16. Differential notation. The derivative of a sum or multiple

If F is a piecewise-smooth function, not of time (as in Lectures 13-15) but of some other variable, e.g., length, mass or temperature, then its derivative F' is still defined by

$$F'(x) \;=\; \text{THAT PART OF } DQ(F,[x,x+h]) \text{ WHICH IS INDEPENDENT OF } h \qquad (16.1a)$$

$$= \lim_{h\to 0} DQ(F,[x,x+h]). \qquad (16.1b)$$

But F' is no longer an instantaneous growth rate, and so in this lecture we reinterpret it. To that end, we introduce some alternative notation for derivatives. We also introduce a formula that greatly reduces the labor of calculating derivatives.

For the sake of definiteness, we begin by supposing that x is temperature and that F(x) is net rate of photosynthetic CO_2 exchange at temperature x in maize. Scaled with respect to its maximum rate, so that $0 \le F(x) \le 1$, F is defined on [12, 51] by

$$F(x) \;=\; m_0(-1153008x + 162144x^2 - 6969x^3 + 134x^4 - x^5) \qquad (16.2)$$

(Lecture 8) with

$$m_0 \;=\; 1.23353\times 10^{-7}. \qquad (16.3)$$

The graph, y = F(x), is like a hill (Figure 1). Traversing this hill from left to right, each horizontal displacement represents a temperature increase while the corresponding increase or decrease of altitude represents a change in rate of photosynthesis. The hill is steep at lower altitudes but much less steep toward the summit. Correspondingly, rate of photosynthesis changes rapidly with respect to temperature at low (near 12°C) or high (near 51°C) temperatures, but relatively slowly at intermediate temperatures. This is all rather vague, however: we need to be more precise.

Accordingly, we focus attention on a generic temperature x, at which net rate of photosynthesis is y. Thus our focal point is the left-hand large dot at (x, y) in Figure 1. The right-hand large dot is a "neighboring point" with coordinates $(x + \delta x, y + \delta y)$; as in Lecture 12, δ denotes an infinitesimal change. The infinitesimal change of altitude between the focal point (x, y) and its neighbor $(x + \delta x, y + \delta y)$ is $y + \delta y - y = \delta y$. The accompanying horizontal displacement is $x + \delta x - x = \delta x$. So the average slope, or average gradient, between these two points is

$$\text{AVERAGE GRADIENT} \;=\; \frac{\text{ALTITUDE CHANGE}}{\text{HORIZONTAL DISPLACEMENT}} \;=\; \frac{\delta y}{\delta x}. \qquad (16.4)$$

This average gradient is the actual gradient of the dashed line in Figure 1. The closer together the two large dots, the more nearly the dashed line coincides with the dotted line, which is called the **tangent** at the focal point, because in this neighborhood it meets the curve precisely once, whereas the dashed line – or **chord** – meets it twice. As a consequence, the closer together the two large dots, the more nearly the average chord gradient $\delta y/\delta x$ coincides with the gradient of the tangent. If we allow $x + \delta x$ to become arbitrarily close to x, i.e., if we allow $\delta x \to 0$, then (because both dots lie on the curve) we automatically ensure that $y + \delta y$ becomes arbitrarily close to y, or $\delta y \to 0$, and hence that $\delta y/\delta x$ becomes arbitrarily close to the gradient of the tangent. A possible mathematical shorthand for this state of affairs would be

$$\text{GRADIENT OF TANGENT AT } (x,y) \;=\; \lim_{\substack{\delta x\to 0\\ \delta y\to 0}} \frac{\delta y}{\delta x}. \qquad (16.5)$$

But the left-hand side is not too short and the right-hand side is clumsy, especially given that $\delta x \to 0$ is impossible without $\delta y \to 0$ as well. So mathematicians use the notation "dy/dx" for the gradient of the tangent at the focal point and simply take for granted that $\delta x \to 0$ implies $\delta y \to 0$. Then

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \qquad (16.6)$$

defines the gradient of the tangent, which measures the steepness of the hill at the focal point. That is, the gradient of a hill at any point is defined to be the gradient of the tangent at that point, which in turn is determined by the right-hand side of (6).

Note the difference between $\delta y/\delta x$ and dy/dx. Both are numbers. But whereas $\delta y/\delta x$ means δy divided by δx, neither dx nor dy means anything at all by itself. On the contrary, dy/dx is a single, indecomposable notation for a single number, namely, the gradient at the focal point. It looks like a ratio but it isn't one (although it is the limit of a ratio, and the symbology is intended to evoke the limit).

Everything we have said about Figure 1 applies with equal force to Figure 2, where the graph of F appears again. In particular, the left-hand large dot is still the focal point, the right-hand large dot is still its neighbor, δx is still the change in x and δy is still the change in y, even though δy is negative. Because δx must be positive, $\delta y/\delta x$ is negative; and so, from (6), dy/dx is also negative – unless the focal point is precisely at the summit. What happens in this special case is that, although $\delta y/\delta x$ is strictly negative, it gets arbitrarily close to zero as the two large dots coalesce; and so, from (6), $dy/dx = 0$ (and the tangent is horizontal).[1] Thus $dy/dx > 0$ going up the hill, $dy/dx = 0$ at the top, and $dy/dx < 0$ going down the other side. Similar considerations apply to traversing a valley (always from left to right): $dy/dx < 0$ going down, $dy/dx = 0$ at the bottom, and $dy/dx > 0$ going up the other side.

None of this is news, however, except for the notation. Because both large dots lie on the graph, $y = F(x)$ and $y + \delta y = F(x + \delta x)$. So $\delta y = y + \delta y - y = F(x+\delta x) - F(x)$, implying $\delta y/\delta x = \{F(x+\delta x) - F(x)\}/(x + \delta x - x) = DQ(F, [x, x + \delta x])$. Thus, from (1) with h replaced by δx, we have

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} DQ(F,[x,x+\delta x]) = F'(x). \qquad (16.7)$$

That is, dy/dx and $F'(x)$ are merely alternative notations for exactly the same thing, namely, the gradient at the focal point. We will refer to dy/dx as differential notation or Leibniz notation (because Leibniz used it), and to $F'(x)$ as standard notation. The two notations are compared in Table 1. Each has advantages and disadvantages, but differential notation is more widely used. Indeed taking a function's derivative is widely known as **differentiating** the function, and henceforth we will also use this terminology.

In the case of Figure 1, the gradient $F'(x)$ tells us how rapidly photosynthetic rate in maize increases with temperature, precisely at temperature x. An explicit expression for $F'(x)$ will emerge in due course; see (20). With its help, the graphs of F' and F on [12, 51] have been sketched as solid curves in the upper two panels of Figure

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[1] Thus, $\delta y/\delta x < 0$ *usually* means $dy/dx < 0$, but because $\delta y/\delta x < 0$ *can* mean only that $dy/dx = 0$ (e.g., at a summit), all that we can deduce *in general* from $\delta y/\delta x < 0$ in the limit as $\delta x \to 0$ is that $dy/dx \le 0$. Similarly, with regard to (4.5) and (4.9), all we can deduce from $K(u) < 1$ in the limit as $u \to \infty$ is that $K(\infty) \le 1$. These results illustrate a more general result, that limiting processes weaken strong inequalities.

3. We see that $F'(x)$ increases from $F'(12) = 0.0679$ to $F'(13.9) = 0.07$ with the first 2˚C increase of temperature, but then decreases from 0.07 to $F'(51) = -0.212$. Thus F is concave up on [12, 13.9] and concave down on [13.9, 51] with an inflection point at 13.9˚C. Moreover, $F'(x) > 0$ on [12, 37.1] and $F'(x) < 0$ on [37.1, 51], with $F'(37.1) = 0$. Thus F is increasing on [12, 37.1] and decreasing on [37.1, 51], with a maximum at 37.1˚C. The dotted curves show where $y = F(x)$ and $y = F'(x)$ would lie on [0, 12] if formulae (2)-(3) and (20) were used to define them there. Because photosynthesis rate is zero below 12˚C, however, the dotted curves have no biological significance; they simply enable us to see more clearly that 14˚C is an inflection temperature.

Because F' is smooth, it has its own derivative, whose graph is sketched in the bottom panel of Figure 3. We interpret $F''(x)$ as the gradient at temperature x of F' (traversed from left to right in the middle panel of Figure 3). F' is increasing if $F''(x) > 0$, whereas F' is decreasing if $F''(x) < 0$. Thus, as remarked in Lecture 14, $y = F(x)$ is concave up where $F''(x) > 0$ and concave down where $F''(x) < 0$ with an inflection point, marked with a dot in Figure 3, where $F''(x) = 0$. Note, however, that $F''(x) = 0$ cannot identify inflection points where F'' changes discontinuously from positive to negative, or vice versa; see, e.g., Exercise 15.10.

	DIFFERENTIAL	STANDARD
INDEPENDENT VARIABLE	x	x
INDEPENDENT CONSTANT	c	c
CHANGE IN INDEPENDENT VARIABLE	δx	h
DEPENDENT VARIABLE	y	$F(x)$
DEPENDENT CONSTANT	$y\big\vert_{x=c}$	$F(c)$
CHANGE IN DEPENDENT VARIABLE	δy	$F(x+h) - F(x)$
AVERAGE GRADIENT BETWEEN FOCAL AND NEIGHBORING POINT	$\frac{\delta y}{\delta x}$	$DQ(F, [x, x + h])$
GRADIENT AT FOCAL POINT	$\frac{dy}{dx}$	$F'(x)$
GRADIENT AT FIXED POINT	$\frac{dy}{dx}\big\vert_{x=c}$	$F'(c)$

Table 16.1 Some different notations for the same thing

In practice, the most useful notation for derivatives is often a mixture of differential and standard notations. To illustrate, we obtain formulae for the derivative of a sum or a multiple. First, suppose that F and G are smooth on [a, b] with sum S, i.e.,

$$S(x) = F(x) + G(x). \tag{16.8}$$

Then $S(x + h) = F(x + h) + G(x + h)$, implying

$$\begin{aligned} DQ(S, [x, x+h]) &= \frac{S(x+h) - S(x)}{h} = \frac{F(x+h) + G(x+h) - \{F(x) + G(x)\}}{h} \\ &= \frac{F(x+h) - F(x)}{h} + \frac{G(x+h) - G(x)}{h} \\ &= DQ(F, [x, x+h]) + DQ(G, [x, x+h]) \\ &= F'(x) + O[h] + G'(x) + O[h]. \end{aligned} \tag{16.9}$$

But $O[h] + O[h] = O[h]$, by (13.27), and so

$$DQ\big(S,[x,x+h]\big) \;=\; F'(x) + G'(x) \;+\; O[h]. \tag{16.10}$$

Extracting the leading term, we have

$$S'(x) \;=\; F'(x) + G'(x). \tag{16.11}$$

If the graphs of F, G and S on [a, b] are denoted by $y = F(x)$, $z = G(x)$ and $u = S(x)$, respectively, we can write (11) purely in differential notation as

$$\frac{du}{dx} \;=\; \frac{dy}{dx} + \frac{dz}{dx} \tag{16.12}$$

In practice, however, it is usually best written in mixed notation:

$$\frac{d}{dx}\big\{F(x)+G(x)\big\} \;=\; F'(x) + G'(x). \tag{16.13}$$

The symbol S becomes unnecessary.

Second, suppose that the multiple Z is defined on [a, b] by

$$Z(x) \;=\; kF(x), \tag{16.14}$$

where k is a constant. Then $Z(x + h) = kF(x + h)$, implying

$$\begin{aligned} DQ\big(Z,[x,x+h]\big) &= \frac{Z(x+h)-Z(x)}{h} = \frac{kF(x+h) - kF(x)}{h} \\ &= k\left\{\frac{F(x+h) - F(x)}{h}\right\} = k\,DQ\big(F,[x,x+h]\big) \\ &= k\{F'(x) + O[h]\} = kF'(x) + kO[h] \qquad (16.15) \\ &= kF'(x) + kO[h] \\ &= kF'(x) + O[h], \end{aligned}$$

because $kO[h] = O[h]$ by (13.29). Extracting the leading term, we have

$$Z'(x) \;=\; kF'(x), \tag{16.16}$$

which we can rewrite independently of Z as

$$\frac{d}{dx}\big\{kF(x)\big\} \;=\; kF'(x). \tag{16.17}$$

From (13) and (17), we see that mixed notation allows us to dispense with Z and S as soon as we have found the desired result.

Our pair of formulae for derivatives of sums and multiples is most useful in practice if first rewritten in terms of a single equation. If q is another constant and G another (smooth) function, then (17) automatically implies

$$\frac{d}{dx}\big\{qG(x)\big\} \;=\; qG'(x), \tag{16.18}$$

whereas (13) automatically implies

$$\frac{d}{dx}\big\{kF(x) + qG(x)\big\} \;=\; \frac{d}{dx}\big\{kF(x)\big\} + \frac{d}{dx}\big\{qG(x)\big\}. \tag{16.19}$$

So, from (17)-(19),

$$\frac{d}{dx}\big\{kF(x) + qG(x)\big\} \;=\; kF'(x) + qG'(x). \tag{16.20}$$

Note that (13) and (17) are the special cases in which $k = 1 = q$ and $q = 0$, respectively. This result not only facilitates considerable expansion of the list of functions whose derivatives we consider known, but it also enables us to obtain our results more efficiently. For example, from (20) and Table 2 we obtain

$$\begin{aligned}\frac{d}{dx}\{Ax + Bx^2\} &= \frac{d}{dx}\{Ax\} + \frac{d}{dx}\{Bx^2\}\\ &= A\frac{d}{dx}\{x\} + B\frac{d}{dx}\{x^2\}. \qquad (16.21)\\ &= A\cdot 1 + B\cdot 2x\\ &= A + 2Bx\end{aligned}$$

more efficiently than in Exercise 13.11.

Restrictions on domain [a, b] of F	$F(x)$	$F'(x)$	SOURCE
	x	1	Exercise 13.1
	x^2	$2x$	Exercise 13.1
	x^3	$3x^2$	Exercise 13.1
	x^4	$4x^3$	Exercise 13.1
	x^5	$5x^4$	Exercise 13.1
	C (constant)	0	Exercise 13.2
$a > 0$	$\frac{1}{x}$	$-\frac{1}{x^2}$	Exercise 13.12
$a > 0$	$1 - \frac{C}{x^2}$	$\frac{2C}{x^3}$	Exercise 13.13
$a > C$	$\frac{1}{x-C}$	$-\frac{1}{(x-C)^2}$	Exercise 13.15
$a \geq 0$	$\frac{C-x}{C+x}$	$-\frac{2C}{(C+x)^2}$	Exercise 13.16
$a > 0$	$\frac{C}{x^3}$	$-\frac{3C}{x^4}$	Exercise 13.17

Table 16.2 Some derivatives considered known at the end of Lecture 13

In particular, we can use our new formula to obtain the gradient with respect to temperature of net rate of photosynthesis in maize. For with Table 2's help, successive applications of (18) to (2) yield

$$\begin{aligned}F'(x) &= \frac{d}{dx}\left\{m_0(-1153008x + 162144x^2 - 6969x^3 + 134x^4 - x^5)\right\}\\ &= m_0\frac{d}{dx}(-1153008x + 162144x^2 - 6969x^3 + 134x^4 - x^5)\\ &= m_0\left\{-1153008\frac{d}{dx}(x) + 162144\frac{d}{dx}\left(x^2\right) - 6969\frac{d}{dx}\left(x^3\right) + 134\frac{d}{dx}\left(x^4\right) - \frac{d}{dx}\left(x^5\right)\right\}\\ &= m_0\left\{-1153008\cdot 1 + 162144\cdot 2x - 6969\cdot 3x^2 + 134\cdot 4x^3 - 5x^4\right\}\end{aligned}$$

$$= -m_0(5x^4 - 536x^3 + 20907x^2 - 324288x + 1153008) \qquad (16.22)$$

Similarly, for the second derivative of F, we have

$$\begin{aligned}
F''(x) &= \frac{d}{dx}\{F'(x)\} = -m_0 \frac{d}{dx}\left\{5x^4 - 536x^3 + 20907x^2 - 324288x + 1153008\right\} \\
&= -m_0\left(5\frac{d}{dx}\left\{x^4\right\} - 536\frac{d}{dx}\left\{x^3\right\} + 20907\frac{d}{dx}\left\{x^2\right\} - 324288\frac{d}{dx}\{x\} + \frac{d}{dx}\{1153008\}\right) \\
&= -m_0\left(5\cdot 4x^3 - 536\cdot 3x^2 + 20907\cdot 2x - 324288\cdot 1 + 0\right) \\
&= -2m_0(10x^3 - 804x^2 + 20907x - 162144). \qquad (16.23)
\end{aligned}$$

Note that F is a quintic, F is a quartic and F is a cubic. Our calculations illustrate a more general result, namely, that the derivative of a polynomial of order m is a polynomial of order m–1 (see Appendix 17C).

Exercises 16

16.1 A function F is defined on $[1, \infty)$ by

$$F(x) = \frac{10}{x} - \frac{7}{x^2} + 4x^5.$$

Using *only* (18) and Table 2, find expressions for $F'(x)$ and $F''(x)$.

16.2 In a photosynthetic reaction, rate of assimilation (as a proportion of saturation rate) at concentration x (measured in units of half-saturation concentration) is K(x), where K is the join defined on $[0, \infty)$ by

$$K(x) = \begin{cases} \frac{4x}{3} - \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 1 - \frac{1}{6x^2} & \text{if } 1 \le x < \infty \end{cases}$$

(a) Verify that K is identical to the function defined in Exercise 4.5.

(b) Using *only* (18) and Table 2 to find K' and K'', show that K is smooth.

16.3 Show that the function defined in Exercise 2.8, i.e., f defined on [0, 1] by

$$f(x) = x^5 - \frac{1}{2}x + \frac{1}{2},$$

is concave up throughout its domain, and find its global minimum.

16.4 A function W is defined on [1, 3] by

$$W(t) = \begin{cases} 4t^3 + 52 & \text{if } 1 \le t < 2 \\ 3t^2 + 36t & \text{if } 2 \le t \le 3 \end{cases}$$

Is W is smooth? Why, or why not?

16.5 A function W is defined on [0, 2] by

$$W(t) = \begin{cases} 3t^2 - 4t & \text{if } 0 \le t < 1 \\ 4t^3 - 5t^2 & \text{if } 1 \le t \le 2 \end{cases}$$

Is W is smooth? Why, or why not?

16.6 A function W is defined on [0, 6] by

$$W(t) = \begin{cases} 4t^3 + 6t^2 + 2t + 240 & \text{if } 0 \le t < 3 \\ 3t^2 + 128t - 3 & \text{if } 3 \le t < 6 \end{cases}$$

Is W is smooth? Why, or why not?

Appendix 16: Newton's root-finding method

The interpretation of the derivative as gradient is the basis of Newton's method for solving the equation

$$G(t) \;=\; 0, \qquad (16.A1)$$

where G is any smooth function. The method finds roots to any degree of precision.

Let t^* be the exact value of the root; i.e., $G(t^*) = 0$, as in Figure 4, where a possible graph $y = G(t)$ is drawn. Let t_0 be any rough approximation to the root, and construct a triangle with vertices at (t_0, y_0), $(t_0, 0)$ and $(t_1, 0)$, with $y_0 = G(t_0)$. Then (t_0, y_0) lies on the graph of G and, for any t_1, the triangle has a right-angle at $(t_0, 0)$; see Figure 4, in which (t_0, y_0) is denoted by A. The triangle has height $y_0 = G(t_0)$ and base is $t_0 - t_1$, so that the slope of its hypotenuse is $y_0/(t_0 - t_1)$. Now choose t_1 to make the tangent to the graph at A coincide with the triangle's hypotenuse. Then, because the slope of the tangent at A is is $G'(t_0)$, we have

$$\frac{y_0}{t_0 - t_1} \;=\; G'(t_0), \qquad (16.A2)$$

implying

$$t_1 \;=\; t_0 \;-\; \frac{y_0}{G'(t_0)} \;=\; t_0 \;-\; \frac{G(t_0)}{G'(t_0)}. \qquad (16.A3)$$

If t_0 is an approximation to t^*, however, then t_1 is a better one; see Figure 4, where B denotes (t_1, y_1), with $y_1 = G(t_1)$. In other words, with F defined by

$$F(t) \;=\; t \;-\; \frac{G(t)}{G'(t)}, \qquad (16.A4)$$

if t_0 is an approximation to t^* then $t_1 = F(t_0)$ is a better one. Drawing a few pictures will convince you that this result holds not only regardless of whether G is concave up (as in Figure 4) or concave down, but also of whether $G(t_0)$ is positive (as in Figure 4) or negative.

If $t_1 = F(t_0)$ is a better approximation to t^* than t_0, however, then $t_2 = F(t_1)$ is an even better one still; again, see Figure 4, where C denotes (t_2, y_2), with $y_2 = G(t_2)$. And so on. Continuing as above, we recursively define a sequence $\{t_k\}$ on $[0, \infty)$ by

$$t_{k+1} \;=\; F(t_k), \quad k \geq 0 \qquad (16.A5)$$

and it is clear from Figure 4 that the sequence converges to t^*. That is,

$$\lim_{k \to \infty} t_k \;=\; t^* \qquad (16.A6)$$

Suppose, for example, that we require the time during systole in Figure 1.3 at which ventricular volume is 100 ml. By (12.19), this time is when $V(t) = 100$, where

$$V(t) \;=\; \frac{43895}{432} + \frac{2450}{3}t - \frac{96250}{9}t^2 + \frac{980000}{27}t^3 - \frac{350000}{9}t^4. \qquad (16.A7)$$

We must therefore solve $G(t) = 0$ with

$$G(t) \;=\; V(t) - 100 \;=\; \frac{695}{432} + \frac{2450}{3}t - \frac{96250}{9}t^2 + \frac{980000}{27}t^3 - \frac{350000}{9}t^4. \qquad (16.A8)$$

From (14.13), or by the method of this lecture, (A8) implies

$$G'(t) \;=\; V'(t) \;=\; \frac{2450}{3} - \frac{192500}{9}t + \frac{980000}{9}t^2 - \frac{1400000}{9}t^3, \qquad (16.A9)$$

from which (A4) yields

$$F(t) \;=\; t \;-\; \frac{G(t)}{G'(t)} \;=\; \frac{10080000t^4 - 6272000t^3 + 924000t^2 + 139}{3360\{4000t^3 - 2800t^2 + 550t - 21\}}. \qquad (16.A10)$$

Figure 1.3 suggests that t^* is not much bigger than 0.1. Therefore, we take $t_0 = 0.1$, from which (A5) and (A10) yield $t_1 = F(t_0) = 0.12247024$, $t_2 = F(t_1) = 0.12059359$ and $t_{k+1} = F(t_k) = 0.12058665$ for $k \geq 2$. That is, to eight significant figures, the sequence $\{t_k\}$ converges in three steps to $t^* = 0.12058665$.

Newton's method is especially useful for finding maxima or minima. For example, the global maximizer t^* of ventricular outflow

$$f(t) = -2450/3 + 192500t/9 - 980000\,t^2/9 + 1400000\,t^3/9 \qquad (16.A11)$$

in Figure 1.3 satisfies $f'(t^*) = 0$. Therefore, to determine t^*, we must solve $G(t) = 0$ with $G(t) = f'(t)$, hence $G'(t) = f''(t)$. From (14.13), or by the method of this lecture, (A.11) implies

$$\begin{aligned} f'(t) &= 192500/9 - 1960000t/9 + 1400000\,t^2/3 \\ &= 17500(11 - 112t + 240\,t^2)/9, \end{aligned} \qquad (16.A12a)$$

hence

$$f''(t) = 17500(-112 + 480t)/9. \qquad (16.A12b)$$

Thus, from (A4), (A5) and (A12), the approximating sequence is defined recursively by

$$t_{k+1} = F(t_k), \quad k \geq 0 \qquad (16.A13a)$$

with

$$F(t) = t - \frac{f'(t)}{f''(t)} = t - \frac{11 - 112t + 240t^2}{480t - 112} = \frac{240t^2 - 11}{16(30t - 7)}. \qquad (16.A13b)$$

See Exercise 5.14.

Answers and Hints for Selected Exercises

16.1 Using Rows 5-8 of Table 2 with C = 1 in Row 6 and C = 7 in Row 8, (18) implies

$$\begin{aligned} F'(x) &= 10\frac{d}{dx}\left\{\frac{1}{x}\right\} + \frac{d}{dx}\left\{\frac{-7}{x^2}\right\} + 4\frac{d}{dx}\left\{x^5\right\} \\ &= 10\frac{d}{dx}\left\{\frac{1}{x}\right\} + 0 + \frac{d}{dx}\left\{\frac{-7}{x^2}\right\} + 4\frac{d}{dx}\left\{x^5\right\} \\ &= 10\frac{d}{dx}\left\{\frac{1}{x}\right\} + \frac{d}{dx}\{1\} + \frac{d}{dx}\left\{\frac{-7}{x^2}\right\} + 4\frac{d}{dx}\left\{x^5\right\} \\ &= 10\frac{d}{dx}\left\{\frac{1}{x}\right\} + \frac{d}{dx}\left\{1 - \frac{7}{x^2}\right\} + 4\frac{d}{dx}\left\{x^5\right\} \\ &= 10\left(-\frac{1}{x^2}\right) + \frac{14}{x^3} + 4\cdot 5x^4 \\ &= 2\left(10x^4 - \frac{5}{x^2} + \frac{7}{x^3}\right) \end{aligned}$$

We can rewrite this result as

$$F'(x) = 1 - \frac{10}{x^2} + \frac{14}{x^3} + 20x^4 - 1.$$

So (18) implies

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left\{1 - \frac{10}{x^2}\right\} + 14\frac{d}{dx}\left\{\frac{1}{x^3}\right\} + 20\frac{d}{dx}\left\{x^4\right\} - \frac{d}{dx}\{1\} \\ &= \frac{20}{x^3} - \frac{42}{x^4} + 20\cdot 4x^3 - 0 \\ &= 2\left(40x^3 - \frac{21}{x^4} + \frac{10}{x^3}\right) \end{aligned}$$

on using Rows 4, 6, 8 and 11 of Table 2.

16.2 Using (18) in conjunction with Rows 1, 2 and 8 of Table 2, we have

$$K'(x) = \begin{cases} \frac{4}{3} - x & \text{if } 0 \le x < 1 \\ \frac{1}{3x^3} & \text{if } 1 \le x < \infty \end{cases}$$

So $K'(1-) = 4/3 - 1 = 1/3 = K'(1+)$, implying that K is continuous. Further, using (18) in conjunction with Rows 1, 6 and 11 (with C = 1/3) of Table 2, we have

$$K''(x) = \begin{cases} -1 & \text{if } 0 \le x < 1 \\ -\frac{1}{x^4} & \text{if } 1 \le x < \infty \end{cases}$$

So $K''(1-) = -1 = K''(1+)$, implying that K is smooth.

16.3 We have $f'(x) = 5x^4 - 1/2$ and $f''(x) = 20x^3$, which is positive on (0, 1]. So f must be concave up on [0, 1]. Also, $f'(x) = 0$ where $5x^4 = 1/2$ or $x = 1/\sqrt{\sqrt{10}}$ = 0.562, which must be a global minimizer, because f is concave up. So the global minimum is

$$f(1/\sqrt{\sqrt{10}}) = \left(\frac{1}{\sqrt{\sqrt{10}}}\right)^5 - \frac{1}{2}\cdot\frac{1}{\sqrt{\sqrt{10}}} + \frac{1}{2}$$
$$= \frac{1}{10}\cdot\frac{1}{\sqrt{\sqrt{10}}} - \frac{1}{2}\cdot\frac{1}{\sqrt{\sqrt{10}}} + \frac{1}{2}$$
$$= \frac{1}{2} - \frac{2}{5}\cdot\frac{1}{\sqrt{\sqrt{10}}} = 0.275$$

16.4 By Table 16.2 and our results for derivatives of joins and sums of multiples,

$$W'(t) = \begin{cases} 12t^2 & \text{if } 1 \le t < 2 \\ 6t + 36 & \text{if } 2 \le t < 3 \end{cases}$$

Therefore

$$W(2-) = 4 \times 2^3 + 52 = 32 + 52 = 84, \quad W(2+) = 3 \times 2^2 + 36 \times 2 = 12 + 72 = 84,$$

implying that W is continuous on [1, 3], and

$$W'(2-) = 12 \times 2^2 = 48, \quad W'(2+) = 6 \times 2 + 36 = 48,$$

implying that W is smooth.

16.5 By Table 16.2 and our results for derivatives of joins and sums of multiples,

$$W'(t) = \begin{cases} 6t - 4 & \text{if } 0 \le t < 1 \\ 12t^2 - 10t & \text{if } 1 \le t < 2 \end{cases}$$

Therefore

$$W(2-) = 3 \times 1^2 - 4 = -1, \quad W(2+) = 4 \times 1^3 - 5 \times 1^2 = -1,$$

implying that W is continuous on [0, 2], and

$$(2-) = 6 - 4 = 2, \quad W'(2+) = 12 \times 1^2 - 10 = 2,$$

implying that W is smooth.

16.6 By Table 16.2 and our results for derivatives of joins and sums of multiples,

$$W'(t) = \begin{cases} 12t^2 + 12t + 2 & \text{if } 0 \le t < 3 \\ 6t + 128 & \text{if } 3 \le t < 6 \end{cases}$$

Therefore

$$W(3-) = 108 + 54 + 6 + 240 = 408, \quad W(3+) = 27 + 384 - 3 = 408,$$

implying that W is continuous on [0, 6]; and

$$W'(3-) = 108 + 36 + 2 = 146, \quad W'(2+) = 18 + 128 = 146,$$

implying that W is smooth.