## 2. Ordinary functions: an algebraic perspective. Sums, products and joins

There were almost no equations in Lecture 1, because we defined functions only in terms of their graphs. But functions can also be defined in terms of formulae.

Suppose, for example, that a patch of weeds is growing vegetatively on flat terrain by expanding uniformly in all directions. We can define an ordinary function A by labelling each possible age t in months of this clone with the area A(t) that it covers. In Figure 1, the spokes all emanate from the point where the clone began its life, at time $t = 0$, as a single plant covering an area so small that we can set $A(0) = 0$ with negligible loss of accuracy. Initially, the patch assumes a shape at random; subsequently its shape does not change, because it expands uniformly outwards from the initial point. That is, in Figure 1, which shows the patch at ages 1, 2, 3 and 4, every spoke at $t = 2$, 3 or 4 is t times as long as the corresponding spoke at $t = 1$. Therefore, A(2) is $2^2 = 4$ times as large as A(1), A(3) is $3^2 = 9$ times as large A(1) and A(4) is $4^2 = 16$ times as large as A(1). More generally, A(t) is $t^2$ times as large as A(1). So, denoting A(1) by α, the function A is defined algebraically by

$$A(t) = \alpha t^2. \qquad (2.1)$$

A value for α can sometimes be estimated from field data. For example, Cousens and Mortimer (1995, p. 76) suggest that, in the absence of interfering vegetation, a clone of the herbaceous perennial *Cyperus esculentis* is approximately circular. Their Figure 3.12 suggests that, during a clone's first six months, its radius increases at a rate of about two ninths of a meter per month. Thus a representative value for area at age 1 month is $\alpha = \pi(2/9)^2 = 0.155$, yielding

$$A(t) = 4\pi t^2 / 81 = 0.155t^2 \qquad (2.2)$$

square meters.

Defining a function algebraically has no effect on its graph. The graph of A is still the set of all possible (t, A(t)) pairs or, if y is an arbitrary area, the set of all possible (t, y) pairs such that $y = A(t) = \alpha t^2$; for example, the graph with equation $y = 0.155t^2$ is sketched in Figure 2. Note that three symbols appear in (1), namely, A, α, and t. The first, A, is the name of the function. The other two, α and t, represent quantities on which area depends: if you change either, then you change the area. But there is an important difference between α and t: For any given patch, the value of α is fixed. Thus a given patch can have different areas only by virtue of t being different, whereas different patches can have different areas either because they are older or younger (different t) or because they grow faster or slower (different α, even if t is the same). Accordingly, we need to distinguish between a quantity that can vary for the subject of interest (here, a patch of weeds) and a quantity that is fixed for the subject of interest, but which in principle could take different values. We call the first a **variable**, and the second a **parameter**. Thus α is a parameter and t is a variable. Of course, changing t changes A(t), so A(t) is also a variable. We distinguish between THING-variables and LABEL-variables by calling t the **independent** variable and A(t) the **dependent** variable.

The function A defined by (1) is a special case of a **power function**. A more general power function f is defined on $[0, \infty)$ by the formula

$$f(t) = \alpha t^{\beta}, \qquad (2.3a)$$

where the two parameters α and β are called the **coefficient** and **exponent**, respectively. Power functions are often used to model relationships among biological variables. For example, the functions h, q, g and r, defined graphically in Figures 1.1-1.2, are all power functions; they can instead be defined algebraically, by the formulae in Table 1. Note that, because a function is neither a thing nor a label, but rather a rule for assigning labels to things, it does not matter in the least what symbol we use for the generic THING in a formula: the rule will be the same for every such symbol. Thus the function f defined by $f(t) = \alpha t^{\beta}$ is *identical* to the function f defined by

$$f(x) = \alpha x^{\beta} \tag{2.3b}$$

or $f(y) = \alpha y^{\beta}$ because all three formulae are merely different ways of saying that, for any THING in the domain of f,

$$f(\text{THING}) = \alpha \text{THING}^{\beta}. \tag{2.3c}$$

When body size is the independent variable, power-function relationships are usually called **allometric** laws (see Lectures 21-22). Table 2 provides a partial list.

| NAME OF FUNCTION | COEFFICIENT ($\alpha$) | EXPONENT ($\beta$) | EQUIVALENT DEFINITIONS |
|---|---|---|---|
| h (Figure 1.1) | $\frac{229}{60}$ | $-\frac{1}{4}$ | $h(x) = \frac{229}{60x^{1/4}} = \frac{3.817}{\sqrt{\sqrt{x}}}$ |
| q (Figure 1.1) | $\frac{56}{5}$ | $\frac{3}{4}$ | $q(x) = \frac{56}{5}(x^{1/4})^3 = 11.2\left(\sqrt{\sqrt{x}}\right)^3$ |
| g (Figure 1.2) | $\left(\frac{229}{60}\right)^4$ | $-4$ | $g(y) = \left(\frac{229}{60y}\right)^4 = 212.2y^{-4}$ |
| r (Figure 1.2) | $\left(\frac{5}{56}\right)^{4/3}$ | $\frac{4}{3}$ | $r(y) = \left(\frac{5y}{56}\right)^{4/3} = 0.04y^{4/3}$ |

Table 2.1 Algebraic definitions of power functions defined graphically in Lecture 1

| DEPENDENT VARIABLE | COEFFICIENT ($\alpha$) | EXPONENT ($\beta$) | REFERENCE |
|---|---|---|---|
| mammalian heart rate (1/sec) | 3.817 | -1/4 | Lecture 1 |
| mammal surface area ($m^2$) | 0.0103 | 2/3 | Reiss (1989) |
| mammalian oxygen consumption rate (ml/min) | 11.2 | 3/4 | Lecture 1 |
| heat production of resting mammal (kcal/min) | 0.0538 | 3/4 | Reiss (1989) |
| cardiac output (average blood flow) of resting mammal (ml/min) | 224 | 3/4 | Reiss (1989) |
| mammal empty heart weight (kg) | 0.0043 | 1 | Reiss (1989) |
| mammalian total blood weight (kg) | 0.056 | 1 | Reiss (1989) |

Table 2.2 Coefficients and exponents for some allometric laws. The independent variable is body mass (kg)

Another function often used to model relationships among biological variables is defined by the formula

$$f(x) = \alpha x + \beta, \qquad (2.4)$$

where α and β are parameters. In this case f is called a **linear function** because its graph is always a straight line. For examaple, Thompson (1942, p. 209) used a linear function to model how mandible length y increases with total body length x in *Cyclommatus tarandus,* the reindeer beetle. His model was based on data from Huxley (1932, p. 59), given in Table 3. The "best-fit" linear model, in a sense made precise in Appendix 2A, is

$$y = f(x) = 0.5829x - 8.085. \qquad (2.5)$$

The function f is graphed alongside the data in Figure 3. It predicts, e.g., mandible length $f(60) = 0.583 \times 60 - 8.06 = 26.9$ mm for reindeer beetles with body length 60 mm.[1]

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Mandible length, y (mm) | 3.9 | 10.7 | 14.1 | 19.9 | 24.0 | 30.7 | 34.5 |
| Total body length, x (mm) | 20.4 | 33.1 | 38.4 | 47.3 | 54.2 | 66.1 | 74.0 |

Table 2.3 Variation of mandible length with total body length in the reindeer beetle

Although, as we have just demonstrated, linear functions or power functions can sometimes be useful in their own right as simple two-parameter models of relationships between biological variables, their real importance in mathematics is that more complex models can be constructed from them. For example, the functions we used in Lecture 1 to model ventricular volume or outflow are such **combinations** of linear or power functions. Accordingly, we now define three different categories of combination, namely, sum, product and join. A fourth category, quotient, will be introduced in Lecture 3, and a fifth category, composition, will be introduced in Lecture 5.

First, if g and h have the same domain, then their **sum** is the function s = g + h defined by s(t) = g(t) + h(t). For example, with g and h defined by g(t) = 192500t/9 and h(t) $= -980000t^2/9$ on [0.05, 0.35], s is defined by $s(t) = 192500t/9 + 980000t^2/9$ on [0.05, 0.35]. The corresponding graphs of g, h and s are shown in the left-hand column of Figure 4. The dashed lines indicate how functions are added graphically. At top left we have g(0.3) = 6416.67, at middle left we have h(0.3) = –9800, and at bottom left we have s(0.3) = 6416.67 – 9800 = –3383.33. The graph at top right is that of w defined by $w(t) = 1400000t^3/9$. The graph at middle right is that of s + w. The dashed lines indicate how to determine s(0.225) + w(0.225) = –700 + 1771.87 = 1071.87 graphically (however, the height of the dashed line at middle right is actually only 271.87, because the vertical scale begins at 800). Finally, the graph at bottom right in Figure 4 is that of f defined by

$$f(t) = -2450/3 + 192500t/9 - 980000t^2/9 + 1400000t^3/9, \qquad (2.6)$$

i.e., the sum of s, w and the constant – 2450/3. The dashed line has height f(0.225) = s(0.225) + w(0.225) – 2450/3 = 1071.87 – 816.667 = 255.208. The final graph should look familiar: it is that of systolic blood flow in our cardiac cycle (Figure 1.3). Thus blood flow can be represented by a sum of power functions.

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[1] Huxley (1932, p. 58) believed mandible and body lengths to be related by a power law, but analysis shows that a linear law yields a better fit.

Second, if g and h have the same domain, then their **product** is the function $p = g \bullet h$ defined by $p(t) = g(t)h(t)$.[2] If, for example, $g(t) = 20t - 1$ and $h(t) = 3 - 10t$, then $p(t) = (20t - 1)(3 - 10t)$. The corresponding graphs of g, h and p on [0.05, 0.35] are shown in the left-hand column of Figure 5. The dashed lines indicate how functions are multiplied graphically: At top left we have $g(0.15) = 2$, at middle left we have $h(0.15) = 1.5$, and at bottom left we have $p(0.15) = 2 \times 1.5 = 3$. At top right is the graph of w defined by $w(t) = 7 - 20t$, and below it is the graph of $p \bullet w$. The dashed lines indicate how $p(0.225)w(0.225) = 2.625 \times 2.5 = 6.5625$ is determined graphically. Finally, at bottom right in Figure 5 is the graph of f defined by

$$f(t) = 350(20t - 1)(3 - 10t)(7 - 20t)/9, \tag{2.7}$$

i.e., the product of p, w and the constant 350/9. The dashed line now has height $f(0.225) = p(0.225) \times w(0.225) \times 350/9 = 6.5625 \times 350/9 = 255.208$. You can easily verify that (7) is equivalent to (6); see Exercise 3. Thus ventricular outflow on subdomain [0.05, 0.35] can be represented either by a sum of power functions or by a product of linear ones.

Now, from Figure 1.3, there is no flow at all during the first 0.05 seconds of the cardiac cycle, i.e., $f(t) = 0$ on subdomain [0, 0.05]. With f now defined on both [0, 0.05] and [0.05, 0.35], we can extend its domain to [0, 0.35] by writing

$$f(t) = \begin{cases} 0 & \text{if } 0 \le t \le 0.05 \\ -\frac{2450}{3} + \frac{192500}{9}t + \frac{980000}{9}t^2 + \frac{1400000}{9}t^3 & \text{if } 0.05 \le t \le 0.35 \end{cases} \tag{2.8}$$

This function is an example of a join. More generally, if F has domain [a, c] and G has domain [c, b], then the **join** of F and G is W defined on [a, b] by

$$W(t) = \begin{cases} F(t) & \text{if } a \le t \le c \\ G(t) & \text{if } c \le t \le b \end{cases} \tag{2.9}$$

We refer to F and G as **components** of W, and we assume that

$$F(c) = G(c). \tag{2.10}$$

Otherwise W(c) would be ambiguous. For example, with W = f, (8) becomes the special case of (9) in which a = 0, b = 0.35, c = 0.05, F(t) = 0 and

$$G(t) = -\frac{2450}{3} + \frac{192500}{9}t + \frac{980000}{9}t^2 + \frac{1400000}{9}t^3, \tag{2.11}$$

so that (9) requires G(0.05) = F(0.05) or

$$-\frac{2450}{3} + \frac{192500}{9} \times 0.05 + \frac{980000}{9} \times 0.05^2 + \frac{1400000}{9} \times 0.05^3 = 0. \tag{2.12}$$

You can easily verify that this condition is satisfied.

Note that a sum or product combines functions in parallel, whereas a join combines functions serially. In other words, $g + h$ and $g \bullet h$ both have the same domain as g or h, but their join extends the domain of each. Any number of components may be serially combined in this way. For example, a join S of three components on consecutive subdomains $[a, c_1]$, $[c_1, c_2]$ and $[c_2, b]$ would have the form

$$W(t) = \begin{cases} F(t) & \text{if } a \le t \le c_1 \\ G(t) & \text{if } c_1 \le t \le c_2 \\ H(t) & \text{if } c_2 \le t \le b \end{cases} \tag{2.13}$$

with

[2] Note that, for reasons to emerge in Lecture 4, we do *not* write p = gh.

$$F(c_1) \;=\; H(c_1) \tag{2.14a}$$

and

$$H(c_2) \;=\; G(c_2). \tag{2.14b}$$

In particular, we can extend the domain of ventricular outflow f from [0, 0.05] to [0, 0.4] by writing

$$f(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 \le t \le 0.05 \\ -\frac{2450}{3} + \frac{192500}{9}t + \frac{980000}{9}t^2 + \frac{1400000}{9}t^3 & \text{if} \quad 0.05 \le t \le 0.35 \\ 0 & \text{if} \quad 0.35 \le t \le 0.4 \end{cases} \tag{2.15}$$

which is the special case of (13) for which a = 0, b = 0.4, $c_1 = 0.05$, $c_2 = 0.35$, F(t) = 0, H(t) = 0 and G is again defined by (11). For (14b) we require G(0.35) = H(0.35) or

$$-\frac{2450}{3} + \frac{192500}{9} \times 0.35 + \frac{980000}{9} \times 0.35^2 + \frac{1400000}{9} \times 0.35^3 \;=\; 0, \tag{2.16}$$

which is easily seen to be satisfied.

In fact, by adding two more components, we can extend ventricular outflow f from subdomain [0, 0.4] to its entire domain [0, 0.9]:

$$f(t) \;=\; \begin{cases} 0 & \text{if} \quad 0 \le t \le 0.05 \\ -\frac{2450}{3} + \frac{192500}{9}t + \frac{980000}{9}t^2 + \frac{1400000}{9}t^3 & \text{if} \quad 0.05 \le t \le 0.35 \\ 0 & \text{if} \quad 0.35 \le t \le 0.4 \\ \frac{489600}{49} - \frac{57854400}{1127}t + \frac{4080000}{49}t^2 - \frac{48960000}{1127}t^3 & \text{if} \quad 0.4 \le t \le 0.75 \\ 126000 - \frac{4900000}{11}t + \frac{1568000}{3}t^2 - \frac{2240000}{11}t^3 & \text{if} \quad 0.75 \le t \le 0.9 \end{cases} \tag{2.17}$$

Note that each component is a sum of nonnegative integer power functions; specifically,

$$f(t) \;=\; c_0 + c_1 t + c_2 t^2 + c_3 t^3 \tag{2.18}$$

throughout [0, 0.9], but with different parameters $c_0$, $c_1$, $c_2$, $c_3$ on different subdomains. Because such components are so important, we give them a special name. A sum of nonnegative integer power functions is called a **polynomial**, whose highest exponent with nonzero coefficient is called its **order** or **degree**.[3] In particular, (18) defines a third-order or **cubic** polynomial (at least when $c_3 \neq 0$). Similarly, ventricular volume V in Figure 1.3 is a join of fourth-order or **quartic** polynomials, i.e.,

$$V(t) \;=\; c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 \tag{2.19}$$

throughout [0, 0.9], but with different $c_0$, $c_1$, $c_2$, $c_3$, $c_4$ on different subdomains. The values of these coefficients are given in Appendix 2B. A join of several polynomials is called a **piecewise-polynomial** function, and its order is the highest order of any component. So, for example, f in (17) is piecewise-cubic (even though $c_0$, $c_1$, $c_2$, $c_3$ are all identically zero on the first and third subdomains) and ventricular volume V is is piecewise-quartic (even though V has order zero on the first and third subdomains).

Appendix 2C provides alternative definitions of f and V as products of linear and quadratic functions. If a function has two equivalent representations, which of them is better? For example, which is a more more useful representation of ventricular outflow on [0.05, 0.35], (6) or (7)? It depends. Suppose, for example, that you need to know the times at which flow starts or stops. Then you need to know where f has a **zero**, that is, the values of t for which f(t) = 0. These values of t are known as the **roots** of the equation f(t) = 0. From (7) we see at a glance that f(t) = 0 when t = 1/20, t = 3/10 and t = 7/20, whereas

[3] Order is defined only when at least one coefficient is non-zero.

the roots are hardly obvious at a glance from (6). On the other hand, (6) is a more useful representation for other purposes, as we will discover in later lectures.

In this lecture, we have been careful to distinguish between functions and labels by always using different letters. Initially, this is an excellent habit. Once you understand the difference between function and label, however, it rarely causes confusion to use the same letter for both. Henceforward, therefore, we allow ourselves the luxury of saying things like, "Let t denote time, and let V = V(t) be ventricular volume at time t." The first V is technically a label, and the second V is technically a function, but it will be obvious from context which meaning for V is intended. It's rather like using "Darwin" to refer either to a well known Victorian scientist or to one of his many books (usually that which appeared in 1859): Darwin the scientist is not the same as Darwin the book, but it is always obvious from context which meaning is intended.

We conclude by introducing summation notation, which will come in handy later. From (18) and (19), a polynomial is a sum of terms of the form $c_k t^k$, where k is a nonnegative integer and $c_0$, $c_1$, $c_2$, etc., are given parameters. So it is convenient to use

$$\sum_{k=0}^{m} c_k t^k \qquad (2.20)$$

as mathematical shorthand for "the sum of all terms of the form $c_k t^k$, for values of k between zero and m," where m is the order of the polynomial. With this shorthand, we can write (18) much more concisely as

$$f(t) = \sum_{k=0}^{3} c_k t^k \qquad (2.21)$$

and (19) much more concisely as

$$V(t) = \sum_{k=0}^{4} c_k t^k, \qquad (2.22)$$

for relevant values of the coefficients $c_0$, $c_1$, $c_2$, ... (which are given in Appendix 2B). See Exercises 6-7 for practice.

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## Exercises 2

**2.1** A tumor grows by expanding uniformly in all directions from a single cell of negligible volume. If $\varsigma$ is its volume at age 1, what is its volume at age t?

2.2* Note that A in Figure 2 is concave up. More generally, the power function defined by (3) is concave up or down on $[0, \infty)$ according to whether $\alpha\beta(\beta-1)$ is positive or negative. Confirm this result by graphing the power function on [0, 2] for many different pairs of values of the parameters $\alpha$ and $\beta$. Hint: You could begin, for example, with $\alpha = -1/2$, $\beta = -1/6$; $\alpha = -1/3$, $\beta = 3/7$; $\alpha = -1/4$, $\beta = 13/11$; $\alpha = 1/5$, $\beta = -2/3$; $\alpha = 4/7$, $\beta = 2/5$; $\alpha = 3/2$, $\beta = 4/3$; and so on.

2.3* Verify that (7) and (6) define the same function.

**2.4** What is a simpler name for a first-order polynomial?

**2.5** Why is it necessary to decompose domains into subdomains? Give at least two good reasons.

**2.6** Write out the following polynomials in full:

(i) $\sum_{k=1}^{3}\{2k-1\}t^k$ (ii) $\sum_{k=0}^{4}\{2k^2-7k\}t^k$ (iii) $\sum_{k=1}^{4}\{2k^3+k^2\}t^{k-1}$

**2.7*** Write out the following polynomials in full:

(i) $\sum_{k=0}^{4}\{3k+k^2\}t^k$ (ii) $\sum_{k=1}^{3}\{k^3-10k-2\}t^k$ (iii) $\sum_{k=2}^{5}\{2k^3+k^2\}t^{k-2}$

**2.8*** Individuals of the eusocial wasp *Polistes dominulus* Christ vary in strength, which can be measured by an index between 0 (weakest) and 1 (strongest). According to Theraulaz, Bonabeau and Deneubourg (1995, p. 315), as wasps move around the nest, encounters between pairs of individuals occur more or less randomly. The probability, y, that such an individual is willing to engage in a fight depends upon its strength, x. The strongest individuals are much more likely to engage in a fight than the weakest ones, but individuals of intermediate strength have an even lower engagement probability because of spatial considerations (they do most of the foraging, and so are frequently away from the nest, where encounters are more likely to be agonistic). Theraulaz et al (1995, pp. 316-317) have found empirically that y = f(x), where f is defined on [0, 1] as a quintic polynomial with coefficients $c_0 = 1/2$, $c_1 = -1/2$, $c_2 = 0$, $c_3 = 0$, $c_4 = 0$ and $c_5 = 1$.

(i) Obtain an explicit expression for f(x).
(ii) Plot the graph of f
(iii) Is f increasing, decreasing, or neither?
(iv) What is the global maximum of f? What is the global minimum of f? What is therefore the range of f?

2.9 Skellam (1951, p. 200) used equation (1), i.e., $A(t) = \alpha t^2$, to model the spread of the muskrat, *Ondatra zibethica* L., in central Europe following its introduction in 1905; however, he gave no value for $\alpha$. Estimate this parameter from Skellam's Figure 1 and Figure 2. Hint: The distance from Munich to Breslau (Wroclaw) is about 500 kilometers.

## Appendix 2A: The best-fit linear approximation to Huxley's reindeer beetle data

The purpose of this appendix is to show how the model in Figure 3 was obtained. If the model $f(x) = \alpha x + \beta$ is fitted to the data in Table 3, then squares of differences between model prediction and observation, or squared errors, are as shown in the table below. Adding and simplifying the final column, we find that the sum of squared errors is

$$S(\alpha,\beta) = 3433.26 - 15599\alpha + 18006.5\alpha^2 - 275.6\beta + 667\alpha\beta + 7\beta^2 \qquad (2.A1)$$

This expression defines a bivariate function S with global minimum where $\alpha = 0.5829$, $\beta = -8.084$. See Lecture 25.

| x | OBSERVED y | PREDICTED y (= $\alpha x + \beta$) | SQUARED ERROR |
|---|---|---|---|
| 20.4 | 3.9 | $20.4\alpha + \beta$ | $(20.4\alpha + \beta - 3.9)^2$ |
| 33.1 | 10.7 | $33.1\alpha + \beta$ | $(33.1\alpha + \beta - 10.7)^2$ |
| 38.4 | 14.1 | $38.4\alpha + \beta$ | $(38.4\alpha + \beta - 14.1)^2$ |
| 47.3 | 19.9 | $47.3\alpha + \beta$ | $(47.3\alpha + \beta - 19.9)^2$ |
| 54.2 | 24.0 | $54.2\alpha + \beta$ | $(54.2\alpha + \beta - 24.0)^2$ |
| 66.1 | 30.7 | $66.1\alpha + \beta$ | $(66.1\alpha + \beta - 30.7)^2$ |
| 74.0 | 34.5 | $74.0\alpha + \beta$ | $(74.0\alpha + \beta - 34.5)^2$ |

## Appendix 2B: Functions introduced in Lecture 1 as joins and products of polynomials

### POLYNOMIAL REPRESENTATION

$$P(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4$$

| FUNCTION P NAME | DOMAIN | ORDER m | COEFFICIENTS $c_0$ | $c_1$ | $c_2$ | $c_3$ | $c_4$ |
|---|---|---|---|---|---|---|---|
| P = f | [0, 0.05] | | 0 | 0 | 0 | 0 | 0 |
| | [0.05, 0.35] | 3 | –2450/3 | 192500/9 | –980000/9 | 1400000/9 | 0 |
| | [0.35, 0.4] | | 0 | 0 | 0 | 0 | 0 |
| | [0.4, 0.75] | 3 | 489600/49 | –57854400/1127 | 4080000/49 | –48960000/1127 | 0 |
| | [0.75, 0.9] | 3 | 126000 | –4900000/11 | 1568000/3 | –2240000/11 | 0 |
| P = V | [0, 0.05] | 0 | 120 | 0 | 0 | 0 | 0 |
| | [0.05, 0.35] | 4 | 43895/432 | 2450/3 | –96250/9 | 980000/27 | –350000/9 |
| | [0.35, 0.4] | 0 | 50 | 0 | 0 | 0 | 0 |
| | [0.4, 0.75] | 4 | 1620894/1127 | –489600/49 | 28927200/1127 | –1360000/49 | 12240000/1127 |
| | [0.75, 0.9] | 4 | 293892/11 | –126000 | 2450000/11 | –1568000/9 | 560000/11 |

### PRODUCT REPRESENTATION

| FUNCTION NAME | SUBDOMAIN | DEFINITION |
|---|---|---|
| f | [0, 0.05] | $f(t) = 0$ |
| | [0.05, 0.35] | $f(t) = 350(20t-1)(3-10t)(7-20t)/9$ |
| | [0.35, 0.4] | $f(t) = 0$ |
| | [0.4, 0.75] | $f(t) = 81600(30\,t-23)(5\,t-2)(3-4\,t)/1127$ |
| | [0.75, 0.9] | $f(t) = 14000(12\,t-11)(4\,t-3)(9-10\,t)/33$ |
| V | [0, 0.05] | $V(t) = 120$ |
| | [0.05, 0.35] | $V(t) = 38888.9(0.465137-t)(0.0622462+t)(0.0902426-0.530443t+t^2)$ |
| | [0.35, 0.4] | $V(t) = 50$ |
| | [0.4, 0.75] | $V(t) = 10860.7(0.858516-1.80895t+t^2)(0.15425-0.746601t+t^2)$ |
| | [0.75, 0.9] | $V(t) = 50909.1(1.07035-2.04811t+t^2)(0.490312-1.37411t+t^2)$ |

**Answers and Hints for Selected Exercises**

2.1 $V(t) = \varsigma t^3$.

2.4 A linear function

2.5 A function may be uninvertible on its domain but invertible on several subdomains. Also, different formulae may be needed on different subdomains.

2.6 (i) $t + 3t^2 + 5t^3$ (ii) $-5t - 6t^2 - 3t^3 + 4t^4$ (iii) $3 + 20t + 63t^2 + 144t^3$

2.7 (i) $4t + 10t^2 + 18t^3 + 28t^4$

2.8 (i) $f(x) = x^5 - \frac{1}{2}x + \frac{1}{2}$.

(iii) The function is neither increasing or decreasing on [0, 1]; however, it is decreasing on subdomain [0, c] and increasing on subdomain [c, 1], where c is the global minimizer of f. From your graph, the value of c is roughly 0.56.

(iv) The global minimum is f(c) = 0.275. The global maximum is f(1) = 1. Hence the range is [f(c), f(1)] = [0.275, 1].