentration. The greater the imbalance between higher and lower concentration, the faster the flow. In other words, the steeper the *concentration gradient*, the faster the flow. By tradition, concentration of oxygen is expressed in terms of its partial pressure, which is the fraction of the total gas pressure attributable to oxygen; for example, the partial pressure of oxygen in air at atmospheric pressure is 0.21 atm (because air by volume is about 21% oxygen and 78% nitrogen, with 1% trace elements). Accordingly, let y be the partial pressure of oxygen at depth x, and let y_0 be the partial pressure at the dorsal surface. The higher the value of $|dy/dx|$, the faster the flow of oxygen, i.e., the higher the value of $|F|$. Moreover, if $dy/dx > 0$ then $F < 0$, because if oxygen concentration is higher at lower levels then oxygen diffuses upward; whereas if $dy/dx < 0$ then $F > 0$, because if concentration is higher at higher levels then oxygen diffuses downward. It is consistent with these observations to assume that the downward flux of oxygen per unit area is a constant times the concentration gradient or

$$F \;=\; -q\frac{dy}{dx}, \qquad (18.23)$$

where q is called the diffusion coefficient. This proportional relationship between flux and concentration, satisfied remarkably well in practice, is commonly known as Fick's

---

[2] Alexander's (1990) Section 4.3 (pp. 119-122) is the primary source for the remainer of this lecture. Note, however, some errors on p.121 of Alexander's discussion; in particular, he confuses dorsal and mid-section partial pressures in the mathematical analysis that leads to his equation (4.2).

law.[3] Alexander (1990, pp. 120-121) suggests that a suitable value for flatworm tissue is $q = 2 \times 10^{-5}$ mm^2 atm^{-1} s^{-1}.

Let us assume that oxygen is supplied to the upper half of our flatworm's body by diffusion across its dorsal surface, and to its lower half by diffusion across its ventral (lower) surface. Then the cuboid shaded in Figure 2 must receive its oxygen across its upper surface CC′D′D (and hence ultimately across speckled area AA′B′B of the dorsal surface). Let CC′D′D (and hence AA′B′B) have area ε mm^2. Then the cuboid's volume is just ε times its height, or $\varepsilon(s - x)$. Thus if m is the rate per unit volume at which oxygen is consumed by flatworm tissue (in mm^3/sec), then $m\varepsilon(s - x)$ mm^3 of oxygen must be supplied across CC′D′D every second. But the rate of oxygen supply across unit surface area is εF. Hence, from (23), we require

$$\varepsilon\left(-q\frac{dy}{dx}\right) = \varepsilon m(s-x), \tag{18.24}$$

or

$$\frac{dy}{dx} = \frac{m}{q}(x-s). \tag{18.25}$$

Note that, because $x \le s$ inside the cuboid, $dy/dx \le 0$.

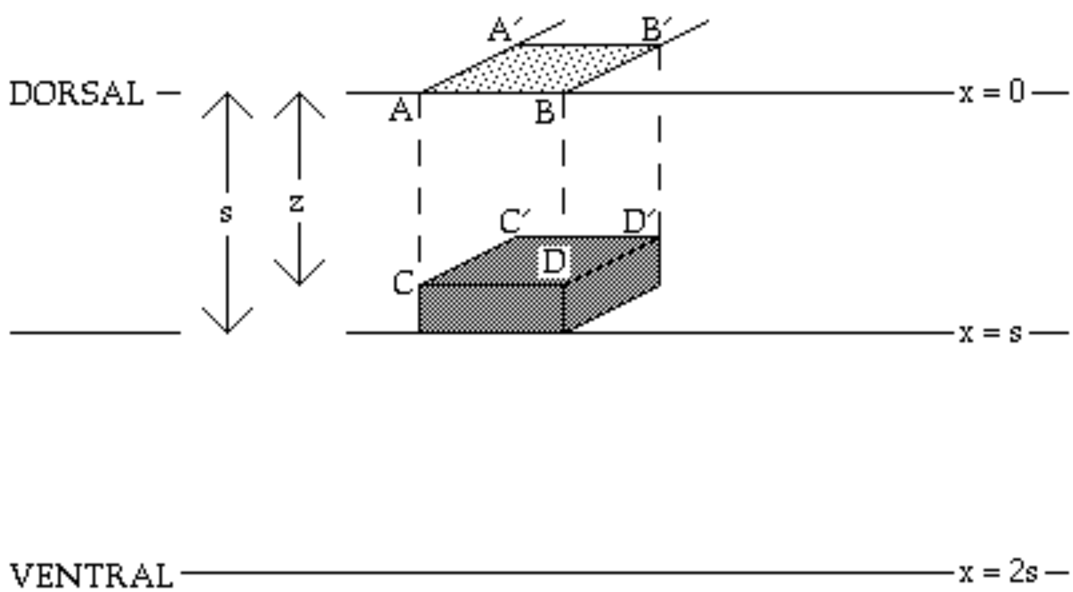

Figure 18.2

How must partial pressure vary with depth to satisfy (25)? The fundamental theorem immediately provides the answer; from (9) with a = 0, we have

$$\frac{dy}{dx} = \frac{m}{q}(x-s) \quad \Rightarrow \quad y = C + \int_0^x \frac{m}{q}(t-s)\,dt. \tag{18.26}$$

Setting $x = 0$ yields

$$y\big|_{x=0} = C + \int_0^0 \frac{m}{q}(t-s)\,dt = C + 0 = C, \tag{18.27}$$

[3] After Adolf Fick, a 19th-century professor of physiology at Wurzburg.

implying $C = y|_{x=0} = y_0$ and hence

$$y \;=\; y_0 \;+\; \int_0^x \frac{m}{q}(t-s)\,dt. \qquad (18.28)$$

So, on using (12.25)-(12.26),

$$\begin{aligned} y \;&=\; y_0 \;+\; \frac{m}{q}\int_0^x t\,dt \;-\; \frac{ms}{q}\int_0^x 1\,dt \\ &=\; y_0 \;+\; \frac{m}{q}\left(\tfrac{1}{2}x^2 - \tfrac{1}{2}0^2\right) \;-\; \frac{ms}{q}(x-0) \\ &=\; y_0 \;+\; \frac{mx}{2q}(x-2s). \end{aligned} \qquad (18.29)$$

Thus the partial pressure of oxygen at at the flatworm's midsection is

$$y|_{x=s} \;=\; y_0 + \frac{ms}{2q}(s-2s) \;=\; y_0 - \frac{ms^2}{2q}. \qquad (18.30)$$

But partial pressure cannot be negative; hence $y_0 \geq ms^2/2q$, or $s^2 \leq 2qy_0/m$. This means $4s^2 \leq 8qy_0/m$, or

$$2s \;\leq\; \sqrt{\frac{8qy_0}{m}}\,. \qquad (18.31)$$

The right-hand side of this inequality is an upper bound on the thickness of the flatworm.

According to Alexander (1990, p.119), flatworms consume oxygen at a rate in excess of 0.1 cm^3 per hour per gram of body tissue. Since the density of flatworm tissue is about 1 gram per cm^3 and an hour is 3600 seconds, the volume rate of consumption exceeds 0.1 cm^3 of oxygen per 3600 seconds per cm^3 of flatworm; or, which of course is the same thing, 0.1 mm^3 of oxygen per 3600 seconds per mm^3 of flatworm. Thus $m > 0.1/3600$, implying that $1/m < 3600/0.1 = 3.6 \times 10^4$. We have already seen that $y_0 = 0.21$ and $q = 2 \times 10^{-5}$. So $8qy_0 = 3.36 \times 10^{-5}$, implying $8qy_0/m < 3.36 \times 10^{-5} \times 3.6 \times 10^4 = 1.2096$. Hence (31) implies $2s < \sqrt{1.2096} = 1.1$ mm. Thus, according to our analysis, a flatworm couldn't possibly be more than 1.1 millimeters thick. In practice, a flatworm's thickness is never more than about 0.5 mm, less than half of our upper bound. Note, however, that if our flatworm could receive oxygen only through its dorsal surface, then our analysis would predict an upper bound of 0.55 mm on thickness instead, because in that case the ventral surface in Figure 2 would be at $x = s$ (and the worm would have thickness s, instead of $2s$).

We conclude by noting that, although versions (9) and (20) of the fundamental theorem are equivalent in principle, there's a subtle difference in practice: in essence, (9) represents integral thinking, whereas (20) represents antiderivative thinking. To understand the diffence better, it will be instructive to revisit the result that

$$\int_0^x \frac{m}{q}(t-s)\,dt \;=\; \frac{mx}{2q}(x-2s). \qquad (18.32)$$

We obtained it in (28)-(29) by using integral thinking, i.e., by noting that (12.25) implies

$$\int_0^x \frac{m}{q}(t-s)\,dt \;=\; \frac{m}{q}\int_0^x t\,dt \;-\; \frac{ms}{q}\int_0^x 1\,dt \qquad (18.33)$$

and using (12.26) to finish the job. To obtain the result by antiderivative thinking, we instead observe that (16.20) implies

$$\begin{aligned}\frac{d}{dt}\left\{\frac{mt^2}{2q} - \frac{mst}{q}\right\} &= \frac{m}{2q}\frac{d}{dt}\left\{t^2\right\} - \frac{ms}{q}\frac{d}{dt}\left\{t\right\} \\ &= \frac{m}{2q}\cdot 2t - \frac{ms}{q}\cdot 1 = \frac{m}{q}(t-s),\end{aligned} \tag{18.34}$$

i.e., that $V(t) = mt^2/2q - mst/q$ implies $V'(t) = m(t-s)/q$, and then use (20b):

$$\begin{aligned}\int_0^x \frac{m}{q}(t-s)\,dt &= \int_0^x \frac{d}{dt}\left\{\frac{mt^2}{2q} - \frac{mst}{q}\right\}dt = \left(\frac{mt^2}{2q} - \frac{mst}{q}\right)\Bigg|_0^x \\ &= \frac{mx^2}{2q} - \frac{msx}{q} - (0 - 0) = \frac{mx}{2q}(x-2s).\end{aligned} \tag{18.35}$$

Although we could have used (33) to obtain (32) as long ago as Lecture 8, (35) is new, because it depends on the fundamental theorem. Either way, the result is the same; and in principle, neither approach is better than the other. In practice, however, it is often more convenient to calculate integrals by using antiderivative thinking, and we will do so extensively henceforward.

Reference

Alexander, R.McN. (1990). Animals. Cambridge University Press

Exercises 18

18.1 Use the fundamental theorem to show that $\int_0^t x\sqrt{x}\,dx = \frac{2}{5}t^2\sqrt{t}$ for $t \geq 0$.

Hint: Use Exercise 17.5.

18.2 Use the fundamental theorem to show that $\int_1^t x^{-3}\,dx = \frac{1}{2}\left(1-\frac{1}{t}\right)\left(1+\frac{1}{t}\right)$ for $t \geq 1$.

Hint: Use Exercise 13.13 with $C = 1/2$.

18.3 Use the fundamental theorem to show that $\int_1^t x^{-4}\,dx = \frac{(t-1)(t^2+t+1)}{3t^3}$ for $t \geq 1$.

Hint: Use Exercise 13.17 with $C = -1/3$.

18.4 Use the fundamental theorem to show that $\int_2^t \frac{1}{(2x+1)^2}\,dx = \frac{1}{5}\frac{t-2}{2t+1}$ for $t \geq 2$

(i) by using Exercise 13.15 with $b = -1/2$
(ii) by using Exercise 13.16 with $c = 1/2$.

18.5 Use the fundamental theorem to show that $\int_1^t \frac{x(6-x)}{(3-x)^2}\,dx = \frac{(t-1)(2t+3)}{2(3-t)}$

for $1 \leq t < 3$. Hint: Use Exercise 15.17 with $b = 3$.

18.6 Use the fundamental theorem to show that

$$\int_1^t \frac{1}{x^3\sqrt{x}}\,dx = \frac{2}{5}\left(1-\frac{1}{\sqrt{t}}\right)\left(1+\frac{1}{\sqrt{t}}+\frac{1}{t}+\frac{1}{t\sqrt{t}}+\frac{1}{t^2}\right)$$

for $t \geq 1$. Hint: Use Exercise 17.8

18.7 Use the fundamental theorem to show that $\int_5^t \frac{10}{(x+10)^2}\,dx = \frac{2(t-5)}{3(t+10)}$

for $t \geq 5$. Hint: Use Exercise 15.16.

18.8 Show that $\int_2^t 6(x-1)(x-2)\,dx = (t-2)^2(2t-1)$ for $t \geq 2$

(i) with the help of Exercise 14.3
(ii) by another method

18.9 Show that $\int_3^t 6(x-2)(x-5)\,dx = (t-3)(2t^2-15t+15)$ for $t \geq 3$

(i) with the help of Exercise 14.4
(ii) by another method

18.10 Use the fundamental theorem to show that $\int_a^t x^{n-1}\,dx = \frac{1}{n}t^n + \text{const}$ on $[a, \infty)$,

where n is any positive integer. Hint: Use (11.B1).

18.11 Use the fundamental theorem to show that $\int_a^t x^{-n-1}\,dx = -\frac{1}{n}t^{-n} + \text{const}$ on $[a, \infty)$,

where n is any positive integer and $a > 0$. Hint: Use (11.B2).

18.12 The function G defined on $[0, \infty)$ by

$$G(t) = \left(\frac{t^3-1}{t^3+1}\right)^2$$ is known to satisfy

$$\frac{G(t+h)-G(t)}{h} = \frac{12t^2(t^3-1)}{(t^3+1)^3} + O[h].$$

What must be the value of

$$\int_1^2 \frac{12x^2(x^3-1)}{(x^3+1)^3}\,dx?$$

Write your answer as simply as possible.

18.13 The function G defined on $[0, \infty)$ by

$$G(t) = \frac{(t+1)(t+2)}{(t+3)(t+4)}$$

is known to satisfy

$$\frac{G(t+h)-G(t)}{h} = \frac{2(11+10t+2t^2)}{(t^2+7t+12)^2} + O[h].$$

What must be the value of

$$\int_{1/3}^2 \frac{(11+10x+2x^2)}{(x^2+7x+12)^2}\,dx?$$

Write your answer as simply as possible.

18.14 The function G defined on $[0, \infty)$ by $G(t) = \dfrac{(t+1)(t+3)}{(t+2)(t+4)}$ is known to satisfy

$$\frac{G(t+h)-G(t)}{h} = \frac{2(t^2+5t+7)}{(t+2)^2(t+4)^2} + O[h]$$

as $h \to 0+$. What must be the value of

$$\int_0^2 (x+2)^{-2}(x+4)^{-2}(x^2+5x+7)\,dx?$$

Write your answer as simply as possible.

18.15 The function G defined on $[0, \infty)$ by $G(t) = \dfrac{(t+2)(t+3)}{(t+1)(t+4)}$ is known to satisfy

$$\frac{G(t+h)-G(t)}{h} = -\frac{2(2t+5)}{(t+1)^2(t+4)^2} + O[h]$$

as $h \to 0+$. What must be the value of

$$\int_0^1 (x+1)^{-2}(x+4)^{-2}(2x+5)\,dx?$$

Write your answer as simply as possible.

Appendix 18: The fundamental theorem of the calculus

To establish (5), we must establish both that

$$V(t) \;=\; C \;+\; \mathrm{Int}\big(v,[a,t]\big), \qquad a \le t \le b \tag{18.A1}$$

implies

$$V'(t) \;=\; v(t), \qquad a \le t < b \tag{18.A2}$$

and that (A2) implies (A1). We begin with the former.

To show that (A1) implies (A2), we assume (A1) and deduce (A2). Accordingly, suppose that (A1) holds. Then, provided h is small enough to ensure $a \le t + h \le b$, it must also hold that

$$V(t+h) \;=\; C \;+\; \mathrm{Int}\big(v,[a,t+h]\big). \tag{18.A3}$$

Subtracting (A1) from (A3) and using (5.22), we obtain

$$\begin{aligned} V(t+h) - V(t) \;&=\; \mathrm{Int}\big(v,[a,t+h]\big) - \mathrm{Int}\big(v,[a,t]\big) \\ &=\; \mathrm{Int}\big(v,[t,t+h]\big), \end{aligned} \tag{18.A4}$$

so that division by h implies $DQ(V, [t, t + h]) = \mathrm{Int}(v, [t, t + h])/h$ and hence

$$\operatorname*{Lim}_{h\to 0} DQ(V,[t,t+h]) \;=\; \operatorname*{Lim}_{h\to 0} \frac{\mathrm{Int}(v,[t,t+h])}{h} \tag{18.A5}$$

By (9.20), the left-hand side of the above equation equals $V'(t)$. So it remains to show that the right-hand side equals v(t).

Now, no matter how much the function v may increase or decrease on a large domain, if h is positive but very small, then either v is strictly increasing throughout [t, t + h], or v is strictly decreasing throughout [t, t + h], or else v is constant throughout [t, t + h]. Moreover, if h is sufficiently small, then either $v(t) \ge 0$ throughout [t, t + h] or $v(t) \le 0$ throughout [t, t + h]. These six possibilities are sketched in Figure 3, where in each case the shaded region is $\{(x, y) \mid t \le x \le t + h, 0 \le |y| \le |v(x)|\}$. We consider each possibility in turn.

First, suppose that v is increasing and $v(t) \ge 0$ (top left). Then, because the shaded area must lie between the areas of the dashed rectangles, we have

$$h \cdot v(t) \;<\; \mathrm{Int}(v,[t,t+h]) \;<\; h \cdot v(t+h). \tag{18.A6}$$

Second, if v is constant and $v(t) \ge 0$ (middle left), then

$$h \cdot v(t) \;=\; \mathrm{Int}(v,[t,t+h]) \;=\; h \cdot v(t+h). \tag{18.A7}$$

Third, suppose that v is decreasing and $v(t) \ge 0$ (bottom left). Then, again because the shaded area must be bounded by the areas of the rectangles,

$$h \cdot v(t+h) \;<\; \mathrm{Int}(v,[t,t+h]) \;<\; h \cdot v(t). \tag{18.A8}$$

Fourth, suppose that v is decreasing and $v(t) \le 0$ (top right). Then, because the signed shaded area must be bounded by the signed areas of the rectangles, (A8) is satisfied again. Fifth, if v is constant with $v(t) \le 0$ (middle right), then (A7) is satisfied again. Finally, if v is increasing with $v(t) \le 0$ (bottom right) then (A6) is satisfied again. Thus, on dividing by h (>0), one of the following must be true when h is sufficiently small:

$$v(t) \;<\; \frac{\mathrm{Int}(v,[t,t+h])}{h} \;<\; v(t+h) \tag{18.A9}$$

$$v(t) \;=\; \frac{\mathrm{Int}(f,[t,t+h])}{h} \;=\; v(t+h) \tag{18.A10}$$

$$v(t+h) \;<\; \frac{\mathrm{Int}(v,[t,t+h])}{h} \;<\; v(t). \tag{18.A11}$$

Now let $h \to 0$. Then, in all three cases, $v(t + h)$ becomes arbitrarily close to $v(t)$. But also in all three cases, $\{\mathrm{Int}(v, [t, t+h])\}/h$ is sandwiched between $v(t + h)$ and $v(t)$. The only way this can happen is if

$$\lim_{h\to 0} \frac{\mathrm{Int}(v,[t,t+h])}{h} = v(t). \tag{18.A12}$$

So the right-hand side of (A5) is $v(t)$, as required. Note that inequality (A9) weakens to

$$v(t) \le \lim_{h\to 0} \frac{\mathrm{Int}(v,[t,t+h])}{h} \le v(t) \tag{18.A13}$$

in the limit as $h \to 0$, and similarly for (A11); see the footnote to Lecture 11.

To show that (A2) implies (A1), we assume (A2) and deduce (A1). So suppose that (A2) holds, and *define* H on [a, b] by

$$H(t) \;=\; \mathrm{Int}\big(v,[a,t]\big). \tag{18.A14}$$

Then, by the above argument with H in place of V, we find that $H'(t) = v(t)$, $a \le t < b$. But (A2) implies $V'(t) = v(t)$. Therefore $V'(t) = H'(t)$, $a \le t < b$. Define Q on [a, b] by

$$Q(t) \;=\; V(t) - H(t). \tag{18.A15}$$

Then, in view of (11.13) and (11.14), we have

$$Q'(t) \;=\; \frac{d}{dt}\big\{V(t) - H(t)\big\} \;=\; V'(t) - H'(t) \;=\; 0. \tag{18.A16}$$

But (13.21) guarantees the existence of a constant K such that $|DQ(Q, [t, t + h]) - Q'(t)| \le Kh$ for any positive h, however small. So (A.16) implies $|DQ(Q, [t, t + h])| \le Kh$, hence $|DQ(Q, [t, t + h])| < (K+1)h$, no matter how small h is. The only way this can possibly be true is if $DQ(Q, [t, t + h]) = 0$, implying $Q(t+h) = Q(t)$. So Q is constant, say $Q(t) = C$. Then (A.15) implies $V(t) = C + H(t)$, as required.

Answers and Hints for Selected Exercises

18.1 By Exercise 17.5, $\frac{d}{dx}\left\{x^2\sqrt{x}\right\} = \frac{5}{2}x\sqrt{x}$. Thus, by (16.17) with $k = 2/5$,

$$\frac{d}{dx}\left\{\frac{2}{5}x^2\sqrt{x}\right\} = \frac{2}{5}\cdot\frac{5}{2}x\sqrt{x} = x\sqrt{x}.$$

Use version (20) of the fundamental theorem, i.e.,

$$\int_a^t V'(x)\,dx = V(x)\Big|_a^t = V(t) - V(a).$$

Setting $V(t) = 2t^2\sqrt{t}/5$ and $a = 0$, we have

$$\int_0^t x\sqrt{x}\,dx = \int_0^t \frac{d}{dx}\left\{\frac{2}{5}x^2\sqrt{x}\right\}dx$$

$$= \frac{2}{5}x^2\sqrt{x}\Big|_0^t = \frac{2}{5}t^2\sqrt{t} - \frac{2}{5}0^2\sqrt{0} = \frac{2}{5}t^2\sqrt{t}$$

as required.

18.3 By Exercise 13.17 with $C = -1/3$, $\frac{d}{dt}\left\{\frac{-1}{3t^3}\right\} = \frac{1}{t^4}$. Thus, by version (20) of the fundamental theorem,

$$\int_1^t x^{-4}\,dx = \int_1^t \frac{1}{x^4}\,dx = \int_1^t \frac{d}{dx}\left\{\frac{-1}{3x^3}\right\}dx$$

$$= \frac{-1}{3x^3}\Big|_1^t = \frac{-1}{3t^3} - \left\{\frac{-1}{3\cdot 1^3}\right\}$$

$$= \frac{t^3-1}{3t^3} = \frac{(t-1)(t^2+t+1)}{3t^3}$$

as required. Why must the value be zero if $t = 1$?

18.5 By Exercise 15.17 with $b = 3$, $\frac{d}{dt}\left\{\frac{t^2}{3-t}\right\} = \frac{t(6-t)}{(3-t)^2}$. Thus, by version (20) of the fundamental theorem,

$$\int_1^t \frac{x(6-x)}{(3-x)^2}\,dx = \int_1^t \frac{d}{dx}\left\{\frac{x^2}{3-x}\right\}dx = \frac{x^2}{3-x}\Big|_1^t$$

$$= \frac{t^2}{3-t} - \frac{1^2}{3-1} = \frac{2t^2+t-3}{2(3-t)} = \frac{(t-1)(2t+3)}{2(3-t)}$$

as required.

18.9 (i) By Exercise 14.4, we have

$$(t-2)(t-5) = \frac{d}{dt}\left\{\frac{1}{6}(2t^2-19t+41)(t-1)\right\},$$

implying

$$6(x-2)(x-5) = \frac{d}{dx}\left\{(2x^2-19x+41)(x-1)\right\}.$$

So

$$\begin{aligned}\int_3^t 6(x-2)(x-5)\,dx &= \int_3^t \frac{d}{dx}\left\{(2x^2-19x+41)(x-1)\right\}dx \\ &= \left((2x^2-19x+41)(x-1)\right)\Big|_3^t \\ &= (2t^2-19t+41)(t-1) \;-\; (2\cdot 3^2-19\cdot 3+41)(3-1) \\ &= 2t^3-21t^2+60t-45 \;=\; (t-3)(2t^2-15t+15)\end{aligned}$$

(ii) Alternatively, by the fundamental theorem we have

$$\begin{aligned}\int_3^t 6(x-2)(x-5)\,dx &= \int_3^t (6x^2-42x+60)\,dx \;=\; \int_3^t \frac{d}{dx}\left\{2x^3-21x^2+60x\right\}dx \\ &= \left(2x^3-21x^2+60x\right)\Big|_3^t \;=\; 2t^3-21t^2+60t \;-\; (2\cdot 3^3-21\cdot 3^2+60\cdot 3) \\ &= 2t^3-21t^2+60t-45 \;=\; (t-3)(2t^2-15t+15)\end{aligned}$$

as before.

18.12 Extracting the leading term of the difference quotient, we have

$$G'(t) \;=\; \frac{12t^2(t^3-1)}{(t^3+1)^3}.$$

Hence, by the fundamental theorem,

$$\begin{aligned}\int_1^2 \frac{12x^2(x^3-1)}{(x^3+1)^3}\,dx &= \int_1^2 G'(x)\,dx \;=\; G(2)-G(1) \\ &= \left(\frac{2^3-1}{2^3+1}\right)^2 - \left(\frac{1^3-1}{1^3+1}\right)^2 = \left(\frac{7}{9}\right)^2 - 0 = \frac{49}{81}\end{aligned}$$

18.13 Extracting the leading term of the difference quotient, we have

$$G'(t) \;=\; \frac{2(11+10t+2t^2)}{(t^2+7t+12)^2},$$

implying

$$\frac{(11+10x+2x^2)}{(x^2+7x+12)^2} \;=\; \frac{1}{2}G'(x).$$

So, on using the fundamental theorem,

$$\begin{aligned}\int_{1/3}^2 \frac{(11+10x+2x^2)}{(x^2+7x+12)^2}\,dx &= \frac{1}{2}\int_{1/3}^2 G'(x)\,dx \\ &= \frac{1}{2}\{G(2)-G(1/3)\} \;=\; \frac{1}{2}\left\{\frac{2}{5}-\frac{14}{65}\right\} \;=\; \frac{6}{65}.\end{aligned}$$

18.14 $\{G(2)-G(0)\}/2 = (5/8-3/8)/2 = 1/8$

18.15 $-\{G(1)-G(0)\}/2 \;=\; -(6/5-3/2)/2 = 3/20$