8. Index functions. Area and signed area

A function, as you know, is a rule that labels things unambiguously. The labels are always numbers (at least in this course); and for ordinary functions or sequences, the things are numbers, too. But in general things need not be numbers. In particular, things may be pairs consisting of an ordinary function and a possible subdomain. The function is then an **index** function. Thus an index function's job is to assign labels to function-subdomain pairs. In other words, the domain of an index function is a set of all possible pairs of the form (F, [a, b]), where F is a function and [a, b] is an interval.[1,2] As always, new concepts are best grasped in terms of examples. Through them, we will discover that although index functions are a brand new concept in some ways, in other ways they merely formalize what we have known all along, namely, that changing a function's domain can change its properties.

DATE OF READINGS	SAMPLE SIZE	MEAN WIDTH (mm)	DATE OF READINGS	SAMPLE SIZE	MEAN WIDTH (mm)
November 16, 1967	15	2.0	June 19, 1968	16	2.5
December 18, 1967	13	4.0	July 10, 1968	15	2.2
January 19, 1968	18	5.7	August 19, 1968	15	2.3
February 16, 1968	18	7.0	September 18 1968	15	2.6
March 18 1968	18	5.8	October 17 1968	15	3.9
April 18, 1968	16	4.9	November 13, 1968	22	5.9
May 17, 1968	16	3.7			

Table 8.1   Mean testicular size in the European starling

Our first example of an index function is **Diff**. Diff labels a function-subdomain pair with the net change in function value over that subdomain. That is, Diff is defined by

$$\mathrm{Diff}\big(F,[a,b]\big) \;=\; F(b)-F(a). \qquad (8.1)$$

In particular, if F is a function of time, then Diff(F, [a, b]) is net growth of F(t) between t = a and t = b. For example, if W denotes mean testicular width from Lecture 3 then, from Table 3.1 (reproduced here as Table 8.1), net growth of W over the full 12-month period is Diff(W, [0, 12]) = W(12) – W(0) = 5.9 – 2.0 = 3.9. Similarly, winter net growth is given by Diff(W, [0, 3]) = W(3) – W(0) = 7.0 – 2.0 = 5.0; and net "growth" over the remaining nine months is given by Diff(W, [3, 12]) = W(12) – W(3) = 5.9 – 7.0 = –1.1, corresponding to a decrease of 1.1 mm. This example shows that changing subdomains usually changes index values (even though the function remains exactly the same). But changing subdomains does not invariably change index values because, e.g., Diff(W, [1, 2]) = W(2) – W(1) = 5.7 – 4.0 = 1.7 = 3.9 – 2.2 = W(11) – W(8) = Diff(W, [8, 11]). For further practice with Diff, see Exercise 1.

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[1] Mathematicians sometimes refer to index functions as functionals and to ordinary functions as point functions. But for our purposes, functional is not very functional, and point function rather pointless.

[2] We say "a set of all possible pairs" rather than "the set of all possible pairs" because in principle the set may be different for different index functions. For example, the domain of Max consists of functions with maxima paired with possible subdomains, the domain of Min consists of functions with minima paired with possible subdomains, and these two domains are not the same; in particular, for K defined in Lecture 2, (K,[0,1]) belongs to the domain of Min, but not to that of Max.

Our second example of an index function is **Max**. It labels a function-subdomain pair with the function's largest value on that subdomain. So Max is defined by

$$\mathrm{Max}(F,[a,b]) \;=\; F(t_{max}), \tag{8.2}$$

where t_{max} is any global maximizer of F on [a, b], i.e.,

$$F(t_{max}) \;\geq\; F(t) \quad \text{for all } t \in [a,b]. \tag{8.3}$$

For example, Figure 1 depicts net photosynthesis as a function of temperature in wheat and maize. The figure is adapted from Figure 5.2 of Fitter and Hay (1987, p. 190), which in turn is based on work by I.F. Wardlaw. Note that the net rate of CO_2 exchange has been scaled with respect to the maximum value for maize. The upper graph is that of the quintic polynomial ϕ_W defined on [0, 51] by

$$\phi_W(T) \;=\; w_0(4356675T + 60741T^2 - 8476T^3 + 161T^4 - T^5), \tag{8.4}$$

where T is temperature in degrees Centigrade and

$$w_0 \;=\; 6.8441\times 10^{-9} \tag{8.5}$$

is just a constant. The lower graph is that of the piecewise-quintic ϕ_M defined by

$$\phi_M(T) \;=\; \begin{cases} 0 & \text{if } 0 \leq t \leq 12 \\ m_0(-1153008T + 162144T^2 - 6969T^3 + 134T^4 - T^5) & \text{if } 12 \leq t \leq 51 \end{cases} \tag{8.6}$$

where

$$m_0 \;=\; 1.23353\times 10^{-7} \tag{8.7}$$

is again a constant. From Exercise 2, the maximum of ϕ_W is 0.464, which occurs at T_{max} = 23.5; and the maximum of ϕ_M occurs at T_{max} = 37.1 (and is 1 by definition). Thus

$$\mathrm{Max}(\phi_W,[0,51]) \;=\; \phi_W(T_{max}) \;=\; \phi_W(23.5) \;=\; 0.464 \tag{8.8}$$

and

$$\mathrm{Max}(\phi_M,[0,51]) \;=\; \phi_M(T_{max}) \;=\; \phi_M(37.1) \;=\; 1.0. \tag{8.9}$$

Note that, because all rates have been scaled with respect to the maximum for maize, which is 38.2 milligrams per square decimeter per hour, a scaled optimum of 0.464 for wheat implies that the actual maximum net rate of photosynthesis for wheat is 0.464 × 38.2 = 17.7 mg $dm^{-2}h^{-1}$. The minimum, maximum and optimum temperatures at which photosynthesis can occur are often called its cardinal temperatures (e.g., by Fitter and Hay, 1987, p. 188). Thus, according to Figure 1, the three cardinal temperatures of photosynthesis for wheat are 0, 51 and 23.5, whereas those for maize are 12, 51 and 37.1.

It won't surprise you in the least to know that Max has a first cousin **Min** defined by

$$\mathrm{Min}(F,[a,b]) \;=\; F(t_{min}), \tag{8.10}$$

where t_{min} is a global minimizer of F on [a, b], i.e.,

$$F(t_{min}) \;\leq\; F(t) \quad \text{for all } t \in [a,b]. \tag{8.11}$$

Figure 1 illustrates that neither t_{min} nor t_{max} need be unique, although $F(t_{min})$ and $F(t_{max})$ are both unique; for example, T_{min} for wheat is 0 or 51 and T_{min} for maize is 12 or 51, but $F(T_{min}) = 0$ in either case. Thus

$$\mathrm{Min}(\phi_W,[0,51]) \;=\; 0 \;=\; \mathrm{Min}(\phi_M,[0,51]). \tag{8.12}$$

In a sense, of course, this is nothing new. We discovered as long ago as Lecture 1 that global extrema are properties of both a function and its domain. But Max and Min formalize this idea, by making the dependence on domain explicit. Both index functions arise in defining measures of physiological condition. For example, if V denotes ventricular volume in Lecture 1's cardiac cycle, then the stroke volume is Max(V, [0.05, 0.35]) – Min(V, [0.05, 0.35]) = V(0.05) – V(0.3) = 120 – 49.1 = 70.9 ml. Again, if somebody tells you that your blood pressure is 120/80, what they really mean is that Max(p, [0, 0.9]) = 120 and Min(p, [0, 0.9]) = 80, where p(t) denotes arterial pressure at time t. See Exercise 3 for further practice with Max or Min.

We remarked above that, if W denotes mean testes size for Schwab's starlings, then Diff(W, [1, 2]) = 1.7 = Diff(W, [8, 11]): Testes, on average, grew by the same amount between mid-December and mid-January as between mid-July and mid-October. Does this mean that testes grew at the same rate in winter as in late summer and early fall? No, of course not: 1.7 mm over a month represents much faster growth than 1.7 mm over three months. Thus Diff does not provide an adequate measure of how rapidly things change over an interval of time. We therefore introduce a new index function, **Difference Quotient** or simply **DQ**, defined by

$$\mathrm{DQ}(F,[a,b]) \;=\; \frac{F(b)-F(a)}{b-a}. \qquad (8.13)$$

This index function divides net change over an interval by that interval's length to yield the average net rate of change over the interval. In particular, if F is a function of time, then DQ(F, [a, b]) is average net growth rate between t = a and t = b. For example,

$$\mathrm{DQ}(W,[8,11]) \;=\; \frac{W(11)-W(8)}{11-8} \;=\; \frac{3.9-2.2}{11-8} \;=\; 0.57, \qquad (8.14)$$

whereas

$$\mathrm{DQ}(W,[1,2]) \;=\; \frac{W(2)-W(1)}{2-1} \;=\; \frac{5.7-4.0}{2-1} \;=\; 1.7, \qquad (8.15)$$

showing that testes grew three times as fast on average in winter as in late summer and early fall. For further practice with DQ, see Exercise 4.

Our next example of an index function is **Area**, defined only for nonnegative functions, i.e., for functions f satisying f(x) ≥ 0. Area labels a function-subdomain pair with the "scaled" area of the two-dimensional region below the graph of the function, above the horizontal axis and between the ends of the subdomain. By scaled area we mean that both horizontal and vertical units of measurement are determined by scales on the axes; for example, each grid rectangle in Figure 1 has scaled area 10 (horizontal) × 0.2 (vertical) = 5 square units, whereas each grid rectangle in Figure 3 has scaled area 0.05 × 2.5 = 0.125 square units. Formally,

$$\mathrm{Area}(f,[a,b]) \;=\; \text{Scaled area of region } a \le x \le b,\ 0 \le y \le f(x). \qquad (8.16)$$

See Figure 2, where Area(f, [a, b]) is total shaded scaled area. We assume henceforth that area means scaled area.

Now, for any c satisfying $a \le c \le b$, the region $a \le x \le b$, $0 \le y \le f(x)$ is the union of the regions $a \le x \le c$, $0 \le y \le f(x)$ and $c \le x \le b$, $0 \le y \le f(x)$, whose areas can be summed to yield that of the whole. So a general result is that

$$\mathrm{Area}(f,[a,c]) \;+\; \mathrm{Area}(f,[c,b]) \;=\; \mathrm{Area}(f,[a,b]) \qquad (8.17)$$

for any c satisfying $a \le c \le b$. See Figure 2, where Area(f, [a, c]) is the lighter shaded area and Area(f, [c, b]) is the darker one.

For example, we will discover in Lecture 12 that the stroke volume of a cardiac cycle is the area enclosed by the graph of ventricular outflow during systole. Figure 3 shows a crude approximation for a human subject. It is a moot point whether stroke volume should include the backflow that closes the aortic valve; but the percentage error is small, and Figure 3 anyhow ignores the backflow. Then the stroke volume is Area(f, [0.05, 0.3]), i.e., total shaded area in the diagram. This area is easily calculated, because it is the sum of areas of two triangles and a rectangle. The first triangle has base $0.1 - 0.05 = 0.05$ s and height 465 ml/s, hence area $0.5 \times 0.05 \times 465 = 11.625$ ml; the rectangle has base $0.15 - 0.1 = 0.05$ s and hence area $0.05 \times 465 = 23.25$ ml; and the second triangle, with base $0.3 - 0.15 = 0.15$ s, has area $0.5 \times 0.15 \times 465 = 34.875$ ml. Thus stroke volume is $11.625 + 23.25 + 34.875 = 69.75$ ml (excluding backflow).

A major application of Area in biology is to distributions of probability. For example, in Exercise 6.1 and Figure 6.1 we found discrete probability density functions for clutch size in Arctic passerines and leaf thickness in an annual plant, *Dicerandra linearifolia.* A sensible clutch size is always an integer. But there is no reason at all why leaf thickness X should be a multiple of one sixtieth of a millimeter, as in Lecture 5; rather, leaf thickness may be any positive number not exceeding about a quarter of a millimeter. So a better model of leaf thickness variation in *D. linearifolia* than Figure 6.1 is the ordinary function in Figure 4, where total area under the graph is 1, and where Area(f, [a, b]) is interpreted as the probability that a randomly chosen leaf has thickness between a and b mm. We write $\text{Prob}(a \le X \le b) = \text{Area}(f, [a, b])$. For example, the darker shaded area is $\text{Prob}(0.12 \le X \le 0.15) = \text{Area}(f, [0.12, 0.15]) = 0.423$, and other probabilities are given in Table 2.[3]

Leaf thickness now has a **continuous distribution** (as opposed to a discrete one). We say that X is a **continuous random variable**, and we call f the **probability density function** or **p.d.f.** of X. Like many distributions in nature, the distribution of leaf thickness is bell-shaped or **unimodal**, with probability (= area) concentrated near a unique global maximizer, the **mode**. From Figure 4 the mode is 0.149 mm.

PROBABILITY	AREA UNDER GRAPH	THEORETICAL VALUE	NUMERICAL VALUE
$0 \le X \le 0.09$	UNSHADED ON LEFT	Area(f, [0, 0.09])	0.025
$0.09 \le X \le 0.12$	LIGHTER SHADING ON LEFT	Area(f, [0.09, 0.12])	0.148
$0.12 \le X \le 0.15$	DARKER SHADING	Area(f, [0.12, 0.15])	0.423
$0.15 \le X \le 0.18$	LIGHTER SHADING ON LEFT	Area(f, [0.15, 0.18])	0.365
$0.18 \le X < \infty$	UNSHADED ON RIGHT	Area(f, [0.18, ∞))	0.0389

Table 8.2 Probabilities associated with thickness X of randomly chosen *D. linearifolia* leaf

Our last example of an index function, namely, **Int**, is in practice the most important of all. For example, we will show in Lecture 12 how Int determines arterial discharge from, and venous recharge into, a ventricle. Here, however, we satisfy ourselves with a purely mathematical definition. Int is defined in terms of Area. But Int(f, [a, b]) has meaning when f takes negative values, whereas Area(f, [a, b]) has meaning only if $f \ge 0$. So we require a way to express any function in terms of nonnegative components.

Accordingly, let f be defined on [a, b], and define f_{pos} and f_{neg} on [a, b] by

$$f_{pos}(t) \quad = \quad \begin{cases} f(t) & \text{if } f(t) \ge 0 \\ 0 & \text{if } f(t) < 0 \end{cases} \qquad (8.18a)$$

and

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[3] See Exercise 19.5.

$$f_{neg}(t) \quad = \quad \begin{cases} 0 & \text{if } f(t) > 0 \\ f(t) & \text{if } f(t) \le 0. \end{cases} \qquad (8.18b)$$

Equivalent definitions are

$$f_{pos}(t) \quad = \quad \begin{cases} |f(t)| & \text{if } f(t) \ge 0 \\ 0 & \text{if } f(t) < 0 \end{cases} \qquad (8.19a)$$

and

$$f_{neg}(t) \quad = \quad \begin{cases} 0 & \text{if } f(t) > 0 \\ -|f(t)| & \text{if } f(t) \le 0 \end{cases} \qquad (8.19b)$$

(where | • | denotes the magnitude, or absolute value, of •). For example, if f denotes ventricular outflow in Lecture 1's cardiac cycle, then the graphs of f, f_{pos} and f_{neg} are as shown in Figure 5. We see that f_{pos} filters out any negative labels while f_{neg} filters out any positive labels to make f a sum of nonnegative and nonpositive components:

$$f(t) \quad = \quad f_{pos}(t) \quad + \quad f_{neg}(t) \qquad (8.20)$$

for any t in [a, b]. But if f_{neg} is a nonpositive function then $-f_{neg}$ must be a nonnegative function. It is therefore legitimate to define

$$\begin{aligned} \text{Int}(f,[a,b]) \quad &= \quad \text{Area}\big(f_{pos},[a,b]\big) \quad - \quad \text{Area}\big(-f_{neg},[a,b]\big) \qquad (8.21a) \\ &= \quad \text{Area}\big(\big|f_{pos}\big|,[a,b]\big) \quad - \quad \text{Area}\big(\big|f_{neg}\big|,[a,b]\big). \qquad (8.21b) \end{aligned}$$

We refer to Int(f, [a, b]) as the **integral** of f over the interval [a, b], and to the process of obtaining this index of f as **integration**.

From (21), if *signing* area means giving area a positive sign above the axis but a negative sign below it, then Int(f, [a, b]) is just the *signed* area of a two-dimensional region bounded above or below by the horizontal axis and the graph of f, and to the left and right by the ends of the subdomain [a, b]. For example, Figure 6 shows ventricular outflow f and inflow v = –f for the cardiac cycle from Lecture 1, with regions between graph and axis shaded. Numbers on the shaded regions denote their unsigned areas (calculated by a method to be introduced in Lecture 12). From signing these areas, we find that Int(f, [0, 0.05]) = 0, Int(f, [0.05, 0.3]) = 70.9, Int(f, [0.3, 0.35]) = –0.9, Int(f, [0.35, 0.4]) = 0, Int(f, [0.4, 0.75]) = –59.5 and Int(f, [0.75, 0.9]) = –10.5. Correspondingly, Int(v, [0, 0.05]) = 0, Int(v, [0.05, 0.3]) = –70.9, Int(v, [0.3, 0.35]) = 0.9, Int(v, [0.35, 0.4]) = 0, Int(v, [0.4, 0.75]) = 59.5 and Int(v, [0.75, 0.9]) = 10.5.

Note that the stroke volume is Int(f, [0.05, 0.3]) = Area(f, [0.05, 0.3]) = 70.9. But repeated application of (22) or (17) yields Int(v, [0.3, 0.9]) = Area(v, [0.3, 0.9]) = Area(v, [0.3, 0.35]) + Area(v, [0.35, 0.4]) + Area(v, [0.4, 0.75]) + Area(v, [0.75, 0.9]) = 0.9 + 0 + 59.5 + 10.5 = 70.9, which is again the stroke volume. As we will discover in Lecture 12, these two numbers are equal because shaded area above the horizontal axis in Figure 6(b) measures stroke volume as blood refills the ventricle, whereas the area above the axis in Figure 6(a) measures stroke volume as blood discharges into the aorta. See Exercises 5-7 for further practice.

Because Int generalizes Area, any result that is true for Int is inevitably also true for Area, although it need not be true that a result true for Area is also true for Int. Nevertheless, (17) does still hold with Int in place of Area. To see this, observe that (21) implies both

$$\text{Int}(f, [a, c]) \; = \text{Area}(\, f_{pos}\,, [a, c]) \; - \text{Area}(-f_{neg}, [a,c]) \qquad (8.22)$$

and

$$\mathrm{Int}(f, [c, b]) = \mathrm{Area}(f_{pos}, [c, b]) - \mathrm{Area}(-f_{neg}, [c, b]) \tag{8.23}$$

for any c such that $a \le c \le b$. Adding the two equations yields

$$\begin{aligned} \mathrm{Int}(f,[a,c]) + \mathrm{Int}(f,[c,b]) &= \mathrm{Area}(f_{pos}, [a, c]) + \mathrm{Area}(f_{pos}, [c, b]) \\ &\quad - \{\mathrm{Area}(-f_{neg}, [a,c]) + \mathrm{Area}(-f_{neg}, [c, b])\} \\ &= \mathrm{Area}(f_{pos}, [a, b]) - \mathrm{Area}(-f_{neg}, [a,b]), \end{aligned} \tag{8.24}$$

on applying (17) to different nonnegative functions, namely, $f = f_{pos}$ and $f = -f_{neg}$. It now follows immediately from (21) and (24) that

$$\mathrm{Int}(f,[a,c]) + \mathrm{Int}(f,[c,b]) = \mathrm{Int}(f,[a,b]) \tag{8.25}$$

for any c satisfying $a \le c \le b$.

Finally, a word about notation. Recall that we use the notation $\{f_n(x)\}$ in place of $\{f_n(x) \mid L \le n \le M, a \le x \le b\}$ for a function sequences when both [L...M] and [a, b] are obvious from context. Likewise, we replace Area(f, [a, b]) or Int(f, [a, b]) by Area(f) or Int(f) when [a, b] is obvious — but *only* if it is obvious; in particular, if Area or Int appears several times in the same equation, then the identity of the subdomain can be suppressed only if it is the same in every case. For example, it is legitimate to replace (22)-(23) by the statements that

$$\mathrm{Int}(f) = \mathrm{Area}(f_{pos}) - \mathrm{Area}(-f_{neg}) \tag{8.26}$$

on [a, c] and

$$\mathrm{Int}(f, [c, b]) = \mathrm{Area}(f_{pos}, [c, b]) - \mathrm{Area}(-f_{neg}, [c, b]) \tag{8.27}$$

on [c, b]. We cannot, however, replace (17) by the statement that Area(f) + Area(f) = Area(f), because (17) is a statement about three different domains. Indeed Area(f) + Area(f) = Area(f) would imply Area(f) = 0, which is false unless f = 0.

References

Fitter, A. H. & R. K. M. Hay (1987). Environmental Physiology of Plants. Academic Press, New York.

Thompson, D'Arcy W (1942). On Growth and Form. Cambridge University Press.

Exercises 8

8.1* For the functions V and W defined in Appendices 2B and 3, find:

(i) Diff(W, {1, 3}) (ii) Diff(V, {0.1, 0.2}) (iii) Diff(W, {3, 6})
(iv) Diff(V, {0.2, 0.3}) (v) Diff(W, {6, 12}) (vi) Diff(V, {0.3, 0.55})

8.2* Use mathematical software to find the maxima on [0, 51] of the functions ϕ_W, ϕ_M defined by (4)-(7).
Hint: Recall from Lecture 1 that any maximum of a function f is a minimum of -f, and vice versa.

8.3 For the functions f, V and W defined in Appendices 2B and 3, find:

(i) Max(f, {0.05, 0.35}) (ii) Min(W, {2, 4}) (iii) Min(f, {0.05, 0.65})
(iv) Min(V, {0.2, 0.55}) (v) Max(W, {3, 12}) (vi) Max(V, {0.25, 0.45})

8.4 For the functions f, V and W defined in Appendices 2B and 3, find:

(i) DQ(f, {0.05, 0.35}) (ii) DQ(W, {2, 4}) (iii) DQ(f, {0.05, 0.65})
(iv) DQ(V, {0.2, 0.55}) (v) DQ(W, {3, 12}) (vi) DQ(V, {0.25, 0.45})

8.5 For f in Figure 4, find:

(i) Area(f, [0, 0.12]) (ii) Area(f, [0.12, 0.18]) (iii) Area(f, [0.15, ∞))
(iv) Area(f, [0.09, 0.15]) (v) Area(f, [0, 0.18])

8.6 For f and v in Figure 6, find:

(i) Int(f, [0.3, 0.75]) (ii) Int(f, [0.4, 0.9]) (iii) Int(f, [0.05, 0.75])
(iv) Int(f, [0, 0.9]) (v) Int(v, [0, 0.9])

8.7 What is stroke volume in Figure 6 if backflow is interpreted as a negative contribution to systolic discharge?

8.8

TIME (days)	HEIGHT (mm)	TIME (days)	HEIGHT (mm)
0	1	5	57.5
1	2.8	6	72.0
2	6.5	7	79.0
3	24.0	8	79.0
4	40.5		

Thompson (1942, p. 115-16) attributes the above data on growth in height of a beanstalk to Sachs. If Z(t) mm is height after t days, calculate the sequence $\{z_n\}$ defined on [0...7] by z_n = DQ(Z, [n, n + 1]). When is growth fastest?

8.9

TIME (days)	WEIGHT (grams)	TIME (days)	WEIGHT (grams)
6	1	53	42
18	4	60	62
30	9	74	71
39	17	93	74
46	26		

Thompson (1942, p. 115-16) attributes the above data on growth in weight of maize to "Gustav Backman, after Stefanowska." If W(t) gm is weight after t days and the sequence $\{t_n\}$ on [0 ... 8] is defined by $\{t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8\}$ = {6, 18, 30, 39, 46, 53, 60, 74, 93}, calculate the sequence $\{w_n\}$ defined on [0...7] by w_n = DQ(W, $[t_n, t_{n+1}]$). When is growth fastest?

Appendix 8: Functions introduced in Lecture 8 as joins and products of polynomials

FUNCTION P

POLYNOMIAL REPRESENTATION

$$P(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + c_5 t^5$$

NAME	SUBDOMAIN	ORDER	COEFFICIENTS					
		m	c_0	c_1	c_2	c_3	c_4	c_5
$P = \phi_W$	[0,51]	5	0	$4356675\ w_0$	$60741\ w_0$	$-8476\ w_0$	$161\ w_0$	$-w_0$
$P = \phi_M$	[0,12]		0	0	0	0	0	0
	[12,51]	5	0	$-1153008m_0$	$1\ 162144m_0$	$-6969m_0$	$134m_0$	$-m_0$

FUNCTION

PRODUCT REPRESENTATION

NAME	SUBDOMAIN	DEFINITION
ϕ_W	[0,51]	$\phi_W(T) = w_0(17+T)\,T\,(51-T)(5025 - 127T + T^2)$
ϕ_M	[0,12]	$\phi_M(T) = 0$
	[12,51]	$\phi_M(T) = m_0 T(T-12)(51-T)(1884 - 71T + T^2)$

Answers and Hints for Selected Exercises

8.6 (i) Int(f, [0.3, 0.75]) = Int(f, [0.3, 0.35]) + Int(f, [0.35, 0.4]) + Int(f, [0.4, 0.75]) = –0.9 + 0 – 59.5 = – 60.4

(ii) – 59.5 – 10.5 = –70 (iii) 70.9 – 0.9 + 0 – 59.5 = 10.5

(iv) 70.9 – 0.9 + 0 – 59.5 – 10.5 = 0 (v) –70.9 + 0.9 + 0 + 59.5 + 10.5 = 0

8.7 70.9 – 0.9 = 70 ml