11. Arterial discharge: the area under a polynomial

In this lecture we show that the area enclosed by any nonnegative polynomial is readily calculated by adapting the methods of Lecture 10. In particular, we can use these methods to calculate arterial discharge during the systolic phase of Lecture 1's cardiac cycle.

Figure 1 shows the graph of ventricular outflow f defined on [0.05, 0.3] by

$$f(x) \quad = \quad -\frac{2450}{3}+\frac{192500}{9}x-\frac{980000}{9}x^2+\frac{1400000}{9}x^3. \qquad (11.1)$$

The shaded area is F(t) = Area(f, [0.05, t]), which, as we will demonstrate in Lecture 12, is the volume of blood discharged into aorta during [0.05, t]. To calculate this volume, it will be convenient first to define constants c_0, c_1, c_2, c_3 by

$$c_0 = -\tfrac{2450}{3}, \quad c_1 = \tfrac{192500}{9}, \quad c_2 = -\tfrac{980000}{9}, \quad c_3 = \tfrac{1400000}{9} \qquad (11.2)$$

and functions g, h, r, w and z by

$$g(x) = 1, \quad r(x) = x, \quad w(x) = x^2, \quad z(x) = x^3 \qquad (11.3)$$

and

$$h(x) \quad = \quad c_1r(x) + c_2w(x) + c_3z(x) \qquad (11.4)$$

for all x or, which is exactly the same thing, $h = c_1r + c_2w + c_3z$. Then, from (1)-(3),

$$f \quad = \quad c_0g + h, \qquad (11.5)$$

and so (9.17) with $k = c_0$, q = 1, a = 0.05 and b = t implies

$$\text{Area}(f) \quad = \quad \text{Area}(c_0g + h) \quad = \quad c_0\,\text{Area}(g) + \text{Area}(h) \qquad (11.6)$$

on [0.05, t]. A similar argument reveals that

$$\begin{aligned}\text{Area}(h) \quad &= \quad \text{Area}(c_1r + c_2w + c_3z)\\ &= \quad c_1\,\text{Area}(r) + \text{Area}(c_2w + c_3z)\\ &= \quad c_1\,\text{Area}(r) + c_2\,\text{Area}(w) + \text{Area}(c_3z)\\ &= \quad c_1\,\text{Area}(r) + c_2\,\text{Area}(w) + c_3\,\text{Area}(z)\end{aligned} \qquad (11.7)$$

on [0.05, t]. Combining (5)-(7), we have

$$\begin{aligned}\text{Area}(f,[0.05,t]) \quad = \quad & c_0\text{Area}(g,[0.05,t]) + c_1\text{Area}(r,[0.05,t]) + \\ & c_2\text{Area}(w,[0.05,t]) + c_3\text{Area}(z,[0.05,t]).\end{aligned} \qquad (11.8)$$

From (2) and (8), we know the shaded area in Figure 1 if we know Area(g), Area(r), Area(w) and Area(z) on [0.05, t]. Now, Area(g, [a, t]) is the darker area in Figure 2(a). This is the area of a rectangle with base t – a and height 1, implying

$$\text{Area}(g,[a,t]) \quad = \quad 1\cdot(t-a) \quad = \quad t-a \qquad (11.9)$$

(as we already know from Lecture 9). In particular,

$$\text{Area}(g,[0.05,t]) \quad = \quad t-0.05. \qquad (11.10)$$

That's one area down, three to go.

Area(r, [a, t]) is the darker area in Figure 2(b), which is total shaded area minus lighter shaded area. Total shaded area is that of a triangle with base t and height t, or $t^2/2$. Lighter shaded area is that of a triangle with base a and height a, or $a^2/2$. Thus

$$\begin{aligned}\text{Area}(r,[a,t]) &= \text{Area}(r,[0,t]) - \text{Area}(r,[0,a]) \\ &= \frac{1}{2}t^2 - \frac{1}{2}a^2 = \frac{1}{2}(t^2 - a^2)\end{aligned} \tag{11.11}$$

(as we already know from Lecture 9). In particular,

$$\text{Area}(r,[0.05,t]) = \frac{1}{2}(t^2 - 0.05^2). \tag{11.12}$$

Two areas down, two to go.

We now turn to Figure 2(c), where the lighter shaded area is Area(w, [0, a]), the darker area is Area(w, [a, t]) and the total shaded area is Area(w, [0, t]). But we already know from (10.24) that Area(w, [0, t]) = $t^3/3$, hence Area(w, [0, a]) = $a^3/3$. Thus

$$\begin{aligned}\text{Area}(w,[a,t]) &= \text{Area}(w,[0,t]) - \text{Area}(w,[0,a]) \\ &= \frac{1}{2}t^3 - \frac{1}{2}a^3 = \frac{1}{3}(t^3 - a^3).\end{aligned} \tag{11.13}$$

In particular,

$$\text{Area}(w,[0.05,t]) = \frac{1}{3}(t^3 - 0.05^3). \tag{11.14}$$

Three areas down, only one more to go.

Finally we turn to Figure 2(d), where the lighter shaded area is Area(z, [0, a]), the darker area is Area(z, [a, t]) and the total shaded area is Area(z, [0, t]), so that

$$\text{Area}(z,[a,t]) = \text{Area}(z,[0,t]) - \text{Area}(z,[0,a]). \tag{11.15}$$

We define a new function Z by

$$Z(t) = \text{Area}(z,[0,t]). \tag{11.16}$$

Then (15) implies that

$$\text{Area}(z,[a,t]) = Z(t) - Z(a). \tag{11.17}$$

In particular,

$$\text{Area}(z,[0.05,t]) = Z(t) - Z(0.05). \tag{11.18}$$

So, from (8), (10), (12), (14) and (18), the shaded area in Figure 1 is

$$\begin{aligned}\text{Area}(f,[0.05,t]) = \; & c_0(t - 0.05) + \tfrac{1}{2}c_1(t^2 - 0.05^2) + \tfrac{1}{3}c_2(t^3 - 0.05^3) \\ & + c_4\{Z(t) - Z(0.05)\}.\end{aligned} \tag{11.19}$$

But we don't yet have an expression for Z(t).

We do, however, know how to derive one. We obtain Z(t) as the limit of a sequence of approximations, just as we found G(t) in Lecture 10. In Figure 3, the n-th approximation to Z(t), denoted $Z_n(t)$, is the sum of n trapeziums, each of width t/n. The base of the k-th trapezium stretches from $x = (k-1)t/n$ to $x = kt/n$, as in Lecture 10. So the k-th trapezium has minimum height $z((k-1)t/n)$ and maximum height $z(kt/n)$. Its area is therefore t/n times $\{z((k-1)t/n) + z(kt/n)\}/2$. The total shaded area is obtained by summing over all such trapeziums, i.e., by summing over k from $k = 1$ to $k = n$. Thus

$$\begin{aligned} Z_n(t) &= \sum_{k=1}^{n} \frac{1}{2}\frac{t}{n}\{z((k-1)t/n) + z(kt/n)\} \\ &= \frac{1}{2}\frac{t}{n}\sum_{k=1}^{n}\left\{((k-1)t/n)^3 + (kt/n)^3\right\} \\ &= \frac{1}{2}\frac{t}{n}\sum_{k=1}^{n}\frac{t^3}{n^3}\left\{(k-1)^3 + k^3\right\} \\ &= \frac{1}{2}\frac{t^4}{n^4}\sum_{k=1}^{n}\left\{(k-1)^3 + k^3\right\} \\ &= \frac{1}{2}\frac{t^4}{n^4}\left\{\sum_{k=1}^{n}(k-1)^3 + \sum_{k=1}^{n}k^3\right\}. \end{aligned} \tag{11.20}$$

By analogy with (10.17), we have

$$\begin{aligned} \sum_{k=1}^{n}(k-1)^3 &= (1-1)^3 + (2-1)^3 + (3-1)^3 + \ldots + (n-1)^3 \\ &= 0^3 + 1^3 + 2^3 + \ldots + (n-1)^3 \\ &= 0^3 + \sum_{k=1}^{n-1}k^3 \\ &= \sum_{k=1}^{n-1}k^3, \end{aligned} \tag{11.21}$$

which reduces (20) to

$$Z_n(t) = \frac{1}{2}\frac{t^4}{n^4}\left\{\sum_{k=1}^{n-1}k^3 + \sum_{k=1}^{n}k^3\right\}. \tag{11.22}$$

From Exercise 6.3, we have

$$\sum_{k=1}^{M}k^3 = \frac{1}{4}M^2(M+1)^2 \tag{11.23}$$

for any positive integer M. Setting $M = n$ yields

$$\sum_{k=1}^{n}k^3 = \frac{1}{4}n^2(n+1)^2, \tag{11.24}$$

whereas setting $M = n - 1$ yields

$$\sum_{k=1}^{n-1}k^3 = \frac{1}{4}(n-1)^2(n-1+1)^2 = \frac{1}{4}(n-1)^2n^2. \tag{11.25}$$

Substituting from (24)-(25) into (22), we find that

$$\begin{aligned} Z_n(t) &= \frac{1}{2}\frac{t^4}{n^4}\left\{\sum_{k=1}^{n} k^2 \quad + \quad \sum_{k=1}^{n-1} k^2\right\} \\ &= \frac{1}{2}\frac{t^4}{n^4}\left\{\frac{1}{4}n^2(n+1)^2 \;+\; \frac{1}{4}n^2(n-1)^2\right\} \\ &= \frac{1}{2}\frac{t^4}{n^4}\cdot\frac{1}{4}n^2\left\{(n+1)^2 \;+(n-1)^2\right\} \\ &= \frac{1}{8}\frac{t^4}{n^2}\left\{2n^2+2\right\} \\ &= \frac{1}{4}t^4\left\{\frac{4n^2+2}{4n^2}\right\} \\ &= \frac{1}{4}\left\{1+\frac{1}{n^2}\right\}t^4. \end{aligned} \tag{11.26}$$

On letting n become indefinitely large in (26), we obtain

$$Z(t) \;=\; \lim_{n\to\infty} Z_n(t) \;=\; \frac{1}{4}t^4. \tag{11.27}$$

From (19) and (27), we finally deduce that the shaded area in Figure 1 is

$$\begin{aligned} F(t) &= \text{Area}(f,[0.05,t]) \\ &= c_0(t-0.05)+\tfrac{1}{2}c_1(t^2-0.05^2)+\tfrac{1}{3}c_2(t^3-0.05^3)+\tfrac{1}{4}c_4(t^4-0.05^4) \\ &= c_0(t-0.05)+\tfrac{1}{2}c_1(t^2-0.05^2)+\tfrac{1}{3}c_2(t^3-0.05^3)+\tfrac{1}{4}c_4(t^4-0.05^4) \\ &= \tfrac{35}{432}(20t-1)^2(227-1000t+1200t^2) \end{aligned} \tag{11.28}$$

after much simplification, and on using (2); see Exercise 1. In particular, stroke volume (neglecting backflow) is $F(0.3) = 70.9$ ml.

This technique is readily adapted to yield the integral of any integer power function (and hence of any polynomial). For if we replace $z(x) = x^3$ by

$$z(x) \;=\; x^s, \tag{11.29}$$

where s is a positive integer, then replacing third powers by s-th powers in (20) yields

$$\begin{aligned} Z_n(t) &= \frac{1}{2}\frac{t^{s+1}}{n^{s+1}}\left\{\sum_{k=1}^{n}(k-1)^s + \sum_{k=1}^{n} k^s\right\} \\ &= \frac{1}{2}\frac{t^{s+1}}{n^{s+1}}\left\{\sum_{k=1}^{n-1} k^s + \sum_{k=1}^{n} k^s\right\} \\ &= \frac{1}{2}\frac{t^{s+1}}{n^{s+1}}\left\{\sum_{k=1}^{n-1} k^s + \sum_{k=1}^{n-1} k^s + n^s\right\} \\ &= t^{s+1}\left\{\frac{1}{n^{s+1}}\sum_{k=1}^{n-1} k^s \;+\; \frac{1}{2n}\right\}, \end{aligned} \tag{11.30}$$

in place of (26). But $1/2n$ approaches zero as n approaches ∞. So, in place of (27), we get

$$Z(t) \;=\; \lim_{n\to\infty} Z_n(t) \;=\; t^{s+1}\left\{\lim_{n\to\infty}\frac{1}{n^{s+1}}\sum_{k=1}^{n-1}k^s\right\}. \tag{11.31}$$

This limit can be calculated by using formulae like those in Exercises 2-4.

Suppose, for example, that s = 4 or

$$z(x) \;=\; x^4. \tag{11.32}$$

Then

$$\text{Area}(z,[0,t]) \;=\; Z(t) \;=\; \lim_{n\to\infty} Z_n(t) \;=\; t^5\left\{\lim_{n\to\infty}\frac{1}{n^5}\sum_{k=1}^{n-1}k^4\right\}. \tag{11.33}$$

From Exercise 6.4, however, we have

$$\sum_{k=1}^{M}k^4 \;=\; \frac{1}{30}M(M+1)(2M+1)(3M^2+3M-1) \tag{11.34}$$

$$\begin{aligned}\Rightarrow\quad \sum_{k=1}^{n-1}k^4 \;&=\; \frac{1}{30}(n-1)n(2n-1)(3n^2-3n-1)\\ &=\; \frac{n^5}{30}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\left(3-\frac{3}{n}-\frac{1}{n^2}\right)\end{aligned} \tag{11.35}$$

and, substituting into (33), we obtain

$$Z(t) \;=\; t^5\lim_{n\to\infty}\frac{1}{30}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\left(3-\frac{3}{n}-\frac{1}{n^2}\right) \;=\; t^5\,\frac{1\cdot 2\cdot 3}{30} \;=\; \frac{1}{5}t^5. \tag{11.36}$$

Similarly, it is shown in Exercises 2-4 that $z(t) = t^5$ implies Area(z, [0, t]) = $t^6/6$, that $z(t) = t^6$ implies Area(z, [0, t]) = $t^7/7$ and that $z(t) = t^7$ implies Area(z, [0, t]) = $t^8/8$. These results suggest very strongly that

$$z(t) \;=\; t^s \qquad\Rightarrow\qquad \text{Area}(z,[0,t]) \;=\; \frac{t^{s+1}}{s+1} \tag{11.37}$$

for any positive integer s. In fact, we will discover in a later lecture that (37) holds for any integer, positive or negative, except s = –1.

Exercises 11

11.1* Verify (28).

11.2* Use the discrete c.d.f. defined by

$$P_n \;=\; \begin{cases}\dfrac{n^2(n+1)^2(2n^2+2n-1)}{M^2(M+1)^2(2M^2+2M-1)} & \text{if } 0\le n\le M\\ 1 & \text{if } M+1\le n<\infty\end{cases}$$

to establish (by the method of Lecture 6) that

$$\sum_{n=1}^{M}n^5 \;=\; \frac{1}{12}M^2(M+1)^2(2M^2+2M-1).$$

Hence show that $z(t) = t^5$ implies Area(z, [0, t]) = $t^6/6$.

11.3 Use the discrete c.d.f. defined by

$$P_n = \begin{cases} \dfrac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{M(M+1)(2M+1)(3M^4+6M^3-3M+1)} & \text{if } 0 \le n \le M \\ 1 & \text{if } M+1 \le n < \infty \end{cases}$$

to establish that

$$\sum_{n=1}^{M} n^6 = \frac{1}{42} M(M+1)(2M+1)(3M^4+6M^3-3M+1).$$

Hence show that $z(t) = t^6$ implies Area(z, [0, t]) = $t^7/7$

11.4 Use the discrete c.d.f. defined by

$$P_n = \begin{cases} \dfrac{n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)}{M^2(M+1)^2(3M^4+6M^3-M^2-4M+2)} & \text{if } 0 \le n \le M \\ 1 & \text{if } M+1 \le n < \infty \end{cases}$$

to establish that

$$\sum_{n=1}^{M} n^7 = \frac{1}{24} M^2(M+1)^2(3M^4+6M^3-M^2-4M+2).$$

Hence show that $z(t) = t^7$ implies Area(z, [0, t]) = $t^8/8$.

11.5 The probability density function of a size distribution for minnows is defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 5 \\ \alpha(x+3)(125-30x+x^2)^2 & \text{if } 5 \le x \le 25 \\ 0 & \text{if } 25 \le x \le 27 \end{cases}$$

What is the value of α? Assume the result of Exercise 2.

11.6 The probability density function of a size distribution for minnows is defined by

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x \le 10 \\ \alpha x^2(200-30x+x^2)^2 & \text{if } 10 \le x \le 20 \\ 0 & \text{if } 20 \le x \le 27 \end{cases}$$

What is the value of α? Assume the results of Exercises 2-3.

Answers and Hints for Selected Exercises

11.3 First observe that

$$P_n = \begin{cases} Q_n / Q_M & \text{if } 0 \le n \le M \\ 1 & \text{if } M+1 \le n < \infty \end{cases}$$

where $Q_n = n(n+1)(2n+1)(3n^4+6n^3-3n+1)$. For $n = M$ we have $P_n = P_M = Q_M / Q_M = 1$. For $n \ge M+1$ we have $P_n = 1$. So for $n \ge M$ we have $P_n = 1$, hence $P_{n-1} = 1$ for $n \ge M+1$. So for $n \ge M+1$ we have $P_n - P_{n-1} = 1 - 1 = 0$, and (6.16) implies

$$1 = \sum_{n=1}^{M}\{P_n - P_{n-1}\} = \sum_{n=1}^{M}\left\{\frac{Q_n}{Q_M} - \frac{Q_{n-1}}{Q_M}\right\} = \frac{1}{Q_M}\sum_{n=1}^{M}\{Q_n - Q_{n-1}\},$$

or

$$\sum_{n=1}^{M}\{Q_n - Q_{n-1}\} = Q_M.$$

But straightforward expansion (for which mathematical software is highly recommended) yields

$$Q_n = n - 7n^3 + 21n^5 + 21n^6 + 6n^7$$

and hence

$$\begin{aligned} Q_{n-1} &= \{n-1\} - 7\{n-1\}^3 + 21\{n-1\}^5 + 21\{n-1\}^6 + 6\{n-1\}^7 \\ &= n - 7n^3 + 21n^5 - 21n^6 + 6n^7, \end{aligned}$$

so that

$$Q_n - Q_{n-1} = 42n^6.$$

Thus

$$\sum_{n=1}^{M} 42n^6 = Q_M = M(M+1)(2M+1)(3M^4+6M^3-3M+1),$$

from which the first result is immediate.

For the second result, set s = 6 in (29) and use (31) to yield

$$\text{Area}(z,[0,t]) = Z(t) = t^7\left\{\lim_{n\to\infty}\frac{1}{n^7}\sum_{k=1}^{n-1}k^6\right\}.$$

Now, setting M = n – 1 above, we have

$$\begin{aligned} \sum_{k=1}^{n-1}k^6 &= \frac{1}{42}Q_{n-1} = \frac{1}{42}\{n - 7n^3 + 21n^5 - 21n^6 + 6n^7\} \\ &= \frac{n^7}{7}\left\{\frac{1}{6n^6} - \frac{7}{6n^4} + \frac{7}{2n^2} - \frac{7}{2n} + 1\right\}. \end{aligned}$$

So

$$\begin{aligned} Z(t) &= t^7\left\{\lim_{n\to\infty}\frac{1}{n^7}\frac{n^7}{7}\left(\frac{1}{6n^6} - \frac{7}{6n^4} + \frac{7}{2n^2} - \frac{7}{2n} + 1\right)\right\} \\ &= \frac{t^7}{7}\left\{\lim_{n\to\infty}\left(\frac{1}{6n^6} - \frac{7}{6n^4} + \frac{7}{2n^2} - \frac{7}{2n} + 1\right)\right\} = \frac{t^7}{7}\{0+1\} = \frac{t^7}{7}, \end{aligned}$$

as required.

11.5 Straightforward expansion yields

$$(x+3)(30x-125-x^2)^2 \quad = \quad 46875-6875x-4050x^2+970x^3-57x^4+x^5.$$

So, with functions g, r, w, z defined by (3) and p, s, ϕ by $p(x) = x^4$, $s(x) = x^5$ and

$$\phi \quad = \quad 46875g-6875r-4050w+970z-57p+s,$$

we have

$$f(x) \quad = \quad \begin{cases} 0 & \text{if } 0 \le x \le 5 \\ \alpha\phi(x) & \text{if } 5 \le x \le 25 \\ 0 & \text{if } 25 \le x \le 27 \end{cases}$$

implying

$$\begin{aligned} \text{Area}(f,[0,27]) &= \text{Area}(f,[0,5]) + \text{Area}(f,[5,25]) + \text{Area}(f,[25,27]) \\ &= 0 + \alpha\,\text{Area}(\phi,[5,25]) + 0 \end{aligned}$$

or

$$\alpha \quad = \quad \frac{\text{Area}(f,[0,27])}{\text{Area}(\phi,[5,25])} \quad = \quad \frac{1}{\text{Area}(\phi,[5,25])}.$$

On [5, 25] we have

$$\begin{aligned} \text{Area}(\phi) \quad = \quad & 46875\text{Area}(g)-6875\text{Area}(r)-4050\text{Area}(w) \\ & +970\text{Area}(z)-57\text{Area}(p)+\text{Area}(s). \end{aligned}$$

Using (9), (11), (13), (16), (17), (27), (29) and (36) with a = 5 and t = 25, we have Area(g, [5, 25]) = 25 – 5 = 20, Area(r, [5, 25]) = $(25^2-5^2)/2 = 300$, Area(w, [5, 25]) = $(25^3-5^3)/3 = 15500/3$, Area(z, [5, 25]) = $(25^4-5^4)/4 = 97500$ and Area(p, [5, 25]) = $(25^5-5^5)/5 = 1952500$, whereas Exercise 2 yields Area(s, [5, 25]) = $(25^6-5^6)/6 = 40687500$. So

$$\begin{aligned} \text{Area}(\phi,[5,25]) \quad &= \quad 46875\times 20-6875\times 300-4050\times 15500/3 \\ &\qquad +970\times 97500-57\times 1952500+40687500 \\ &= \quad 1920000, \end{aligned}$$

implying $\alpha = 1/1920000 \approx 0.52\times 10^{-6}$.

11.6 Straightforward expansion yields

$$x^2(200 - 30x + x^2)^2 \quad = \quad 40000x^2 - 12000x^3 + 1300x^4 - 60x^5 + x^6.$$

So, with w, z defined by (3) and p, s, q, ϕ by $p(x) = x^4$, $s(x) = x^5$, $q(x) = x^6$ and

$$\phi \quad = \quad 40000w - 12000z + 1300p - 60s + q,$$

we have

$$f(x) \quad = \quad \begin{cases} 0 & \text{if } 0 \le x \le 10 \\ \alpha\phi(x) & \text{if } 10 \le x \le 20 \\ 0 & \text{if } 20 \le x \le 27 \end{cases}$$

implying

$$\begin{aligned} \text{Area}(f,[0,27]) &= \text{Area}(f,[0,10]) + \text{Area}(f,[10,20]) + \text{Area}(f,[20,27]) \\ &= 0 + \alpha\,\text{Area}(\phi,[10,20]) + 0 \end{aligned}$$

or

$$\alpha \quad = \quad \frac{\text{Area}(f,[0,27])}{\text{Area}(\phi,[10,20])} \quad = \quad \frac{1}{\text{Area}(\phi,[10,20])}.$$

On [10, 20] we have

$$\text{Area}(\phi) \quad = \quad 40000\,\text{Area}(w) - 12000\,\text{Area}(z) + 1300\,\text{Area}(p) - 60\,\text{Area}(s) + \text{Area}(q).$$

Using (9), (11), (13), (16), (17), (27), (29) and (36) with $a = 10$ and $t = 20$, we have Area(w, [10, 20]) = $(20^3 - 10^3)/3 = 7000/3$, Area(z, [10, 20]) = $(20^4 - 10^4)/4 = 37500$ and Area(p, [10, 20]) = $(20^5 - 10^5)/5 = 620000$, whereas Exercises 2 and 3 yield Area(s, [10, 20]) = $(20^6 - 10^6)/6 = 10500000$ and Area(q, [10, 20]) = $(20^7 - 10^7)/7 = 1270000000/7$. So

$$\begin{aligned} \text{Area}(\phi,[10,20]) &= 40000 \times 7000/3 - 12000 \times 37500 + 1300 \times 620000 \\ &\qquad - 60 \times 10500000 + 1270000000/7 \\ &= 16000000/21, \end{aligned}$$

implying $\alpha = 21/16000000 = 1.3125 \times 10^{-6}$.