14. Smoothness and concavity: an algebraic perspective

We have already established, in Lecture 13, that a function V is growing or decaying at time t according to whether $V'(t) > 0$ or $V'(t) < 0$. What about times when $V'(t) = 0$? Then V is **stationary**, i.e., neither growing nor decaying. Typically, a function becomes stationary because it has reached a local extremum. If $V'(t^*) = 0$ because $V'(t)$ changes sign from negative to positive at $t = t^*$ — in other words, because V stops decreasing and starts increasing at $t = t^*$ — then $V(t^*)$ is a local minimum, e.g., $V(t^*) = 49.1$ at $t^* = 0.3$ in Figure 2(a)-(b). Similarly, if $V'(t^*) = 0$ because $V'(t)$ changes sign from positive to negative at $t = t^*$ — in other words, because V stops increasing and starts decreasing at $t = t^*$ — then $V(t^*)$ is a local maximum. Thus local extrema of a smooth function V can be found precisely by solving the equation $V'(t) = 0$.[1] For example, on using (13.6) and (13.18), ventricular volume V on [0.28, 0.35] in Lecture 13 has derivative $V' = v$ defined algebraically by

$$v(t) \quad = \quad c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 \tag{14.1}$$

where

$$c_0 = \tfrac{43895}{432}, \quad c_1 = \tfrac{2450}{3}, \quad c_2 = -\tfrac{96250}{9}, \quad c_3 = \tfrac{980000}{27}, \quad c_4 = -\tfrac{350000}{9}. \tag{14.2}$$

On simplifying (1)-(2), we readily find that $V'(t) = 350(20t-1)(10t-3)(7-20t)/9$. So the local minimizer in Figure 2(a)-(b) is $t^* = 0.3$.

Now, the inflow $v = V'$ is itself a smooth function; see Figure 1(a). It therefore has its own derivative v' defined by

$$\begin{aligned} v'(t) &= \text{THAT PART OF } DQ(v, [t, t+h]) \text{ WHICH IS INDEPENDENT OF } h && (14.3a) \\ &= \lim_{h \to 0} DQ(v, [t, t+h]), && (14.3b) \end{aligned}$$

Because

$$\begin{aligned} v(t+h) &= c_1 + 2c_2(t+h) + 3c_3(t+h)^2 + 4c_4(t+h)^3 \\ &= c_1 + 2c_2 t + 2c_2 h + 3c_3(t^2 + 2th + h^2) && (14.4) \\ &\quad + 4c_4(t^3 + 3t^2 h + 3th^2 + h^3), \end{aligned}$$

we have

$$v(t+h) - v(t) \quad = \quad 2c_2 h + 6c_3 th + 3c_3 h^2 + 12c_4 t^2 h + 12c_4 th^2 + 4c_4 h^3, \tag{14.5}$$

implying

$$\begin{aligned} DQ(v,[t,t+h]) &= \frac{v(t+h) - v(t)}{h} && (14.6) \\ &= 2c_2 + 6c_3 t + 12c_4 t^2 + h\{12c_4 t + 4c_4 h + 3c_3\}. \end{aligned}$$

So (3), (6) and (2) imply

$$\begin{aligned} v'(t) &= 2c_2 + 6c_3 t + 12c_4 t^2 && (14.7a) \\ &= -\frac{2500}{9}\{77 - 784t + 1680t^2\} && (14.7b) \end{aligned}$$

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[1] Note, however, that V can be stationary where there is no local extremum. For example, in Figure 1.3, V is stationary not only on [0.35, 0.4], where V = 50 implies $V'(t) = 0$ (Exercise 2), but also at $t = 0.75$, where $V'(0.75) = 0$ even though $t = 0.75$ is not a local extremizer. What happens here is that V starts to increase again as soon as it has stopped increasing.

ml/s^2. The graph of v' is plotted in Figure 1(b), directly below the graph of v. Observe from the dashed lines that, e.g., inflow is increasing at 2520 ml/s^2 after 0.29 seconds, even though blood is still being discharged through the aorta at over 22 ml/s, because $v'(0.29) = 2520$ when $v(0.29) = -22.4$. Similarly, $v'(0.32) = 513.3$ ml/s^2 when $v(0.32) = 25.2$ ml/s and $v'(0.34) = -1291$ ml/s^2 when $v(0.34) = 18$ ml/s.

Because v' is the derivative of $v = V'$, we refer to v' as the **second derivative** of V and denote it by V'' (as well as by v'). More generally, we define V'' by

$$V''(t) \;=\; \text{THAT PART OF } DQ(V', [t, t+h]) \text{ WHICH IS INDEPENDENT OF } h \qquad (14.8a)$$

$$= \lim_{h\to 0} DQ(V', [t, t+h]). \qquad (14.8b)$$

Provided V'' is smooth, we can also define the **third derivative** of V by

$$V'''(t) \;=\; \lim_{h\to 0} DQ(V'', [t, t+h]), \qquad (14.9)$$

and so on. But third derivatives are seldom encountered in practice.

On the other hand, second derivatives are often encountered, and are useful in particular for determining concavity. Figure 2(d) shows the elevation of V in Figure 2(b): For a tiny insect who traverses the graph from left to right, $\theta(t)$ degrees is the angle at time t between line of sight (dotted) and horizontal (dashed), e.g., at $t = 0.305$ the elevation is $\theta(0.305) = 21.1$ degrees. From Lecture 1, V is concave up or down according to whether θ is increasing or decreasing, with an inflection point wherever θ has a local extremum of elevation. Thus, comparing Figure 2(b) with Figure 2(d), V has an inflection point at $t = s$, where θ has a local maximum. Comparing Figure 2(d) with Figure 2(b), however, we see that $t = s$ is also where V' has a local maximum. Further comparison reveals that both θ and V' are zero at $t = t^*$. These observations illustrate a more general result, namely, that *even though elevation and derivative are not the same function,* they have common zeroes and common extremizers, i.e.,

$$\theta(t^*) = 0 \quad\Leftrightarrow\quad V'(t^*) = 0 \qquad (14.10a)$$

$$\text{s is a local extremizer of } \theta \quad\Leftrightarrow\quad \text{s is a local extremizer of } V' \qquad (14.10b)$$

Furthermore, if θ is increasing or decreasing, then so is V', and vice versa. In cases where they are not actually smooth (as assumed in this lecture), these two functions even have common corners, i.e.,

$$\theta \text{ has a discontinuity} \quad\Leftrightarrow\quad V' \text{ has a discontinuity.} \qquad (14.10c)$$

We will establish these results in Appendix 28. Taking them on faith until then, we find from (10b) that inflection points can be determined either by finding the stationary points of θ, i.e., by solving the equation $\theta'(s) = 0$, or else by finding the stationary points of V', i.e., by solving

$$V''(s) = 0. \qquad (14.11)$$

The second method is invariably easier.

For example, in Figure 1 we have $v'(t) > 0$ if $t < s$ but $v'(t) < 0$ if $t > s$, where s is the maximizer of v defined by $v'(s) = V''(s) = 0$ or, from (7),

$$77 - 784s + 1680s^2 = 0. \qquad (14.12)$$

This equation is a quadratic equation, whose roots are $s = (14-\sqrt{31})/60 = 0.140537$ and $s = (14+\sqrt{31})/60 = 0.326129$. Only the second root belongs to [0.28, 0.35]. Thus inflow v increases on [0.28, 0.326] but decreases on [0.326, 0.35], with maximum v(s) = 26.8 ml/s. Correspondingly, V has an inflection point where t = s.

In effect, relationship (10b) between elevation and derivative enables us to redefine inflection points as local extremizers of the derivative (as opposed to local extremizers of elevation). Correspondingly, a function V is concave up or concave down according to whether V'' is positive or negative; see Figure 3. These newer definitions are more useful in practice because V'' is invariably easier to calculate than θ', but they were unavailable in Lecture 1 because there derivative had not yet been defined. Similarly, relationship (10c) between elevation and derivative enables us to redefine a smooth function as a continuous function with a continuous derivative (as opposed to a continuous function without corners).

We conclude this lecture by emphasizing that (13.5), (13.18) and (7a) yield explicit expressions for the first and second derivatives of an arbitrary polynomial of order up to 4. That is,

$$\begin{aligned} V(t) &= c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 \quad \Rightarrow \\ V'(t) &= c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 \qquad (14.13) \\ V''(t) &= 2c_2 + 6c_3 t + 12 c_4 t^2, \end{aligned}$$

not only when V means volume and c_0, c_1, c_2, c_3, c_4 are defined by (2), but also for any other values of these parameters and any other interpretation of V. You will find this result extremely useful in Exercises 1-3.

Exercises 14

14.1 A function f is defined on [0, 5] by $f(t) = 17 - 18t + 8t^2 - t^3$.
- (i) Find all local extrema
- (ii) Where is f concave upward? Concave downward?
- (iii) Find both Max(f, [0, 5]) and Min(f, [0, 5]).
- (iv) Using Mathematica or otherwise, sketch the graphs of f and f', one above the other.

Hint: Set V = f in (13), and choose suitable values for c_0, c_1, c_2, c_3, c_4 (e.g., $c_4 = 0$).

14.2* A function f is defined on [0, 4] by $f(t) = 3t^3 - 14t^2 + 9t + 8$.
- (i) Find all local extrema
- (ii) Where is f concave upward? Concave downward?
- (iii) Find both Max(f, [0, 4]) and Min(f, [0, 4]).
- (iv) Sketch the graphs of f and f', one above the other.

14.3 A function f is defined on [0, 3] by $f(t) = t(9t - 2t^2 - 12)/3$.
- (i) Find an expression for $f'(t)$
- (ii) Hence find all local extrema of f
- (iii) Where is f concave upward? Where is f concave downward?
- (iv) Find both Max(f, [0, 3]) and Min(f, [0, 3])
- (v) Sketch the graphs of f and f', one above the other.

Hint: First expand the expression for f(t), then use the hint for Exercise 1.

14.4 The function f is defined on [0, 7] by $f(t) = (2t^2 - 19t + 41)(t - 1)/6$.

- (i) Find an expression for $f'(t)$
- (ii) Hence find all local extrema of f
- (iii) Where is f concave upward? Where is f concave downward?
- (iv) Find both Max(f, [0, 7]) and Min(f, [0, 7])
- (v) Sketch the graphs of f and f', one above the other.

14.5 A function f is defined on [2, 8] by $f(t) = t(27t - 2t^2 - 108)$. Where is it concave up? Concave down? Find its global maximum.

14.6* A smooth function f with domain [0, 10] and range [-1, 5] has global minimizer t = 0, global maximizer t = 3, local minimizer t = 8, an inflection point at t = 5 and f(10) = 4. Find a possible formula for f.

Answers and Hints for Selected Exercises

14.1 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Assignment B, #3)

14.5 Because $f(t) = -108t + 27t^2 - 2t^3$, (13) with $c_0 = 0 = c_4$, $c_1 = -108$, $c_2 = 27$ and $c_3 = -2$ yields

$$f'(t) = -108 + 54t - 6t^2 = 6(9t - t^2 - 18) = 6(t-3)(6-t)$$

and

$$f''(t) = 54 - 12t = 6(9-2t).$$

So $f''(t) < 0$ when $2 \le t < 9/2$ but $f''(t) > 0$ when $9/2 < t \le 8$, implying that f is concave up on [2, 9/2] and concave down on [9/2, 8], with an inflection point where $t = 9/2$. Also, $f'(t) < 0$ if $2 \le t < 3$, $f'(t) > 0$ if $3 < t < 6$, $f'(t) < 0$ if $6 < t \le 8$; that is, f is decreasing on [2, 3], increasing on [3, 6] and decreasing again on [6, 8]. So the global maximizer is either $t = 2$ or $t = 6$. But $f(2) = -124$ and $f(6) = -108$, which is bigger. So the global maximum is -108.

14.6 The function must satisfy (at least) six constraints, namely, $f(0) = -1$, $f(3) = 5$, $f'(3) = 0$, $f''(5) = 0$, $f'(8) = 0$ and $f(10) = 4$. The simplest such function is probably a fifth-order polynomial.