12. From ventricular inflow to volume: integration

Figure 1 shows ventricular inflow in our cardiac cycle at the end of the ejection phase. The graph is that of v defined on [0.28, 0.35] by

$$v(t) = 2450/3 - 192500t/9 + 980000t^2/9 - 1400000t^3/9 \qquad (12.1a)$$

$$= 350(20t - 1)(3 - 10t)(7 - 20t)/9, \qquad (12.1b)$$

i.e., it is the restriction to [0.28, 0.35] of the function v whose graph is sketched in Figure 1.4. On [0.28, 0.3) we have $v(t) < 0$, corresponding to arterial outflow. On (0.3, 0.35) we have $v(t) > 0$, corresponding to arterial backflow, which closes the aortic valve. In 0.07 seconds, inflow increases from $v(0.28) = -50.1$ ml/s to $v(0.3) = 0$ to $v(0.326) = 26.8$ ml/s, before decreasing again to zero at $t = 0.35$ s. For the first fiftieth of a second, blood flows out of the ventricle; but for the next twentieth of a second, blood flows back in. So how much blood flows in or out, overall? In other words, what is *net* transport of blood by the flow? The purpose of this lecture is to answer that question.

First of all, an inflow of –50.1 ml/s at $t = 0.28$ s means that *if* flow continued at this rate for the next 0.01 s then $-50.1 \times 0.01 = -0.501$ ml of blood would flow into the ventricle. In other words, 0.501 ml of blood would be discharged into the aorta. This volume of discharge equals the area of the shaded rectangle below [0.28, 0.29] in Figure 2(a). Its signed area is –0.501 ml, which is the blood volume that would be transported into the ventricle, or ventricular recharge.

Flow does not continue at –50.1 ml/s for a hundredth of a second, however, because by $t = 0.29$ s it has already increased to $v(0.29) = -22.4$ ml/s. If this higher rate were maintained on [0.28, 0.29], then the ventricular recharge would instead be $-22.4 \times 0.01 = -0.224$ ml. In other words, $22.4 \times 0.01 = 0.224$ ml of blood would be discharged into the aorta. This volume equals the area of the shaded rectangle below [0.28, 0.29] in Figure 2(b). Its signed area is –0.224 ml, which is the recharge.

The true volume of blood transported into the ventricle during [0.28, 0.29] must be somewhere in between: It is underestimated by –0.501 ml, but it is overestimated by –0.224. That is,

$$-0.501 < \text{NET TRANSPORT OF BLOOD DURING } [0.28, 0.29] < -0.224$$

A similar analysis applies to [0.29, 0.3], on which flow increases from –22.4 ml/s to zero: recharge is underestimated by –0.224 ml (signed area of shaded rectangle under [0.29, 0.3] in Figure 2(a)) but overestimated by zero (no shaded rectangle over [0.29, 0.3] in Figure 2(b)). In other words,

$$-0.224 < \text{NET TRANSPORT OF BLOOD DURING } [0.29, 0.3] < 0$$

Thus net recharge during [0.28, 0.3], on which v increases with respect to time but flow is never positive, is greater than $-0.501 - 0.224 = -0.725$ ml but less than $-0.224 + 0 = -0.224$ ml. That is,

$$-0.725 < \text{NET TRANSPORT OF BLOOD DURING } [0.29, 0.3] < -0.224$$

On [0.3, 0.3131], v still increases with respect to time but inflow is nonnegative, so that net recharge is underestimated by zero (no shaded rectangle over [0.3, 0.3131] in Figure 2(a)). Moreover, v cannot exceed $v(0.3131) = 19.7$ ml/s, so that $19.7 \times 0.0131 =$

0.258 ml overestimates the net recharge. This overestimate is the area of the shaded rectangle above [0.3, 0.3131] in Figure 2(b), which is also its signed area. Thus

$$0 < \text{NET TRANSPORT OF BLOOD DURING } [0.3, 0.313] < 0.258$$

Similarly, on [0.3131, 0.3261], flow must exceed 19.7 ml/s but cannot exceed v(0.3261) = 26.8 ml/s, so that net recharge is underestimated by 0.258 ml but overestimated by 26.8 $\times$ 0.013 = 0.35 ml. In other words,

$$0.258 < \text{NET TRANSPORT OF BLOOD DURING } [0.313, 0.326] < 0.35$$

So during [0.3, 0.326], when v is nonnegative and increases with respect to time, net recharge exceeds 0 + 0.258 = 0.258 ml but is less than 0.258 + 0.35 = 0.608 ml:

$$0.258 < \text{NET TRANSPORT OF BLOOD DURING } [0.3, 0.326] < 0.608$$

The underestimate is the (signed) shaded area above [0.313, 0.326] in Figure 2(a), and the overestimate is the corresponding (signed) area in Figure 2(b).

Similar considerations apply to [0.3261, 0.35], on which flow is still nonnegative but v decreases with respect to time. Because v(0.3261) = 26.8 and v(0.33805) = 20.37, net recharge on [0.3261, 0.33805] is overestimated by 26.8 $\times$ 0.01195 = 0.320 ml but underestimated by 20.37 $\times$ 0.01195 = 0.243 ml; whereas net recharge on [0.33805, 0.35] is overestimated by 0.243 ml but underestimated by zero. So net recharge during [0.326, 0.35] exceeds 0.243 + 0 = 0.243 ml but is less than 0.320 + 0.243 = 0.563 ml. That is,

$$0.243 < \text{NET TRANSPORT OF BLOOD DURING } [0.326, 0.35] < 0.563$$

Again, the underestimate is the shaded area over [0.326, 0.35] in Figure 2(a), whereas the overestimate is the corresponding area in Figure 2(b).

Thus an underestimate of total net recharge into the ventricle during [0.28, 0.35] is obtained by signing the shaded area in Figure 2(a), and an overestimate is obtained by signing the area in Figure 2(b). These two estimates are –0.725 + 0.258 + 0.243 = –0.22 ml and –0.224 + 0.608 + 0.563 = 0.95 ml, respectively. In other words,

$$-0.22 < \text{NET TRANSPORT OF BLOOD DURING } [0.28, 0.35] < 0.95$$

These are very crude estimates of total net recharge, but we obtained them by dividing each of [0.28, 0.3], [0.3, 0.326] and [0.326, 0.35] into only two subintervals. We can improve our estimates by doubling the number of subintervals, from two to four, but otherwise proceeding as before. The result is shown in Figure 3. We can obtain even more accurate estimates if we double the number of subintervals again, to eight, as shown in Figure 4. In fact, we can improve the accuracy indefinitely, by continually increasing the number of doublings, say n, of the original three subintervals, as Figure 5 illustrates. At each of these doublings, net recharge of blood into the ventricle during [0.28, 0.35] is underestimated by the signed area with darker shading but overestimated by the signed area with lighter shading. In the limit as $n \to \infty$, however, the two signed areas must coincide, with true net recharge sandwiched between them. Thus true net recharge is the limit of both an increasing sequence of underestimates and a decreasing sequence of overestimates; see Tables 1-2 (and Appendix 12). Either way, true net recharge equals signed area between the horizontal axis and the graph of v on [0.28, 0.35]. Of course, the numbers in Tables 1-2 are just toy numbers – most of the "significant figures" have no physiological significance – but with their help we can more readily understand the mathematics behind the physiology.

NUMBER OF SUBINTERVAL DOUBLINGS	RECTANGULAR UNDERESTIMATES OF NET DISCHARGE DURING [0.28, 0.35] Int(v, [0.28, 0.3])	Int(v, [0.3, 0.326])	Int(v, [0.326, 0.35])	Int(v, [0.28, 0.35])
0	-1.002	0	0	-1.002
1	-0.7249	0.2580	0.2430	-0.2239
2	-0.5931	0.3662	0.3437	0.1169
3	-0.5288	0.4152	0.3889	0.2752
4	-0.4971	0.4383	0.4101	0.3514
5	-0.4813	0.4496	0.4205	0.3887
6	-0.4735	0.4552	0.4255	0.4072
7	-0.4695	0.4579	0.4281	0.4164
8	-0.4676	0.4593	0.4293	0.4210
9	-0.4666	0.4600	0.4299	0.4233
10	-0.4661	0.4603	0.4302	0.4244
11	-0.4659	0.4605	0.4304	0.4250
12	-0.4658	0.4606	0.4305	0.4253
13	-0.4657	0.4606	0.4305	0.4254
14	-0.4657	0.4606	0.4305	0.4255
15	-0.4656	0.4606	0.4305	0.4255
16	-0.4656	0.4606	0.4306	0.4256

Table 12.1 Rectangular underestimates of net backflow into ventricle during [0.28, 0.35]

NUMBER OF SUBINTERVAL DOUBLINGS	RECTANGULAR OVERESTIMATES OF NET DISCHARGE DURING [0.28, 0.35] Int(v, [0.28, 0.3])	Int(v, [0.3, 0.326])	Int(v, [0.326, 0.35])	Int(v, [0.28, 0.35])
0	0	0.7000	0.6395	1.340
1	-0.2240	0.6080	0.5627	0.9467
2	-0.3426	0.5412	0.5036	0.7022
3	-0.4036	0.5027	0.4688	0.5679
4	-0.4345	0.4821	0.4501	0.4977
5	-0.4500	0.4715	0.4404	0.4619
6	-0.4578	0.4661	0.4355	0.4438
7	-0.4617	0.4634	0.4330	0.4347
8	-0.4637	0.4620	0.4318	0.4301
9	-0.4647	0.4613	0.4312	0.4279
10	-0.4651	0.4610	0.4309	0.4267
11	-0.4654	0.4608	0.4307	0.4261
12	-0.4655	0.4607	0.4306	0.4259
13	-0.4656	0.4607	0.4306	0.4257
14	-0.4656	0.4607	0.4306	0.4256
15	-0.4656	0.4607	0.4306	0.4256
16	-0.4656	0.4607	0.4306	0.4256

Table 12.2 Rectangular overestimates of net backflow into ventricle during [0.28, 0.35]

We have thus established net recharge of blood during [0.28, 0.35] must equal Int(v, [0.28, 0.35]). But net recharge of blood during [0.28, 0.35] must also equal increase of ventricular volume during the same interval, and so

$$\begin{aligned} \mathrm{Int}\big(v,[0.28,0.35]\big) &= V(0.35) - V(0.28) \\ &= \mathrm{Diff}\big(V,[0.28,0.35]\big). \end{aligned} \qquad (12.2)$$

Furthermore, the arguments above do not depend in any way on choosing [0.28, 0.35] as the subdomain of v: they would apply with equal force to any other subdomain. So, for arbitrary times a and b, net recharge during the interval [a, b] equals Int(v, [a, b]) and

$$\begin{aligned}\mathrm{Int}(v,[a,b]) &= V(b) - V(a) \\ &= \mathrm{Diff}(V,[a,b]),\end{aligned} \tag{12.3}$$

or, equivalently,

$$V(b) = V(a) + \mathrm{Int}(v,[a,b]). \tag{12.4}$$

Because b is arbitrary, we can set b = t to reveal a fundamental relationship between ventricular volume V and inflow v:

$$V(t) = V(a) + \mathrm{Int}(v,[a,t]). \tag{12.5}$$

Physiologically speaking, ventricular volume at the current time equals ventricular volume at any earlier time plus subsequent net recharge.

By interpreting Int(v, [a, b]) as net recharge from inflow v, we can deduce some important properties of the integral. First, if v is net inflow, then –v is net outflow (Figure 1.4). Hence Int(–v, [a, b]) is net discharge from the ventricle during [a, b]. But net discharge on [a, b] must equal V(a) – V(b) = –{V(b) – V(a)}. Therefore, from (3):

$$\mathrm{Int}(-v,[a,b]) = -\mathrm{Int}(v,[a,b]). \tag{12.6}$$

Second, because, e.g., doubling or tripling an inflow will double or triple the associated net recharge, we have Int(2v, [a, b]) = 2 Int(v, [a, b]) and Int(3v, [a, b]) = 3 Int(v, [a, b]). More generally, changing the flow by a factor of k will change the associated recharge by a factor of k or

$$\mathrm{Int}(kv,[a,b]) = k\,\mathrm{Int}(v,[a,b]). \tag{12.7}$$

Third, suppose that two venules converge at C to form a vein, as cartooned in Figure 6. At time t, let u(t) ml/s and v(t) ml/s be the outflows from the venules at C. Then total flow into the vein at C must be u(t) + v(t), because there is nowhere else for blood to go. For the same reason, discharge into the vein during any interval [a, b] must equal total discharge out of the venule, or

$$\mathrm{Int}(u+v,[a,b]) = \mathrm{Int}(u,[a,b]) + \mathrm{Int}(v,[a,b]). \tag{12.8}$$

Note that, even if u or v were not a flow, we could pretend that u and v are flows, and none of their properties could thereby change. Thus (6)-(8) are general properties of integrals. Furthermore, by the method of Lecture 9, they are easily combined into a single result, namely,

$$\mathrm{Int}(ku+qv,[a,b]) = k\,\mathrm{Int}(u,[a,b]) + q\,\mathrm{Int}(v,[a,b]), \tag{12.9}$$

agreeing with (9.17) in the special case where u, v and ku + qv are all nonnegative.

These results enable us to obtain an explicit formula for ventricular volume at any time during our cardiac cycle. Suppose, for example, that $0.05 \le t \le 0.35$, and define constants c_0, c_1, c_2, c_3 and functions g, r, w, z by

$$c_0 = \tfrac{2450}{3}, \quad c_1 = -\tfrac{192500}{9}, \quad c_2 = \tfrac{980000}{9}, \quad c_3 = -\tfrac{1400000}{9} \tag{12.10}$$

and

$$g(x) = 1, \quad r(x) = x, \quad w(x) = x^2, \quad z(x) = x^3, \tag{12.11}$$

so that

$$v = c_0 g + c_1 r + c_2 w + c_3 z \tag{12.12}$$

from (1). Now, by analogy with (11.1)-(11.8), successively applying (9) with b = t yields

$$\begin{aligned}\mathrm{Int}(v,[a,t]) &= \mathrm{Int}(c_0 g + c_1 r + c_2 w + c_3 z,[a,t]) \\ &= c_0\mathrm{Int}(g,[a,t]) + c_1\mathrm{Int}(r,[a,t]) + \\ &\qquad c_2\mathrm{Int}(w,[a,t]) + c_3\mathrm{Int}(z,[a,t]).\end{aligned} \tag{12.13}$$

Because Int = Area for a nonnegative function, however, from Lecture 11 we have

$$\mathrm{Int}(g,[a,t]) = t-a \tag{12.14}$$

$$\mathrm{Int}(r,[a,t]) = \tfrac{1}{2}(t^2-a^2) \tag{12.15}$$

$$\mathrm{Int}(w,[a,t]) = \tfrac{1}{3}(t^3-a^3) \tag{12.16}$$

$$\mathrm{Int}(z,[a,t]) = \tfrac{1}{4}(t^4-a^4). \tag{12.17}$$

So, from (5) and (13)-(17),

$$\begin{aligned}V(t) &= V(a) + \mathrm{Int}\big(v,[a,t]\big) \\ &= V(a) + c_0(t-a) + \tfrac{1}{2}c_1(t^2-a^2) + \tfrac{1}{3}c_2(t^3-a^3) + \tfrac{1}{4}c_3(t^4-a^4) \\ &= V(a) - c_0 a - \tfrac{1}{2}c_1 a^2 - \tfrac{1}{3}c_2 a^3 - \tfrac{1}{4}c_3 a^4 \\ &\qquad + c_0 t + \tfrac{1}{2}c_1 t^2 + \tfrac{1}{3}c_2 t^3 + \tfrac{1}{4}c_3 t^4.\end{aligned} \tag{12.18}$$

For example, because $V(0.05) = 120$ from Figure 3, with $a = 0.05$ we have

$$V(t) = \frac{43895}{432} + \frac{2450}{3}t - \frac{96250}{9}t^2 + \frac{980000}{27}t^3 - \frac{350000}{9}t^4 \tag{12.19}$$

for any $t \in [0.05, 0.35]$. In particular, from (19), net recharge on [0.28, 0.35] is Int(v, [0.28, 0.35]) = V(0.35) – V(0.28) = 50 – 49.5744 = 0.4256ml. Corresponding expressions for the rest of the cardiac cycle are similarly obtained; see Exercise 1.

We conclude by discussing notation. In Figures 2-5 we found Int(v, [a, b]) as the limit of either the sum of signed areas of a large number of overestimating rectangles or the sum of signed areas of a large number of underestimating rectangles, as the number of rectangles became infinitely large. If the number approaches infinity, however, then the width of each rectangle approaches zero. Now, in mathematics, the Greek letter δ is traditionally used to denote "infinitesimal change in" (maybe because it resembles an upside-down tadpole). Thus δx stands for a small change in x, and a typical approximating rectangle has height v(x) and signed area $v(x)\cdot\delta x$. So integration means summing a large number of signed areas of the form $v(x)\cdot\delta x$ and finding the limit of the sum as $\delta x \to 0$. Symbolically, we have

$$\mathrm{Int}(v,[a,b]) = \lim_{\delta x\to 0}\sum_{[a,b]} v(x)\delta x, \tag{12.20}$$

where [a, b] under the Σ sign indicates that every piece of the interval [a, b] must be covered by some δx. It is often useful to have a mathematical shorthand that evokes the right-hand side of (19), and so we define

$$\int_a^b v(x)dx = \lim_{\delta x\to 0}\sum_{[a,b]} v(x)\delta x. \tag{12.21}$$

Immediately, we have an alternative notation for the integral of v over [a, b]. That is,

$$\int_a^b v(x)dx \quad = \quad \mathrm{Int}(v,[a,b]). \qquad (12.22)$$

The left-hand side of (22) is usually read as the "integral of v(x) with respect to x between x = a and x = b" – from which, given that "$\int$" is an integral sign, we read "dx" as "with respect to x." Note, however, that the right-hand side of (22) does not depend in any way on x, and so any letter *except* a or b can be used in lieu of x on the left-hand side of the equation. For example,

$$\int_a^b v(t)dt \quad = \quad \mathrm{Int}(v,[a,b]) \qquad (12.23)$$

is *exactly* the same statement as (22). Henceforeward, we will refer to "$\int$" as Leibniz notation (because Leibniz introduced it) and to "Int" as standard notation.

For every statement in standard notation there is an identical statement in Leibniz notation, and vice versa. For example, (5), (9) and (14)-(17) are identical to

$$V(t) \quad = \quad V(a) \quad + \quad \int_a^t v(x)dx, \qquad (12.24)$$

$$\int_a^b \{k\,u(x) \; + \; q\,v(x)\}dx \quad = \quad k\int_a^b u(x)dx \quad + \quad q\int_a^b v(x)dx, \qquad (12.25)$$

and

$$\begin{aligned} \int_a^t 1\,dx \quad &= \quad t-a, & \int_a^t x\,dx \quad &= \quad \frac{1}{2}(t^2-a^2) \\ \int_a^t x^2\,dx \quad &= \quad \frac{1}{3}(t^3-a^3), & \int_a^t x^3\,dx \quad &= \quad \frac{1}{4}(t^4-a^4), \end{aligned} \qquad (12.26)$$

respectively; and (8.25), i.e., Int(f, [a, b]) = Int(f, [a, c]) + Int(f, [c, b]), becomes

$$\int_a^b f(x)\,dx \quad = \quad \int_a^c f(x)\,dx \quad + \quad \int_c^b f(x)\,dx, \qquad (12.27)$$

for any c satisfying $a \le c \le b$.

Each notation has its advantages and disadvantages. In particular, standard notation makes clearer that Int(v, [a, b]) depends only on v, a and b: it does not depend in any way on t, x, or anything else. On the other hand, with Leibniz notation, we can go straight from v to V without steps (10)-(12): from (24), (1), successive application of (25) and (26)-(27), we have

$$\begin{aligned} V(t) \; &= \; V(0.05) \; + \; \int_{0.05}^t v(x)dx \\ &= \quad 120 \; + \; \int_{0.05}^t \{\tfrac{2450}{3} + \tfrac{192500}{9}x + \tfrac{980000}{9}x^2 + \tfrac{1400000}{9}x^3\}dx \\ &= \quad 120 \; + \; \tfrac{2450}{3}\int_{0.05}^t 1\,dx \; + \; \tfrac{192500}{9}\int_{0.05}^t x\,dx \; + \; \tfrac{980000}{9}\int_{0.05}^t x^2\,dx \; + \; \tfrac{1400000}{9}\int_{0.05}^t x^3\,dx \\ &= \quad 120 \; + \; \tfrac{2450}{3}(t-1) \; + \; \tfrac{96250}{9}(t^2-0.05^2) \; + \; \tfrac{980000}{27}(t^3-0.05^3) \; + \; \tfrac{350000}{9}(t^4-0.05^4), \end{aligned} \qquad (12.28)$$

which reduces to (19).

Exercises 12

12.1 Verify the expression for ventricular volume V(t) in Appendix 2B. Also verify that (19) is consistent with (11.28).

12.2 The functions g and G are defined on [0, 3] by

$$g(t) = \begin{cases} 4-3t & \text{if } 0 \le t \le 1 \\ 2-t^3 & \text{if } 1 \le t < 3 \end{cases} \qquad \text{and} \qquad G(t) = \int_0^t g(x)dx.$$

Find an explicit formula for G(t).

12.3 The functions g and G are defined on [0, 4] by

$$g(t) = \begin{cases} 4-t^2 & \text{if } 0 \le t \le 1 \\ t^3+2 & \text{if } 1 \le t \le 3 \\ 10t-1 & \text{if } 3 \le t \le 4 \end{cases} \qquad \text{and} \qquad G(t) = \int_0^t g(x)dx.$$

Find an explicit formula for G(t).

12.4* The functions ξ and φ are defined on [0, 3] by

$$\xi(t) = \begin{cases} -2 & \text{if } 0 \le t \le 1 \\ 7t-9 & \text{if } 1 \le t \le 2 \\ 3t-1 & \text{if } 2 \le t \le 3 \end{cases}$$

and

$$\phi(t) = \int_0^t \xi(x)dx.$$

Find an explicit formula for φ(t). Plot the graphs of ξ and φ.

Hint: Use Int(ξ, [0, t]) = Int(ξ, [0, a]) + Int(ξ, [a, t]) with appropriate values of a.

12.5 The functions ξ and φ are defined on [0, ∞) by

$$\xi(t) = \begin{cases} 4-2t & \text{if } 0 \le t \le 3 \\ t-5 & \text{if } 3 \le t \le 6 \\ 1 & \text{if } 6 \le t < \infty \end{cases}$$

and

$$\phi(t) = \int_0^t \xi(x)dx.$$

Find an explicit formula for φ(t). Plot the graphs of ξ and φ.

12.6 A piecewise-linear function g is defined on [0, 12] by the graph in Figure 8. A function G is defined on [0, 12] by G(t) = Int(g, [0, t]). Obtain an explicit formula for G(t).

12.7 The functions g and G are defined on [0, 3] by

$$g(t) = \begin{cases} 5 - 8t & \text{if } 0 \le t \le 1 \\ t^3 - 4 & \text{if } 1 \le t \le 2 \\ 2t^2 + 3t - 10 & \text{if } 2 \le t \le 3 \end{cases} \quad \text{and} \quad G(t) = \int_0^t g(x)\,dx.$$

Find an explicit formula for G(t).

12.8 Calculate Int(W, [1, 3]) for W is defined on [1, 3] by

$$W(t) = \begin{cases} 4t^3 + 52 & \text{if } 1 \le t < 2 \\ 3t^2 + 36t & \text{if } 2 \le t \le 3 \end{cases}$$

12.9 Calculate Int(W, [0, 2]) for W is defined on [0, 2] by

$$W(t) = \begin{cases} 3t^2 - 4t & \text{if } 0 \le t < 1 \\ 4t^3 - 5t^2 & \text{if } 1 \le t \le 2 \end{cases}$$

12.10 Calculate Int(W, [2, 4]) for W is defined on [2, 4] by

$$W(t) = \begin{cases} 4t^3 + 6t^2 + 2t + 240 & \text{if } 2 \le t < 3 \\ 3t^2 + 128t - 3 & \text{if } 3 \le t < 4 \end{cases}$$

Appendix 12: Rectangular versus trapezoidal approximation of integrals

Table 1 shows the sequence $\{U_k\}$ of underestimates obtained by summing the darker signed areas in Figures 2-5. The sequence $\{U_k\}$ is seen to be increasing, i.e., $U_{k+1} > U_k$ for all $k \geq 1$. Table 2 shows the sequence $\{O_k\}$ of overestimates obtained by summing the lighter signed areas in Figures 2-5. The sequence $\{O_k\}$ is seen to be decreasing, i.e., $O_{k+1} < O_k$ for all $k \geq 1$. Furthermore, both sequences converge. Thus if O_∞ and U_∞ are, respectively, the greatest lower bound for O_k and the least upper bound for U_k, then

$$U_1 < U_2 < U_3 < U_4 < U_5 < \ldots < U_\infty = O_\infty < \ldots < O_5 < O_4 < O_3 < O_2 < O_1. \qquad (12.A1)$$

That is, regardless of whether we overestimate or underestimate, our approximations converge to Int(f, [0.28, 0.35]). In fact,

$$U_\infty = O_\infty = \tfrac{22981}{54000} = 0.4256\,\text{ml}. \qquad (12.A2)$$

Although Int(-v, [0.28, 0.3]) = 0.4656 ml of blood is discharged into the aorta during the first 0.02 s of the interval, there is a reverse discharge of Int(v, [0.3, 0.35]) = 0.8912 ml during the last 0.05s, and the net effect over 0.07 s is that ventricular volume has increased by 0.4256 ml.

NUMBER OF	TRAPEZOIDAL ESTIMATES OF NET DISCHARGE DURING [0.28, 0.35]			
SUBINTERVAL DOUBLINGS	Int(v, [0.28, 0.3])	Int(v, [0.3, 0.326])	Int(v, [0.326, 0.35])	Int(v, [0.28, 0.35])
0	-0.5009	0.35	0.3198	0.1689
1	-0.4744	0.433	0.4029	0.3614
2	-0.4678	0.4537	0.4236	0.4095
3	-0.4662	0.4589	0.4288	0.4216
4	-0.4658	0.4602	0.4301	0.4246
5	-0.4657	0.4605	0.4304	0.4253
6	-0.4656	0.4606	0.4305	0.4255
7	-0.4656	0.4606	0.4305	0.4256
8	-0.4656	0.4606	0.4306	0.4256

Table 12.3 Trapezoidal estimates of net backflow into ventricle during [0.28, 0.35]

Both $\{U_k\}$ and $\{O_k\}$ converge too slowly to be useful in practice. Nevertheless, we can speed convergence by the simple expedient of averaging the signed areas of the light and dark rectangles in Figures 2 - 5. In other words, to speed convergence we define a sequence $\{T_k\}$ by

$$T_k = \frac{1}{2}\{U_k + O_k\} \qquad (12.A3)$$

and calculate net backflow from

$$\text{Int}(v,[0.28,0.35]) = T_\infty = \lim_{k\to\infty} T_k = \frac{1}{2}\{U_\infty + O_\infty\} = U_\infty = O_\infty \qquad (12.A4)$$

instead. Table 3 illustrates the faster convergence.

Note that $\{T_k\}$ is an underestimating sequence throughout [0.28, 0.35]. Why? Let a function f be either nonnegative or nonpositive, and define ϕ on [L, R] by

$$\begin{aligned}\phi(x) &= f(L) + \left\{\frac{f(R)-f(L)}{R-L}\right\}(x-L)\\ &= \frac{Rf(L)-Lf(R)}{R-L} + \left\{\frac{f(R)-f(L)}{R-L}\right\}x.\end{aligned} \qquad (12.A5)$$

Then, because ϕ is linear, with $\phi(L) = f(L)$ and $\phi(R) = f(R)$, the graph of ϕ is a straight line from $(L, \phi(L))$ to $(R, \phi(R))$, i.e., from $(L, f(L))$ to $(R, f(R))$. Thus

$$\mathrm{Int}(\phi,[L,R]) = \frac{1}{2}(R-L)\{\phi(L)+\phi(R)\} = \frac{1}{2}(R-L)\{f(L)+f(R)\} \qquad (12.A6)$$

is the signed area of a trapezium represented by a lighter shaded area in Figure 7, above or below the axis according to whether f is positive or negative on [L, R]). A darker shaded area represents Int(f, [L, R}). So Int(ϕ, [L, R}) overestimates Int(f, [L, R}) if f is concave up (top half of diagram) but underestimates Int(f, [L, R}) if f is concave down (bottom half), regardless of whether $f \geq 0$ (left-hand column) or $f \leq 0$ (right-hand column). Accordingly, if f is concave up or concave down on [a, b], then Int(f, [a, b]) can be accurately overestimated or underestimated, respectively, by decomposing [a, b] into several subdomains and applying (A6) to each.

Furthermore, in Figures 2-5, both the underestimating rectangle and the overestimating rectangle for Int(f, [L, R]) have base R – L; one has signed altitude f(L), the other has signed altitude f(R), and so their signed areas are (R–L)f(L) and (R–L)f(R), respectively. The average of these two quantities is the right-hand side of (A6). Thus averaging the two rectangular estimates is equivalent to trapezoidal approximation. In particular, (A3) defines a trapezoidal approximation. We can now answer the question that began the previous paragraph: $\{T_k\}$ is an underestimating sequence on [0.28, 0.35] because f is concave down throughout that interval (see Figure 1).

Answers and Hints for Selected Exercises

12.2 Suppose that $0 \leq t \leq 1$. Then, because $0 \leq x \leq t$ implies $g(x) = 4-3x$, we have

$$G(t) = \int_0^t g(x)dx = \int_0^t \{4-3x\}dx = 4\int_0^t 1\,dx - 3\int_0^t x\,dx$$
$$= 4(t-0) - \frac{3}{2}(t^2-0^2) = 4t - \tfrac{3}{2}t^2,$$

on using (25)-(26). In particular, $G(1) = 5/2$. Now suppose that $1 \leq t \leq 3$. Then, because $1 \leq x \leq t$ implies $g(x) = 2-x^3$, we have

$$G(t) = \int_0^t g(x)dx = \int_0^1 g(x)dx + \int_1^t g(x)dx$$
$$G(1) + \int_1^t \{2-x^3\}dx$$
$$= \frac{5}{2} + 2\int_1^t 1dx - \int_1^t x^3\,dx$$
$$= \frac{5}{2} + 2(t-1) - \frac{1}{4}(t^4-1^4) = \frac{3}{4} + 2t - \frac{1}{4}t^4,$$

on using (25)-(26) again. In sum,

$$G(t) = \begin{cases} 4t - \tfrac{3}{2}t^2 & \text{if } 0 \leq t \leq 1 \\ \tfrac{3}{4} + 2t - \tfrac{1}{4}t^4 & \text{if } 1 \leq t \leq 3 \end{cases}$$

12.3 Suppose that $0 \le t \le 1$. Then, because $0 \le x \le t$ implies $g(x) = 4 - x^2$, we have

$$G(t) = \int_0^t g(x)\,dx = \int_0^t \{4 - x^2\}\,dx = 4\int_0^t 1\,dx - \int_0^t x^2\,dx$$

$$= 4(t-0) - \frac{1}{3}(t^3 - 0^3) = \frac{1}{3}t(12 - t^2),$$

on using (25)-(26). In particular, $G(1) = 11/3$. Now suppose that $1 \le t \le 3$. Then, because $1 \le x \le t$ implies $g(x) = x^3 + 2$, we have

$$G(t) = \int_0^t g(x)\,dx = \int_0^1 g(x)\,dx + \int_1^t g(x)\,dx$$

$$G(1) + \int_1^t \{x^3 + 2\}\,dx$$

$$= \frac{11}{3} + \int_1^t x^3\,dx + 2\int_1^t 1\,dx$$

$$= \frac{11}{3} + \frac{1}{4}(t^4 - 1^4) + 2(t-1) = \frac{1}{4}t^4 + 2t + \frac{17}{12},$$

on using (25)-(26) again. In particular, $G(3) = 83/3$. Finally, suppose that $3 \le t \le 4$. Then, because $3 \le x \le t$ implies $g(x) = 10x - 1$, we have

$$G(t) = \int_0^t g(x)\,dx = \int_0^3 g(x)\,dx + \int_3^t g(x)\,dx$$

$$G(3) + \int_3^t \{10x - 1\}\,dx$$

$$= \frac{83}{3} + 10\int_3^t x\,dx - \int_3^t 1\,dx$$

$$= \frac{83}{3} + 10\cdot\frac{1}{2}(t^2 - 3^2) - (t-3) = 5t^2 - t - \frac{43}{3}.$$

In sum,

$$G(t) = \begin{cases} \frac{1}{3}t(12 - t^2) & \text{if } 0 \le t \le 1 \\ \frac{1}{4}t^4 + 2t + \frac{17}{12} & \text{if } 1 \le t \le 3 \\ 5t^2 - t - \frac{43}{3} & \text{if } 3 \le t \le 4 \end{cases}$$

12.7 G is defined on [0, 3] by

$$G(t) = \int_0^t g(x)\,dx = \begin{cases} 5t - 4t^2 & \text{if } 0 \le t \le 1 \\ \frac{1}{4}t^4 - 4t + \frac{19}{4} & \text{if } 1 \le t \le 2 \\ \frac{2}{3}t^3 + \frac{3}{2}t^2 - 10t + \frac{29}{3} & \text{if } 2 \le t \le 3 \end{cases}$$

12.8 $\mathrm{Int}(W, [1, 3]) = \mathrm{Int}(W, [1, 2]) + \mathrm{Int}(W, [2, 3]) =$

$$\int_1^2 (4t^3 + 52)\,dt + \int_2^3 (3t^2 + 36t)\,dt =$$

$$4\int_1^2 t^3\,dt + 52\int_1^2 1\,dt + 3\int_2^3 t^2\,dt + 36\int_2^3 t\,dt =$$

$$4\cdot\frac{1}{4}(2^4 - 1^4) + 52\cdot(2-1) + 3\cdot\frac{1}{3}(3^3 - 2^3) + 36\cdot\frac{1}{2}(3^2 - 2^2) =$$

$$= 15 + 52 + 19 + 90 = 176$$

12.9 $\mathrm{Int}(W, [0, 2]) = \mathrm{Int}(W, [0, 1]) + \mathrm{Int}(W, [1, 2]) =$

$$\int_0^1 (3t^2 - 4t)\,dt + \int_1^2 (4t^3 - 5t^2)\,dt =$$

$$3\int_0^1 t^2\,dt - 4\int_0^1 t\,dt + 4\int_1^2 t^3\,dt - 5\int_1^2 t^2\,dt =$$

$$3\cdot\frac{1}{3}(1^3 - 0^3) - 4\cdot\frac{1}{2}(1^2 - 0^2) + 4\cdot\frac{1}{4}(2^4 - 1^4) - 5\cdot\frac{1}{3}(2^3 - 1^3) =$$

$$= 1 - 2 + 15 - 35/3 = 7/3$$

12.10 $\mathrm{Int}(W, [2, 4]) = \mathrm{Int}(W, [2, 3]) + \mathrm{Int}(W, [3, 4]) =$

$$\int_2^3 (4t^3 + 6t^2 + 2t + 240)\,dt + \int_3^4 (3t^2 + 128t - 3)\,dt =$$

$$4\int_2^3 t^3\,dt + 6\int_2^3 t^2\,dt + 2\int_2^3 t\,dt + 240\int_2^3 1\,dt +$$

$$3\int_3^4 t^2\,dt + 128\int_3^4 t\,dt - 3\int_3^4 1\,dt =$$

$$4\cdot\frac{1}{4}(3^4 - 2^4) + 6\cdot\frac{1}{3}(3^3 - 2^3) + 2\cdot\frac{1}{2}(3^2 - 2^2) + 240\cdot(3-2)$$

$$3\cdot\frac{1}{3}(4^3 - 3^3) + 128\cdot\frac{1}{2}(4^2 - 3^2) - 3\cdot(4-3) =$$

$$= 65 + 38 + 5 + 240 + 37 + 448 - 3 = 830$$