13. From ventricular volume to inflow: the derivative as rate of change

Although we can determine approximately where an ordinary function V increases or decreases by looking at its graph, there are times when greater accuracy is needed. We then use algebraic methods based on the index function DQ. So first recall from Lecture 8 that

$$\mathrm{DQ}(V,[a,b]) \;=\; \frac{V(b)-V(a)}{b-a} \qquad (13.1)$$

is the average net rate of increase of V over the interval [a, b].

For the sake of simplicity, we assume in this lecture that V is a function of time (although in Lecture 16 our method will apply with equal force to functions of other variables, e.g., mass, temperature, or CO_2 concentration). Thus increase or decrease means growth or decay. Moreover, we consider only smooth functions (although in Lecture 15 our method will apply with equal force to piecewise-smooth functions, which are smooth on various subdomains).

A quantity is growing or decaying at any given time if it is bigger or smaller soon afterwards. So V is increasing at time t if V(t + h) > V(t) when h is very small and positive, whereas V is decreasing at time t if V(t + h) < V(t) when h is very small and positive. For example, if V is ventricular volume, then V(t + h) > V(t) if the ventricle is refilling at time t but V(t + h) < V(t) if the ventricle is discharging. Note that h ≠ 0, because we must compare different times to know whether V is growing or decaying.

In other words, V is increasing at time t if

$$\mathrm{Diff}(V,[t,t+h]) \;=\; V(t+h)-V(t) \;>\; 0 \qquad (13.2a)$$

for all positive and very small h, whereas V is decreasing at time t if

$$\mathrm{Diff}(V,[t,t+h]) \;=\; V(t+h)-V(t) \;<\; 0 \qquad (13.2b)$$

for all positive and very small h. Because h > 0, however, Diff(V, [t, t +h]) invariably has the same sign as

$$\frac{\mathrm{Diff}(V,[t,t+h])}{h} \;=\; \frac{V(t+h)-V(t)}{(t+h)-t} \;=\; \mathrm{DQ}(V,[t,t+h]). \qquad (13.3)$$

Hence, from (2), V is increasing at time t if

$$\mathrm{DQ}(V,[t,t+h]) \;>\; 0 \qquad (13.4a)$$

for all positive and very small h, whereas V is decreasing at time t if

$$\mathrm{DQ}(V,[t,t+h]) \;<\; 0 \qquad (13.4b)$$

for all positive and very small h. Moreover, because DQ(V, [t, t + h]) is the average net growth rate over the interval [t, t + h] and h is very small, we regard DQ(V, [t, t + h]) as an approximation to the net growth rate of V at time t. The smaller the value of h, the better this approximation.

For example, suppose that V is ventricular volume during the systolic phase of our cardiac cycle. Then Diff(V, [t, t + h]) is net recharge (positive or negative) between time t and time t + h, so that DQ(V, [t, t + h]) is average rate of recharge over [t, t + h]; the ventricle is refilling if DQ(V, [t, t + h]) > 0 and discharging if DQ(V, [t, t + h]) < 0. But rate of recharge equals inflow. So DQ(V, [t, t + h]) is average inflow over [t, t + h]; and because h is very small, DQ(V, [t, t + h]) approximates inflow at time t. The smaller the value of h, the better this approximation.

For the sake of definiteness, we now restrict our attention to the last 0.07 seconds of the systolic phase. Then V has domain [0.28, 0.35], on which it is defined by

$$V(t) \;=\; c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 \qquad (13.5)$$

with

$$c_0 = \tfrac{43895}{432}, \quad c_1 = \tfrac{2450}{3}, \quad c_2 = -\tfrac{96250}{9}, \quad c_3 = \tfrac{980000}{27}, \quad c_4 = -\tfrac{350000}{9}, \tag{13.6}$$

from (12.19). So

$$\begin{aligned} V(t+h) &= c_0 + c_1(t+h) + c_2(t+h)^2 + c_3(t+h)^3 + c_4(t+h)^4 \\ &= c_0 + c_1 t + c_1 h + c_2(t^2 + 2th + h^2) \\ &\quad + c_3(t^3 + 3t^2h + 3th^2 + h^3) + c_4(t^4 + 4t^3h + 6t^2h^2 + 4th^3 + h^4) \end{aligned} \tag{13.7}$$

and, subtracting (5) from (7),

$$\begin{aligned} V(t+h) - V(t) &= c_1 h + 2c_2 th + c_2 h^2 + 3c_3 t^2 h + 3c_3 th^2 + c_3 h^3 \\ &\quad + 4c_4 t^3 h + 6c_4 t^2 h^2 + 4c_4 th^3 + c_4 h^4. \end{aligned} \tag{13.8}$$

Dividing by h, we find that

$$\begin{aligned} DQ(V,[t,t+h]) &= \frac{V(t+h) - V(t)}{h} \\ &= c_1 + 2c_2 t + c_2 h + 3c_3 t^2 + 3c_3 th + c_3 h^2 \\ &\quad + 4c_4 t^3 + 6c_4 t^2 h + 4c_4 th^2 + c_4 h^3. \end{aligned} \tag{13.9}$$

Rearranging terms, we have

$$\begin{aligned} DQ(V,[t,t+h]) &= c_1 + 2c_2 t + 3c_3 t^2 + 4c_4 t^3 + \\ &\quad h\{3c_3 t + c_3 h + c_2 + 6c_4 t^2 + 4c_4 th + c_4 h^2\}. \end{aligned} \tag{13.10}$$

Observe that DQ(V, [t, t + h]) consists of a first term or **leading** term that is independent of h plus a second term or **error** term that depends on h. This error term has the nice property that it can be neglected if h is sufficiently small. Why? We argue as follows. Because the domain of V is [0.28, 0.35], we have

$$t \le \frac{7}{20}. \tag{13.11}$$

Because h is very small, it is safe to assume that

$$h \le 1 \tag{13.12}$$

is very comfortably satisfied. So

$$3t + h \le 3 \times \frac{7}{20} + 1 = \frac{41}{20} \tag{13.13}$$

and

$$6t^2 + 4th + h^2 \le 6 \times \left(\tfrac{7}{20}\right)^2 + 4 \times \tfrac{7}{20} \times 1 + 1^2 = \frac{627}{200} \tag{13.14}$$

Now, because the magnitude of a sum cannot exceed the sum of the magnitudes, the magnitude of the error term satisfies

$$\begin{aligned} & h\left|3c_3 t + c_3 h + c_2 + 6c_4 t^2 + 4c_4 th + c_4 h^2\right| \\ &= h\left|c_2 + (3t+h)c_3 + (6t^2 + 4th + h^2)c_4\right| \\ &\le h\left\{|c_2| + (3t+h)|c_3| + (6t^2 + 4th + h^2)|c_4|\right\} \\ &\le h\left\{|c_2| + \tfrac{41}{20}|c_3| + \tfrac{627}{200}|c_4|\right\} \\ &\le h\left\{\tfrac{96250}{9} + \tfrac{41}{20}\tfrac{980000}{27} + \tfrac{627}{200}\tfrac{350000}{9}\right\} \\ &= \frac{5589500h}{27} = 207018.5h. \end{aligned} \tag{13.15}$$

Thus the magnitude of the error term in (10) cannot possibly exceed 207019h, and we can make 207019h negligibly small by making h sufficiently small. For example, if numbers less than 10^{-6} are considered neglibly small, then we can neglect 207019h (and anything smaller) if h is less than 4.83×10^{-12}. If, on the other hand, only numbers less than 10^{-12} are considered negligibly small, then we can neglect 207019h if h is less than 4.83×10^{-18}; and so on. A small enough h (> 0) can always be found to make 207019h negligible, no matter how small we consider negligible to mean. If, however, the sign of DQ(V, [t, t+h]) determines whether V is increasing or decreasing and the error term can always be neglected, then the leading term must completely determine whether V is increasing or decreasing at time t. That is, if we define a function v on [0.28, 0.35] by

$$v(t) \quad = \quad c_1 + 2c_2t + 3c_3t^2 + 4c_4t^3, \qquad (13.16)$$

then V is increasing at time t if v(t) > 0 but decreasing at time t if v(t) < 0. The graph of v is plotted in Figure 1, directly below the graph of V. Note that v(t) < 0 if t < 0.3 but v(t) > 0 if t > 0.3. Thus V is increasing on subdomain [0.28, 0.3], but V is decreasing on subdomain [0.3, 0.35].

Furthermore, because DQ(V, [t, t + h]) approximates ventricular inflow at time t, and because this approximation gets better and better as h gets smaller and smaller, if h is so small that the error term can be neglected then whatever is left must completely determine inflow at time t . In other words, v(t) must be the inflow at time t, not approximately, but precisely. As indicated by dashed lines in Figure 1, for example, after 0.29 seconds ventricular volume is 49.2 ml and decreasing at 22.4 ml/s, because V(0.29) = 49.2 and v(0.29) = –22.4. Similarly, after 0.32 s the volume is 49.4 ml and increasing quite rapidly at 25.2 ml/s, because V(0.32) = 49.4 and v(0.32) = 25.2; and after 0.34 s the volume is 49.9 ml and increasing more slowly at 18 ml/s, because V(0.34) = 49.9 and v(0.34) = 18.0.

Our discussion of leading term and error term is perfectly general, in the sense that it applies to any smooth function. It is therefore convenient to introduce some general notation. Accordingly, let V be any smooth function; let $V'(t)$ denote the leading term of DQ(V, [t, t + h]), i.e., the part of the difference quotient that depends only on t; and let $\varepsilon_V(h, t)$ denote the error term of DQ(V, [t, t + h]), i.e., the part of the difference quotient that depends on h but which can be neglected if h is sufficiently small. Then

$$DQ\big(V, [t, t+h]\big) \quad = \quad V'(t) \quad + \quad \varepsilon_V(h,t). \qquad (13.17)$$

For example, in the case of Figure 1, (10) implies that

$$V'(t) \quad = \quad c_1 + 2c_2t + 3c_3t^2 + 4c_4t^3 \qquad (13.18)$$

and

$$\varepsilon_V(h,t) \quad = \quad h(3c_3t + c_3h + c_2 + 6c_4t^2 + 4c_4th + c_4h^2). \qquad (13.19)$$

The notation $\varepsilon_V(h, t)$ is a very precise one, and usually precision is valued in mathematics. There are times, however—and this is one of them—when it is possible to have too much precision. For the purpose of this lecture, we do not really care how the error term depends on h and t; all we care about is that it can be neglected if h is sufficiently small. So we adopt notation that retains no more information than we actually need. We define "big oh" by

$$O[h] \quad = \quad \text{ANYTHING WHOSE MAGNITUDE APPROACHES ZERO AS } h \text{ APPROACHES ZERO} \qquad (13.20)$$

so that $\varepsilon_V(h, t) = O[h]$, implying

$$DQ(V,[t,t+h]) \;=\; V'(t) \;+\; O[h]. \qquad (13.21)$$

Then why bother with ε_V at all? The answer is that on rare occasions we need some extra precision. In this course, the only such time will be in Appendix 20. Accordingly, we will use ε_V in Appendix 20, but O[h] everywhere else.

The function V' defined by

$$V'(t) \;=\; \text{THAT PART OF } DQ(V,[t,t+h]) \text{ WHICH IS INDEPENDENT OF } h \qquad (13.22)$$

is called the **derivative** of V. It determines not only *whether* V is growing or decaying at time t, but also the *rate* at which V is growing or decaying; in other words, $V'(t)$ is the **instantaneous net growth rate**. An alternative definition of V' emerges from (21), in which we can allow h to become arbitrarily close to zero, as long as h never actually reaches zero itself. We call this process taking the limit as h tends to zero. If h approaches zero, however, then O[h] must also approach zero, and so DQ(V, [t, t + h]) = $V'(t)$ + O[h] must approach $V'(t)$. In mathematical shorthand,

$$\lim_{h\to 0} O[h] \;=\; 0 \qquad (13.23)$$

and

$$\lim_{h\to 0} DQ(V,[t,t+h]) \;=\; V'(t). \qquad (13.24)$$

Either (22) or (24) defines a new function, which we have agreed to call the derivative.

No, wait a minute. A function isn't properly defined until we have identified its domain. So what is the domain of V'? It is essentially the same as that of V; but not quite, because DQ(V, [t, t + h]) is well defined only if [t, t + h] is a subdomain of V, which requires

$$a \;\le\; t \;<\; t+h \;\le\; b \qquad (13.25).$$

and hence $t < b$ (because $h > 0$). Even though t may be arbitrarily close to b as $h \to 0$, we must insist on $t \ne b$ because $h \ne 0$. So, strictly speaking, the domain of V' is not [a, b], but rather [a, b), and $V'(b)$ is undefined. On the other hand, $V'(t)$ is well defined for any $t < b$, and all such values become arbitrarily close together as t approaches b from below. Accordingly, we regard their common limit as the definition of $V'(b)$. In other words, we extend the domain of V' from [a, b) to [a, b] by defining

$$V'(b) \;=\; \lim_{t\to b-} V'(t)\,, \qquad (13.26)$$

where $t \to b-$ signifies that t approaches b from below (or, if you prefer, from the left, because t increases horizontally to the right).

We conclude this lecture with a few remarks about big oh. It is important to understand that O is *not* a function, because O[h] does not label h unambiguously; for example, h = O[h] and 2h = O[h] are both true, hence O[h] – O[h] = 0 is false. Rather, O[h] is a catchall for terms that approach zero as $h \to 0$. So big oh yields some strange equations. Quite apart from

$$O[h] \;+\; O[h] \;=\; O[h], \qquad (13.27)$$

which does not imply O[h] = 0, we have

$$\{O[h]\}^2 \;=\; O[h], \qquad (13.28)$$

which implies neither O[h] = 0 nor O[h] = 1. Furthermore, for any Z we have

$$Z(t)O[h] = O[h], \qquad (13.29)$$

which does not imply Z(t) = 1. These strange equations make perfect sense as soon as we learn how to interpret them properly. Because O is not a function, we never write equations of the form O[h] = SOMETHING; rather, we write SOMETHING = O[h], and we interpret this equation to mean that SOMETHING reduces to O[h] if all we are interested in is its behavior as h approaches zero. Thus, intuitively, (27) says that if you add two numbers of negligible magnitude then the result is still of negligible magnitude (even if this magnitude is bigger than that of the numbers you added). Again, intuitively, (28) says that if you if you multiply two numbers of negligible magnitude, then the result is still of negligible magnitude (in fact, its magnitude is even smaller than that of the numbers you multiplied). Finally, the intuitive interpretation of (29) is that you can't prevent something from approaching zero as $h \to 0$ by multiplying it by something independent of h. We will find these results especially useful in Lectures 17 and 20.

Exercises 13

13.1 The power function defined by $f(t) = t^m$ is an increasing function on $[0, \infty)$ for all $m > 0$. Use DQ to establish this result for m = 1, 2, 3, 4 and 5, and in each case obtain an explicit expression for $f'(t)$.

13.2 The function F is defined on [a, b] by F(t) = C, where C is a constant. Prove that $F'(t) = 0, a \le t < b$.

13.3 For f defined on [0.05, 0.35] in Appendix 2B, use Mathematica to find an expression for DQ(f, [t, t+h]). Deduce an expression for $f'(t)$.
<u>Hint</u>: Think of DQ(f, [t, t+h]) as a polynomial in h with coefficients that depend on t, and employ Mathematica's **Coefficient** function in an appropriate way.

13.4* For f defined on [0.4, 0.75] in Appendix 2B, use Mathematica to find an expression for DQ(f, [t, t+h]). Deduce an expression for $f'(t)$.

13.5 For f defined on [0.75, 0.9] in Appendix 2B, use Mathematica to find an expression for DQ(f, [t, t+h]). Deduce an expression for $f'(t)$.

13.6* For V defined on [0.4, 0.75] in Appendix 2B, use Mathematica to find an expression for DQ(V, [t, t+h]). Deduce an expression for $V'(t)$.

13.7 For V defined on [0.75, 0.9] in Appendix 2B, use Mathematica to find an expression for DQ(V, [t, t+h]). Deduce an expression for $V'(t)$..

13.8 For W defined on [0, 3.001] in Appendix 3, use Mathematica to find an expression for DQ(W, [t, t+h]). Deduce an expression for $W'(t)$.

13.9 For W defined on [3.001, 12] in Appendix 3, use Mathematica to find an expression for DQ(W, [t, t+h]). Deduce an expression for $W'(t)$.

13.10 For S defined on [3, 3.002] in Appendix 3, use Mathematica to find an expression for DQ(S, [t, t+h]). Deduce an expression for $S'(t)$.

13.11 For F defined by

$$F(t) = At + Bt^2$$

where A and B are arbitrary parameters, show that

$$DQ(F,[t,t+h]) = A + 2Bt + Bh,$$

and deduce an expression for $F'(t)$.

13.12 For F defined on $[A, \infty)$ by

$$F(t) = \frac{1}{t}$$

with $A > 0$, show that

$$DQ(F,[t,t+h]) = -\frac{1}{t^2} + \frac{h}{t^2(t+h)}.$$

Deduce an expression for $F'(t)$, and show that the magnitude of the error term cannot exceed $A^{-3}h$.

13.13 For G defined on $[2, \infty)$ by

$$G(t) = 1 - \frac{C}{t^2}$$

where C is an arbitrary positive parameter, show that

$$DQ(G,[t,t+h]) = \frac{2C}{t^3} - \frac{Ch(2h+3t)}{t^3(t+h)^2}.$$

Deduce an expression for $G'(t)$, and show that the magnitude of the error term cannot exceed $Ch/4$ on $[2, \infty)$. Hint: Why can you assume $2h \le t$?

13.14 Which of the following equations are true? Why?

(i) $h^2 = O[h]$ (ii) $hO[h] = O[h]$ (iii) $\frac{1}{h} = O[h]$

13.15 For Q defined on $[b + 1, \infty)$ by

$$Q(t) = \frac{1}{t-b},$$

show that

$$DQ(Q,[t,t+h]) = -\frac{1}{(t-b)^2} + \frac{h}{(t+h-b)(t-b)^2}.$$

Deduce an expression for $Q'(t)$, and show that the magnitude of the error term cannot exceed h on $[b+1, \infty)$.

13.16 For Q defined on $[0, \infty)$ by

$$Q(t) = \frac{c-t}{c+t}$$

where c is any positive constant, show that

$$DQ(Q,[t,t+h]) = -\frac{2c}{(c+t)^2} + \frac{2ch}{(t+h+c)(t+c)^2}.$$

Deduce an expression for $Q'(t)$, and show that the magnitude of the error term cannot exceed $2c^{-2}h$ on $[0, \infty)$.

13.17 For R defined on $[1, \infty)$ by

$$R(t) = \frac{C}{t^3},$$

show that

$$DQ(R,[t,t+h]) = -\frac{3C}{t^4} + \frac{Ch(3h^2+8ht+6t^2)}{t^4(t+h)^3}.$$

Deduce an expression for $R'(t)$, and show that the magnitude of the error term cannot exceed 17Ch.

Answers and Hints for Selected Exercises

13.1 Suppose, e.g., that $f(t) = t^3$. Then $f(t+h) = (t+h)^3$ and, on by binomial expansion,

$$DQ(f,[t,t+h]) = \frac{f(t+h)-f(t)}{h} = \frac{(t+h)^3 - t^3}{h} = \frac{t^3+3t^2h+3th^2+h^3-t^3}{h} = \frac{3t^2h+3th^2+h^3}{h}$$

which must be positive for $h > 0$ when $t \geq 0$. So f is increasing on $[0, \infty)$. Further simplification yields $DQ(f, [t, t+h]) = 3t^2 + h(3t+h) = 3t^2 + O[h]$, so that $f'(t) = 3t^2$ on extracting the leading term.

13.13 From $G(t) = 1 - C/t^2$, we have $G(t+h) = 1 - \frac{C}{(t+h)^2}$. Therefore

$$DQ(G,[t,t+h]) = \frac{1}{h}\{G(t+h)-G(t)\} = \frac{1}{h}\left\{\left(1-\frac{C}{(t+h)^2}\right) - \left(1-\frac{C}{t^2}\right)\right\}$$

$$= \frac{1}{h}\left\{1 - \frac{C}{(t+h)^2} - 1 + \frac{C}{t^2}\right\} = \frac{C}{h}\left\{\frac{1}{t^2} - \frac{1}{(t+h)^2}\right\}$$

$$= \frac{C}{h}\left\{\frac{(t+h)^2-t^2}{t^2(t+h)^2}\right\} = \frac{C}{h}\left\{\frac{2th+h^2}{t^2(t+h)^2}\right\} = \frac{C(2t+h)}{t^2(t+h)^2}.$$

Also,

$$\frac{2C}{t^3} - \frac{Ch(2h+3t)}{t^3(t+h)^2} = \frac{2C(t+h)^2 - Ch(2h+3t)}{t^3(t+h)^2}$$

$$= \frac{2C(t^2+2th+h^2)-2Ch^2-3Cht}{t^3(t+h)^2}$$

$$= \frac{2Ct^2+Cht}{t^3(t+h)^2} = \frac{Ct(2t+h)}{t^3(t+h)^2} = \frac{C(2t+h)}{t^2(t+h)^2}.$$

So

$$DQ(G,[t,t+h]) = \frac{2C}{t^3} - \frac{Ch(2h+3t)}{t^3(t+h)^2},$$

as required. Extracting the leading term, $G'(t) = 2C/t^3$.

The magnitude of the error term is

$$\varepsilon = \left|-\frac{Ch(2h+3t)}{t^3(t+h)^2}\right| = \frac{Ch(2h+3t)}{t^3(t+h)^2},$$

because $h > 0$ and $t \geq 2$ (and therefore, in particular, $t > 0$). Because h is so small, we can at least assume $h \leq 1$, hence $2h \leq 2$. But $t \geq 2$. Therefore $2h \leq 2 \leq t$, implying $2h \leq t$. Thus $2h + 3t \leq t + 3t = 4t$, implying

$$\varepsilon = \frac{Ch(2h+3t)}{t^3(t+h)^2} \leq \frac{Ch\cdot 4t}{t^3(t+h)^2} = \frac{4Ch}{t^2(t+h)^2} = \frac{4Ch}{\{t(t+h)\}^2}.$$

But $h > 0$ and $t \geq 2$ implies $t + h > 2$, hence $t(t+h) > 4$. Therefore $\{t(t+h)\}^2 > 16$, implying $1/\{t(t+h)\}^2 < 1/16$ and

$$\varepsilon = \frac{Ch(2h+3t)}{t^3(t+h)^2} \leq \frac{4Ch}{\{t(t+h)\}^2} = \frac{1}{\{t(t+h)\}^2}\cdot 4Ch < \frac{1}{16}\cdot 4Ch = \frac{1}{4}Ch.$$

In other words, the magnitude of the error term cannot exceed $Ch/4$ on $[2, \infty)$.

13.15 From $Q(t) = 1/(t-b)$, we have $Q(t+h) = 1/(t+h-b)$. Therefore

$$DQ(Q,[t,t+h]) = \frac{1}{h}\{Q(t+h)-Q(t)\} = \frac{1}{h}\left\{\frac{1}{t+h-b} - \frac{1}{t-b}\right\}$$
$$= \frac{1}{h}\left\{\frac{t-b-(t+h-b)}{(t+h-b)(t-b)}\right\} = \frac{1}{h}\left\{\frac{t-b-t-h+b}{(t+h-b)(t-b)}\right\}$$
$$= \frac{-1}{(t+h-b)(t-b)}.$$

Also,

$$\frac{h}{(t+h-b)(t-b)^2} - \frac{1}{(t-b)^2} = \frac{h-(t+h-b)}{(t+h-b)(t-b)^2} = \frac{h-t-h+b}{(t+h-b)(t-b)^2}$$
$$= \frac{-(t-b)}{(t+h-b)(t-b)^2} = \frac{-1}{(t+h-b)(t-b)}.$$

So

$$DQ(Q,[t,t+h]) = -\frac{1}{(t-b)^2} + \frac{h}{(t+h-b)(t-b)^2},$$

as required. Extracting the leading term,

$$Q'(t) = -\frac{1}{(t-b)^2}.$$

Also, the magnitude of the error term is

$$\varepsilon = \left|\frac{h}{(t+h-b)(t-b)^2}\right| = \frac{h}{(t+h-b)(t-b)^2}.$$

Because $h > 0$ and $t \geq b+1$, we have $t - b \geq 1$ and $t + h - b > 1$. So the denominator exceeds 1, implying $\varepsilon < h$.

13.16 Go to http://www.math.fsu.edu/~mm-g/QuizBank/mac3311.s97.html (Assignment B, #2)

13.17 From $R(t) = C/t^3$, we have $R(t+h) = C/(t+h)^3$ Therefore

$$\begin{aligned} DQ(R,[t,t+h]) &= \frac{1}{h}\{R(t+h)-R(t)\} = \frac{1}{h}\left\{\frac{C}{(t+h)^3} - \frac{C}{t^3}\right\} \\ &= \frac{C}{h}\left\{\frac{t^3-(t+h)^3}{(t+h)^3t^3}\right\} = \frac{C\left(t^3-\{t^3+3t^2h+3th^2+h^3\}\right)}{h(t+h)^3t^3} \\ &= \frac{C\left(-3t^2h-3th^2-h^3\right)}{h(t+h)^3t^3} = -\frac{C\left(3t^2+3th+h^2\right)}{(t+h)^3t^3}. \end{aligned}$$

Also,

$$\begin{aligned} -\frac{3C}{t^4} + \frac{Ch\left(3h^2+8ht+6t^2\right)}{(t+h)^3t^4} &= \frac{-3C(t+h)^3+Ch\left(3h^2+8ht+6t^2\right)}{t^4(t+h)^3} \\ &= \frac{-3C(t^3+3t^2h+3th^2+h^3)+Ch\left(3h^2+8ht+6t^2\right)}{t^4(t+h)^3} \\ &= \frac{-3Ct^3-9Ct^2h-9Cth^2-3Ch^3+3Ch^3+8Cth^2+6Ct^2h}{t^4(t+h)^3} \\ &= \frac{-3Ct^3-3Ct^2h-Cth^2}{t^4(t+h)^3} = -\frac{C\left(3t^2+3th+h^2\right)}{(t+h)^3t^3}. \end{aligned}$$

So

$$DQ(R,[t,t+h]) = -\frac{3C}{t^4} + \frac{Ch(3h^2+8ht+6t^2)}{t^4(t+h)^3},$$

as required. Extracting the leading term, $R'(t) = -3C/t^4$.

The magnitude of the error term is

$$\varepsilon = \left|\frac{Ch(3h^2+8ht+6t^2)}{t^4(t+h)^3}\right| = \frac{Ch(3h^2+8ht+6t^2)}{t^4(t+h)^3},$$

because $h > 0$ and $t \geq 1$ (and therefore, in particular, $t > 0$). Because h is so small, we can at least assume $h \leq 1$. But $t \geq 1$. Therefore $h \leq t$, implying $h^2 \leq t^2$ and $ht \leq t^2$. Thus $3h^2+8ht+6t^2 \leq 3t^2 + 8t^2 + 6t^2 = 17t^2$. Also, $t \geq 1$ implies $t^2 \geq 1$ and $(t+h)^3 > 1$ (because $h > 0$), hence $t^2(t+h)^3 > 1$. So $t^4(t+h)^3 = t^2\ t^2(t+h)^3 > t^2$, i.e., the denominator exceeds t^2. So

$$\varepsilon = \frac{Ch(3h^2+8ht+6t^2)}{t^4(t+h)^3} < \frac{Ch(3h^2+8ht+6t^2)}{t^2}.$$

But $3h^2+8ht+6t^2$ cannot exceed $17t^2$. Hence, on $[1, \infty)$, the error cannot exceed $17Cht^2/t^2 = 17Ch$.