5. Ordinary sequences. Fibonacci's rapid rabbits

We have dealt with ordinary functions defined on an interval [a, b]. But domains and ranges need not in general be intervals. In particular, a function can have a discrete set of integers for its domain and a discrete set of numbers for its range. The function is then a called a **sequence**. It can still be defined in terms of its graph, which is still the plot of all possible (THING, LABEL) pairs with THING measured along a horizontal axis and LABEL along a vertical one. But the graph is now a discrete set of points – unlike the graph of an ordinary function, which is a (usually continuous) curve.

THICKNESS (mm)	FREQUENCY	THICKNESS (mm)	FREQUENCY
0.067	9	0.167	118
0.083	5	0.183	17
0.1	28	0.2	9
0.117	45	0.233	1
0.133	165	0.25	2
0.15	90		

Table 5.1 Leaf thicknesses in *Dicerandra linearifolia*

For example, leaf-thickness variation in *Dicerandra linearifolia,* an annual plant in the mint family (Lamiaceae) endemic to North Florida, South Georgia and parts of Alabama, is of interest to biologists because different thicknesses may be favored at different temperatures for several reasons (including, e.g., that thicker leaves retard heat loss whereas thinner ones intercept more light, because more of them can be produced). Table 1 shows thicknesses of 489 specimens of *D. linearifolia* leaves, measured by Winn (1996) with an ocular micrometer under a dissecting microscope and corrected for magnification. A consequence of this method is that measurements are recorded as multiples of one sixtieth of a millimeter, with leaf thickness varying between 4 and 15 of these units. Accordingly, using f_k to denote the label assigned to k by the sequence f (to distinguish it from the label f(k) that the ordinary function f would assign), we define a sequence on the discrete set of integers {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15} by

$$f_k \quad = \quad \text{frequency of leaf thickness } k/60 \text{ mm.} \qquad (5.1)$$

For example, $f_6 = 28$, because there are 28 specimens of thickness 0.1 mm, and $f_{13} = 0$, because there is no specimen of thickness zero. The graph is shown in Figure 1.

NUMBER OF CLUTCHES OF SIZE (BOLD)

Species	Locality	**1**	**2**	**3**	**4**	**5**	**6**	**7**	Total
Lapland Longspur	Devon Island, Canada	0	1	4	12	16	17	4	54
Lapland Longspur	Victoria and Jenny Lind Island	0	0	0	9	22	33	0	64
Lapland Longspur	Cape Thompson	1	1	4	23	65	17	0	111
McCown's Longspur	Wyoming	0	0	18	18	2	0	0	38

Table 5.2 Clutch size in arctic passerines. Source: Hussell (1972, p. 325)

A convenient shorthand for the set of all integers between L and M is [L...M], and the corresponding shorthand for a sequence f defined on [L...M] is $\{f_k \mid L \leq k \leq M\}$. For example, Figure 1 is the graph of the sequence $\{f_k \mid 1 \leq k \leq 15\}$ defined by (1), and Figure 2 is the graph of the sequence $\{f_k \mid 1 \leq k \leq 7\}$ defined, using data from Table 2, by

$$f_k \quad = \quad \text{frequency of clutch with k eggs} \tag{5.2}$$

in Lapland Longspur, *Calcarius lapponicus.* If [L...M] is obvious from context, however, then in place of $\{f_k \mid L \leq k \leq M\}$ we can write $\{f_k\}$, or simply f.[1]

Period	Deaths	Period	Deaths	Period	Deaths
$0 \leq t < 1$	167	$3 \leq t < 4$	6	$6 \leq t < 7$	1
$1 \leq t < 2$	48	$4 \leq t < 5$	3	$7 \leq t < 8$	1
$2 \leq t < 3$	23	$5 \leq t < 6$	6	$8 \leq t < 9$	1

Table 5.3 Deaths from malignant melanoma. Source: Gross and Clark (1975)

A common way to generate a sequence is to sample an ordinary function of time at integer times. For example, Table 3 shows McDonald's (1963) data on deaths among 256 males with malignant melanoma and metastasis (spread of disease beyond original site) upon admission to the M.D. Anderson Tumor Clinic between 1944 and 1960. Time t is measured in years from date of admission. We can think of these data as output from some death process associated with the melanoma and represented mathematically by the ordinary function F, defined on $[0, \infty)$ by

$$F(t) \;=\; \text{proportion deceased at time t after diagnosis of metastasis.} \tag{5.3}$$

Although this function is defined for all $t \geq 0$, with aggregated data we can "observe" it only at integer times. In this way we generate the sequence $\{F_k \mid 0 \leq k \leq 9\}$ whose graph is shown in Figure 3; for example, $F_0 = F(0) = 0$, $F_1 = F(1) = 167/256 = 0.652$, $F_2 = F(2) = 215/256 = 0.84$, and so on. We resist any temptation to join the dots until Lecture 10.

The sequences graphed in Figures 1-3 are all finite. But a sequence can also be infinite. In particular, a sequence can be defined for every nonnegative or positive integer, in which case, we denote its domain by $[0...\infty)$ or $[1...\infty)$, respectively. For example, in one of the earliest known examples of biomathematics, dating all the way back to the beginning of the 13th century, Leonardo Fibonacci of Pisa – reputedly the most distinguished mathematician of the Middle Ages – considered the growth of an idealized rabbit population with zero mortality in which every rabbit is paired from birth (until eternity) with a member of the opposite sex. He supposed this population to sprout from a single pair of newborns introduced at time zero. He further assumed that rabbits reach maturity at age one month, and that every adult pair reproduces itself – precisely once – every month. How many pairs of rabbits will there be on December 31 if the initial pair is introduced on New Year's Day?

At the time corresponding to either the end of month k or the beginning of month k+1, let a_k denote the number of adult pairs, let y_k denote the number of young pairs, and let u_k denote the grand total. Then

$$u_k \;=\; a_k \;+\; y_k \tag{5.4}$$

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[1] More generally, we denote the set of all things of type • with property P by {•} if P is obvious from context, by {P} if • is obvious from context, and by {• | P} if neither • nor P is obvious.

for any value of k. Because the population starts with a pair of newborns, on January 1 (which we regard not only as the beginning of month 1, but also as the end of month 0) there are no adults – just a pair of juveniles. Hence

$$a_0 = 0, \quad y_0 = 1. \tag{5.5}$$

At midnight on January 31 (or, if you prefer, zero hours on February 1), Adam & Eve Rabbit reach maturity. There are now no juveniles (because they have just become adults), but we do have a single pair of adults, namely, A & E Rabbit. That is,

$$a_1 = 1, \quad y_1 = 0. \tag{5.6}$$

During the month of February, A & E Rabbit reproduce themselves. So, on February 29 at midnight (it's leap year, what else?), a first pair of young is counted; and A & E Rabbit are still around, so

$$a_2 = 1, \quad y_2 = 1. \tag{5.7}$$

During March, A & E Rabbit reproduce again, so a fresh pair of young is counted in the midnight census of March 31. Only a single pair of young is counted at that time, however, because A & E's first son and daughter have just become adults. On the other hand, we now have two pairs of adults (A & E plus kids). So

$$a_3 = 2, \quad y_3 = 1. \tag{5.8}$$

Continuing in this manner, we find that the number of young at the end of month k is identical to the number of adults at the beginning of month k, which in turn is identical to the number of adults at the end of month k–1. That is,

$$y_k = a_{k-1}. \tag{5.9}$$

This result is illustrated by Figure 4, where time increases downwards, unfilled circles correspond to juveniles, filled circles correspond to adults, and all circles on the same vertical line correspond to the same pair of rabbits; for example, $y_4 = a_3$ because there are two unfilled circles on the level corresponding to time k = 4 and two filled circles on the level above. Similarly, because number of filled circles at any level equals total number of circles (both unfilled and filled) at the level above, number of adults at the end of month k equals number of young at the end of month k–1 plus number of adults at the end of month k–1, i.e.,

$$a_k = y_{k-1} + a_{k-1} \tag{5.10}$$

Because (9) and (10) are true for any k, we can replace k by k + 1 to obtain

$$y_{k+1} = a_k \tag{5.11a}$$

and

$$a_{k+1} = y_k + a_k \tag{5.11b}$$

Now, from (4) with k+1 in place of k, (11), (4), (10) and (4) with k–1 in place of k, we obtain

$$\begin{aligned} u_{k+1} &= a_{k+1} + y_{k+1} \\ &= y_k + a_k + a_k \\ &= u_k + a_k \\ &= u_k + y_{k-1} + a_{k-1} \\ &= u_k + u_{k-1} \end{aligned} \tag{5.12}$$

So, from (4)-(6) and (12), the number of rabbit pairs at time k is defined recursively by

$$u_0 = 1 \qquad (5.13a)$$
$$u_1 = 1 \qquad (5.13b)$$
$$u_{k+1} = u_k + u_{k-1} \quad \text{if} \quad k \geq 1 \qquad (5.13c)$$

yielding $u_2 = u_1 + u_0 = 2$, $u_3 = u_2 + u_1 = 2 + 1 = 3$, $u_4 = 3 + 2 = 5$, $u_5 = 5 + 3 = 8$, and so on, as you can readily verify by counting all circles at a given level in Figure 4. We call (13c) a **recurrence relation** (because recurrent use of it yields the sequence). Figure 5(a) shows the graph of $\{u_k \mid 0 \leq k \leq 10\}$, whereas Table 4 defines $\{u_k \mid 0 \leq k \leq 19\}$ explicitly.

k	y_k	a_k	u_k	k	y_k	a_k	u_k
0	1	0	1	10	34	55	89
1	0	1	1	11	55	89	144
2	1	1	2	12	89	144	233
3	1	2	3	13	144	233	377
4	2	3	5	14	233	377	610
5	3	5	8	15	377	610	987
6	5	8	13	16	610	987	1597
7	8	13	21	17	987	1597	2584
8	13	21	34	18	1597	2584	4181
9	21	34	55	19	2584	4181	6765

Table 5.4 The Fibonacci sequence

Note that u_k gets larger and larger as k gets larger and larger. Indeed there is no number, however large, that u_k cannot exceed, for large enough k. We identify this state of affairs by saying that the sequence $\{u_k\}$ **diverges** to infinity as $k \to \infty$. We write

$$\lim_{k\to\infty} u_k = \infty. \qquad (5.14)$$

Because there is no mortality to hold rabbits in check, the behavior of the sequence is neither surprising nor realistic. Nevertheless, zero mortality may not be unreasonable for a year or so, in which case, we can answer the question we began with: From Table 4, the prediction for midnight on December 31 is $u_{12} = 233$ rabbit pairs.

A more interesting sequence compares the number of rabbit pairs at the end of a month with the number at the end of the previous month. Accordingly, we define the Fibonacci ratio, ϕ_k, to be the ratio between number of rabbit pairs at the end of month k and number at the end of month k–1, i.e., we define

$$\phi_k = \frac{u_k}{u_{k-1}}. \qquad (5.15)$$

This ratio is a measure of how rapidly the population has grown during month k. Note that $k \geq 1$; (15) is meaningless for k = 0, because u_{-1} is undefined, and so the domain of ϕ is $[1...\infty)$, as opposed to $[0...\infty)$ for u. Figure 5(b) shows the graph of $\{\phi_k \mid 1 \leq k \leq 10\}$, and Table 5 defines $\{\phi_k \mid 1 \leq k \leq 20\}$ explicitly (correct to 10 significant figures). For example, from (15) and Table 4, we have $\phi_1 = u_1/u_0 = 1/1 = 1$, $\phi_2 = u_2/u_1 = 2/1 = 2$, $\phi_3 = u_3/u_2 = 3/2$, $\phi_4 = u_4/u_3 = 5/3$, and so on.

Alternatively, dividing (13c) by u_k and using (15), for $k \geq 1$ we have

$$\frac{u_{k+1}}{u_k} = \frac{u_k}{u_k} + \frac{u_{k-1}}{u_k} = 1 + \frac{1}{\frac{u_{k-1}}{u_k}} = \frac{1}{\phi_k}. \qquad (5.16)$$

But replacing k by k+1 in (15) yields

$$\phi_{k+1} = \frac{u_{k+1}}{u_{(k+1)-1}} = \frac{u_{k+1}}{u_k}. \qquad (5.17)$$

Thus, on substituting from (17) into (16), we find that $\{\phi_k\}$ is defined on $[1...\infty)$ by the recurrence relation[2]

$$\phi_1 = 1 \qquad (5.18a)$$

$$\phi_{k+1} = 1 + \frac{1}{\phi_k} \quad \text{if } k \geq 1. \qquad (5.18b)$$

Now, from (16) alone, we have $\phi_2 = 1+1/1 = 2$, $\phi_3 = 1 + 1/2 = 3/2$, $\phi_4 = 1 + 2/3 = 5/3$, etc., agreeing with previous calculations.

k	ϕ_k	k	ϕ_k
1	1	11	1.617977528
2	2	12	1.618055556
3	1.5	13	1.618025751
4	1.666666667	14	1.618037135
5	1.6	15	1.618032787
6	1.625	16	1.618034448
7	1.615384615	17	1.618033813
8	1.619047619	18	1.618034056
9	1.617647059	19	1.618033963
10	1.618181818	20	1.618033999

Table 5.5 The Fibonacci ratio

The behavior of $\{\phi_k\}$ as time k increases is very different from that of $\{u_k\}$. The ratio ϕ_k alternately increases and decreases, by smaller amounts each time, until eventually it settles down to a number somewhere near 1.618. Let the exact value of this number be denoted by ϕ_∞. In fact, it is shown in Appendix 5A that[3]

$$\phi_\infty = \tfrac{1}{2}\left(1+\sqrt{5}\right). \qquad (5.19)$$

Then, as k gets larger and larger, ϕ_k gets closer and closer to ϕ_∞, until for all practical purposes ϕ_k and ϕ_∞ are indistinguishable. We identify this state of affairs by writing

$$\lim_{k\to\infty} \phi_k = \phi_\infty, \qquad (5.20)$$

and we say that the sequence $\{\phi_k\}$ **converges** to ϕ_∞. See Exercises 2-6.

Finally, a remark on terminology. Henceforth, we will sometimes refer to the above sequences as **ordinary sequences** to distinguish them from function sequences, which we introduce in Lecture 7.

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[2] Because (16) defines ϕ_{k+1} in terms of ϕ_k whereas (13) defines u_{k+1} in terms of both u_k and u_{k-1}, (16) is a first-order recurrence relation whereas (13) is a second-order one.

[3] This number is called the golden ratio and plays an important role in studies of phyllotaxis. See, for example, Jean (1994).

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Exercises 5

5.1 (i) Solve (A2) (ii) Verify (A3).

5.2 The sequence $\{s_n \mid n \geq 0\}$ is defined recursively by

$$s_0 = 1$$
$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{9}{s_n}\right), \quad n \geq 0.$$

(i) Using Mathematica or otherwise, find the values of s_1, s_2, s_3 and s_4 correct to six significant figures.

(ii) What is the value of $s_\infty = \lim_{n\to\infty} s_n$, precisely?

5.3* The sequence $\{s_n \mid n \geq 0\}$ is defined recursively by

$$s_0 = 1$$
$$s_{n+1} = \frac{1}{2}\left(s_n + \frac{10}{s_n}\right), \quad n \geq 0.$$

(i) Using Mathematica or otherwise, find the values of s_1, s_2, s_3 and s_4 correct to six significant figures.

(ii) What is the value of $s_\infty = \lim_{n\to\infty} s_n$, precisely?

5.4 From Figure 1.3, arterial flow during the ejection phase of our cardiac cycle reaches a maximum between 0.1 and 0.2 s. From Appendix 16, the exact time of maximum is the limit as $n \to \infty$ of the sequence defined by

$$t_n = \begin{cases} 0.1 & \text{if } n = 0 \\ \dfrac{240t_{n-1}^2 - 11}{16(30t_{n-1} - 7)} & \text{if } n \geq 1. \end{cases}$$

(i) Find t_∞ correct to 6 significant figures.

(ii) What is t_∞ if $t_0 = 0.1$ is replaced by $t_0 = 0.3$? Can you guess its significance?

(iii) Find a quadratic polynomial Q such that $Q(t_\infty) = 0$, not approximately, but precisely. Deduce the exact value of t_∞, and verify that it agrees with (i).

5.5* From Figure 1.4, venous inflow during the diastolic phase of our cardiac cycle reaches a maximum between 0.5 and 0.55 s. By the method of Appendix 16, the exact time of maximum is the limit as $n \to \infty$ of the sequence defined by

$$t_n = \begin{cases} 0.5 & \text{if } n = 0 \\ \dfrac{1800t_{n-1}^2 - 709}{100(36t_{n-1} - 23)} & \text{if } n \geq 1. \end{cases}$$

(i) Find t_∞ correct to 6 significant figures.

(ii) Find a quadratic polynomial Q such that $Q(t_\infty) = 0$, not approximately, but precisely.

(iii) Deduce the exact value of t_∞ and verify that it agrees with (i).

5.6 From Figure 1.4, venous inflow during the diastolic phase of our cardiac cycle reaches a local maximum between 0.8 and 0.85 s. By the method of Appendix 16, the exact time of maximum is the limit as $n \to \infty$ of the sequence defined by

$$t_n = \begin{cases} 0.8 & \text{if } n = 0 \\ \dfrac{15(48t_{n-1}^2 - 35)}{16(90t_{n-1} - 77)} & \text{if } n \geq 1. \end{cases}$$

(i) Find t_∞ correct to 6 significant figures.

(ii) Find a quadratic polynomial Q such that $Q(t_\infty) = 0$, not approximately, but precisely.

(iii) Deduce the exact value of t_∞, and verify that it agrees with (i).

5.7 The Fibonacci sequence is defined recursively, and therefore implicitly, by (13). But the sequence can also be defined explicitly by

$$u_n = \frac{1}{\sqrt{5}}\left\{(\phi_\infty)^{n+1} + \frac{(-1)^n}{(\phi_\infty)^{n+1}}\right\},$$

where ϕ_∞ is defined by (19).

(i) Verify that the above expression satisfies (13), and use it to compute the values of u_0, u_1, u_2, u_3, u_4, u_5 and u_6.

(ii) Deduce an explicit expression for the Fibonacci ratio ϕ_n.

5.8 Show that the sequence $\{U_n\}$ defined by $U_n = \sum_{k=0}^{n} u_k$ is related to the Fibonacci sequence $\{u_n\}$ by $U_n + 1 = u_{n+2}$ (for $n \geq 0$).

5.9 Show that the Fibonacci sequence $\{u_n\}$ satisfies

$$u_{n+1}u_{n-1} + (-1)^n = \{u_n\}^2$$

for all $n \geq 1$.

Appendix 5A: Convergence of the Fibonacci ratio sequence

The purpose of this appendix is to establish (20) and to show that convergence is oscillatory. We first determine what the limit of $\{\phi_k\}$ must be, *if* the sequence converges. So assume there exists some number ϕ_∞, as yet unknown, to which the sequence converges. Then, as k gets larger and larger, ϕ_k gets closer and closer to ϕ_∞, until for all practical purposes ϕ_k and ϕ_∞ are indistinguishable. In particular, ϕ_{k+1} is even closer to ϕ_∞ than ϕ_k. From (18), however, we have

$$\phi_{k+1} \quad = \quad 1+\frac{1}{\phi_k} \tag{5.A1}$$

for all $k \geq 1$. As k gets larger and larger, this equation gets closer and closer to

$$\phi_\infty \quad = \quad 1+\frac{1}{\phi_\infty}, \tag{5.A2}$$

becoming indistinguishable from it in the limit as $k \to \infty$. Multiplying (A2) by ϕ_∞, we have $\phi_\infty^2 = \phi_\infty + 1$ or $\phi_\infty^2 - \phi_\infty - 1 = 0$, a quadratic equation whose only positive solution is (19); see Exercise 1. Subtracting (A2) from (A1), we find (Exercise 1) that

$$\phi_{k+1} - \phi_\infty \quad = \quad -\left(\frac{\phi_k - \phi_\infty}{\phi_\infty \phi_k}\right). \tag{5.A3}$$

In terms of magnitude, therefore, i.e., ignoring sign, we have

$$|\phi_{k+1} - \phi_\infty| \quad = \quad \frac{|\phi_k - \phi_\infty|}{\phi_\infty \phi_k}. \tag{5.A4}$$

Because $\phi_1 > 0$, by (18), (A1) implies $\phi_k > 1$ for all $k \geq 2$. Thus $1/\phi_k < 1$ for all $k \geq 2$, so that (A4) implies

$$|\phi_{k+1} - \phi_\infty| \quad < \quad \frac{|\phi_k - \phi_\infty|}{\phi_\infty}. \tag{5.A5}$$

From (19), however, we have $1/\phi_\infty \; = \; 2/\{1+\sqrt{5}\} = 0.618$. So (A5) implies

$$\left|\phi_{k+1} - \phi_\infty\right| \quad < \quad 0.62\left|\phi_k - \phi_\infty\right| \tag{5.A6}$$

for all $k \geq 2$. That is, the distance between ϕ_k and ϕ_∞ is reduced by at least 38% at each iteration of the recurrence relation, and must therefore eventually approach zero. This establishes (20). Moreover, from (A3), if $\phi_k > \phi_\infty$ then $\phi_{k+1} < \phi_\infty$, and vice versa. So the convergence is oscillatory.

Answers and Hints for Selected Exercises

5.2 (i) $n = 0 \Rightarrow s_1 = \frac{1}{2}\left(s_0 + 9/s_0\right) = \frac{1}{2}\left(1 + 9/1\right) = \frac{10}{2} = 5$

$n = 1 \Rightarrow s_2 = \frac{1}{2}\left(s_1 + 9/s_1\right) = \frac{1}{2}\left(5 + 9/5\right) = \frac{17}{5} = 3.4$

$n = 2 \Rightarrow s_3 = \frac{1}{2}\left(s_2 + 9/s_2\right) = \frac{1}{2}\left(17/5 + 45/17\right) = \frac{257}{85} = 3.02353$

$n = 3 \Rightarrow s_4 = \frac{1}{2}\left(s_3 + 9/s_3\right) = \frac{1}{2}\left(\frac{257}{85} + 765/257\right) = \frac{65537}{21845} = 3.00009$

(ii) In the limit as $n \to \infty$, we have $s_\infty = \frac{1}{2}\left(s_\infty + 9/s_\infty\right) \Rightarrow 2s_\infty = s_\infty + 9/s_\infty$

$\Rightarrow s_\infty = 9/s_\infty \Rightarrow {s_\infty}^2 = 9 \Rightarrow s_\infty = \sqrt{9} = 3.$

5.3 (i) $n = 0 \Rightarrow s_1 = \frac{1}{2}\left(s_0 + 10/s_0\right) = \frac{1}{2}\left(1 + 10/1\right) = \frac{11}{2} = 5.5$

$n = 1 \Rightarrow s_2 = \frac{1}{2}\left(s_1 + 10/s_1\right) = \frac{1}{2}\left(11/2 + 20/11\right) = \frac{161}{44} = 3.65909$

$n = 2 \Rightarrow s_3 = \frac{1}{2}\left(s_2 + 10/s_2\right) = \frac{1}{2}\left(161/44 + 440/161\right) = \frac{45281}{14168} = 3.19601$

$n = 3 \Rightarrow s_4 = \frac{1}{2}\left(s_3 + 10/s_3\right) = \frac{1}{2}\left(\frac{45281}{14168} + 141680/45281\right) = \frac{4057691201}{1283082416} = 3.16246$

(ii) In the limit as $n \to \infty$, we have $s_\infty = \frac{1}{2}\left(s_\infty + 10/s_\infty\right) \Rightarrow s_\infty = \sqrt{10}$

5.5 (i) 0.519348

(ii) t_∞ satisfies $t_\infty = \dfrac{1800t_\infty^2 - 709}{100(36t_\infty - 23)}$ or $Q(t_\infty) = 0$, where $Q(t) = 1800t^2 - 2300t + 709$

(iii) $t_\infty = \dfrac{115 - \sqrt{463}}{180}$ (because the other solution lies outside [0.5, 0.55])

5.6 (i) 0.802559

(ii) $Q(t) = 720t^2 - 1232t + 525$

(iii) $t_\infty = \dfrac{154 - \sqrt{91}}{180}$ (because the other solution lies outside [0.8, 0.85])