1. Ordinary functions: a graphical perspective

The most fundamental concept in calculus is that of a function. We first define the concept (whose meaning in mathematics is very different from its meaning in biology) in general terms, and then proceed immediately to examples.

A **function** is a rule that unambiguously labels things belonging to a given set. That is, each thing has a unique label (although a label can be assigned to more than one thing). The set of all possible things is the function's **domain**, and the set of all possible labels is the function's **range**. Usually, both things and labels are numbers, in which case we call the function an **ordinary** function. An ordinary function is most readily defined in terms of its **graph**, a plot of all possible (THING, LABEL) pairs with THING measured along a horizontal axis and LABEL along a vertical one.

For example, larger mammals are known to have slower heart (or pulse) rates than smaller mammals. For mammals between 1 kg and 10 kg in body mass, this relationship between size and heart rate can be modelled (i.e., approximated) by the upper curve in Figure 1 (see, for example, Dawson, 1991, p. 4). This curve is a graph, a plot of all possible (MASS, RATE) pairs, with body mass measured in kg along the horizontal axis and resting heart rate in beats/sec along the vertical one. The graph defines a function, because each possible body mass is unambiguously labelled by a heart rate. More precisely, each possible x between 1 and 10 is labelled by a y between 2.1 and 3.8 such that (x, y) lies on the graph. See Figure 1, where the dashes run vertically from (x, 0) to (x, y) and horizontally from (x, y) to (0, y).

We give this function a name: we call it h (for heart rate). The domain of h contains all possible numbers between 1 and 10 (on the horizontal axis), whereas the range of h contains all possible numbers between 2.1 and 3.8 (on the vertical axis). More concisely, if [a, b] denotes the set of all numbers between a and b inclusive, then the domain and range of h are [1, 10] and [2.1, 3.8], respectively.

MAMMAL	REPRESENTATIVE SIZE	SOURCE
zorilla	1 kg	Estes 1991, p. 429
rock hyrax	3 kg	op. cit., p. 254
female Kirk's dik-dik	5.5 kg	op. cit., p. 47
jackal	10 kg	op. cit., pp. 398-408

Table 1.1 Some representative African mammal sizes

Different functions can be defined on the same domain. For example, larger mammals have higher rates of oxygen consumption than smaller mammals. For mammals between 1 and 10 kg, this relationship between size and oxygen consumption rate (OCR) is modelled by the lower curve in Figure 1 (see, e.g., Dawson, 1991, p. 5). Again, this curve is a graph, a plot of all possible (MASS, RATE) pairs; and again it defines a function, because each possible body mass x between 1 and 10 kg is uniquely labelled by an OCR z between 11 and 63 ml/min. We give this function a name: we call it q (for quaffing air). The domain is again [1, 10], but the range is now [11, 63].

Let h(x) denote h's label for x. Then the upper graph in Figure 1 is the set of all possible (x, h(x)) pairs with $1 \le x \le 10$. In principle, only two letters are needed to represent a function, one for its name (e.g., h or q) and one for the generic THING (e.g., x). In practice, however, it is often useful to reserve a third letter (e.g., y or z) for a generic LABEL. Then the graph of h can also be described as the set of all possible (x, y) pairs satisfying $y = h(x)$, $1 \le x \le 10$. Likewise, the graph of q is the set of all possible (x, z) pairs such that $z = q(x)$, $1 \le x \le 10$. These two graphs can be used to estimate both heart rate and OCR for any mammal between 1 and 10 kg in size. For example, from Table 1, the model predicts that $h(1) = 2.1$ beats/sec and $h(10) = 3.8$ beats/sec are representative heart rates for zorilla and jackal, respectively; and from Figure 1 we estimate that $q(3) = 25$ ml/min and $q(5.5) = 40$ ml/min are (again, according to the model) representative OCRs for a hyrax and a pygmy antelope .

Note that heart rate keeps dropping as one moves to the right along the upper curve in Figure 1. We say that h is a **decreasing** function. Similarly, because OCR keeps rising as one moves to the right along the lower curve, we say that q is an **increasing** function. Both increasing and decreasing functions have an important property: they are **invertible**, i.e., each possible LABEL belongs to precisely one THING. For example, h is invertible because each possible heart rate between 2.1 and 3.8 is associated with precisely one body mass (according to the model), and q is invertible because each possible OCR between 11 and 63 is associated with precisely one body mass.

If a function is invertible, then we define its **inverse** function by interchanging the roles of domain and range. For example, let g denote the inverse of h in Figure 1. Then we obtain the graph of g by flipping over the graph of h, in such a way that we interchange axes while holding holding (0, 0) – the **origin** – fixed. This maneuver transports the range of h from the vertical axis to the horizontal axis to become the domain of g, and the domain of h from the horizontal axis to the vertical axis to become the range of g. The resulting graph of g is the set of all possible $(y, g(y))$ pairs with $2.1 \le y \le 3.8$ (or, if you prefer, the set of all possible (y, x) pairs such that $x = g(y)$, $2.1 \le y \le 3.8$). It appears at top right in Figure 2. The graph of h is shown at top left for comparison. The inverse of q, which we denote by r, is similarly obtained: it appears at bottom right in Figure 2, with the graph of q at bottom left for comparison. We can use g and r to estimate the body size associated with a given heart rate or OCR. For example, the models associate a heart rate of 3 beats per second with a body size of $g(3) = 2.6$ kg and an OCR of 48 ml/min with a body size of $r(48) = 7$ kg.

Not every function is invertible, however, because not every function is increasing or decreasing. For example, in each cardiac cycle the volume of blood in a human left ventricle decreases from about 120 ml at the end of diastole to about 50 ml at the end of systole and then increases to 120 ml again for the start of the next cycle. In a (resting) heart that beats 67 times a minute, a cycle lasts for 0.9 seconds. If we regard a cycle as beginning when the mitral valve closes to block off the venous return, then the upper graph in Figure 3 is a typical set of (TIME, VOLUME) pairs.[1] This graph defines a function, say V, because each possible time t in the domain [0, 0.9] is unambiguously labelled by a ventricular volume $y = V(t)$. But V is not an invertible function, because each possible volume between 50 ml and 120 ml is reached once during systole and once during diastole. Hence

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[1] A few remarks are perhaps in order. The traces of blood volume and flow in Figures 3 and 4 are merely cartoons of data I abstracted from Folkow and Neil (1971, p. 157) and Levick (1995, pp. 16-20). These cartoons capture all essential aspects of the relationship between ventricular volume and blood flow for pedagogical purposes. For example, the diagrams include (in order) an isovolumetric contraction phase, when both input and output valves are closed; an ejection phase, when the arterial flow rises to a maximum and then decreases to zero as the ventricle empties; a backflow phase, during which the ventricle refills slightly; an isovolumetric relaxation phase, during which both valves are closed again; a primary phase of ventricular filling, when the chamber fills rapidly, at first by suction; and an atrial contraction phase, which forces additional blood into the ventricle. Nevertheless, by virtue of being cartoons, Figures 3-4 are not intended to be accurate in every detail. For example, the relaxation phase at the beginning of diastole is only 0.05 seconds (Levick suggests 0.08); backflow is only about 1% of stroke volume (Levick suggests that it may be closer to 5%); and so on.

every y satisfying $50 < y < 120$ is associated with two different values of t; for example, $V(t) = 100$ both when $t = 0.12$ and when $t = 0.62$. Furthermore, $y = 120$ is associated with every t such that $0 \le t \le 0.05$; this is the isovolumetric contraction phase, when the mitral valve has closed and ventricular blood pressure rapidly rises to open the aortic valve. Similarly, $y = 50$ is associated with every t such that $0.35 \le t \le 0.4$; this is the isovolumetric relaxation phase, when the aortic valve has closed and ventricular blood pressure rapidly falls again to open the mitral valve. Because the graph $y = V(t)$ is flat for $0 \le t \le 0.05$ and again for $0.35 \le t \le 0.4$, we say that V is **constant** both on [0, 0.05] and on [0.35, 0.4].

Now, if you look very carefully at Figure 3, you will see that the end-systolic volume of 50 ml at $t = 0.35$ is not the lowest volume achieved during a cycle; rather, the lowest volume is 49.1 ml, which is achieved at $t = 0.3$. Although the mitral valve is still closed when $0.3 \le t \le 0.35$, ventricular volume increases due to a brief arterial backflow (which closes the aortic valve). Thus the graph of V falls all the way from (0.05, 120) to (0.3, 49.1) but rises thereafter from (0.3, 49.1) to (0.35, 50); it is flat between (0.35, 50) and (0.4, 50), but then rises all the way to (0.9, 120). Accordingly, V is an uninvertible function only if we insist that its domain is [0, 0.9]. If instead we restrict its domain to [0.05, 0.3], then V is decreasing, and therefore invertible. We call this function the **restriction** of V to [0.05, 0.3], and we refer to [0.05, 0.3] as a **subdomain** of V. Likewise, if we restrict V's domain to [0.4, 0.9], then V is increasing, and again invertible. The moral here is that the properties of a function can depend on its domain of definition. In particular, invertibility is not so much a property of a function as a property of both function and domain: V is invertible on [0.05, 0.3] or [0.4, 0.9], but not on [0, 0.9]. For further practice, see Exercise 3.

Another property of both function and domain is that of **extremum**, a generic term for **minimum** or **maximum**. A **global** minimum or maximum is the lowest or highest value a function takes anywhere on its domain, whereas a **local** minimum or maximum is the height of any hilltop or valley floor on its graph. Any domain element corresponding to a maximum or minimum is called a **maximizer** or **minimizer**, respectively, the generic term being **extremizer**. Note that extrema lie in the range, and extremizers in the domain. For example, the lower graph in Figure 3 defines ventricular outflow as a function of time. Let's call this function f, and use O for a typical outflow (so that the graph has equation $O = f(t)$, $0 \le t \le 0.9$). Inspection reveals the global maximum and minimum of f on [0, 0.9] to be $O = 470$ and $O = -296$ ml/s, respectively, with global maximizer $t = 0.14$ and global minimizer $t = 0.52$. There are three local minima, namely, $O = -27$ with minimizer $t = 0.33$, $O = -296$ with minimizer $t = 0.52$, $O = -119$ with minimizer $t = 0.8$; however, there is only one local maximum, namely, 470. In this example, each global extremum is also a local extremum. But a global extremum can also occur at an endpoint; for example, in Figure 3, 0.9 is a maximizer for ventricular volume.[2]

A global extremum is unique on any given domain, but there may be several extremizers; for example, any t such that $0 \le t \le 0.05$ and $t = 0.9$ are all global maximizers for ventricular volume in Figure 3. Nevertheless, changing domains can change extrema. For example, if we were interested only in systolic blood flow, then we would restrict the domain of f to [0, 0.35]; the global maximum would still be 470, but the global minimum would now be -27. Similarly, if interested only in diastole, we would restrict the domain

[2] If, as is usual, a function F defined on [a, b] is either increasing or decreasing at the ends of its domain, then F(a) is either the lowest or highest label assigned in the vicinity of a; and similarly for F(b). Common sense suggests that a and b are then local extremizers, with the advantage that any global extremum must also be a local extremum. The dominant tradition in college calculus, however, appears to be to insist that a local extremizer must lie in a domain's interior. Why? I have no idea. Nevertheless, here I abide by tradition.

of f to [0.35, 0.9]; the global minimum would still be -296, but the global maximum would be zero.

During diastole, outflow is never positive because the ventricle is refilling, and so it is easier to think in terms of inflow. Inflow is just the negative of outflow: If outflow has equation $O = f(t)$, then inflow has equation $I = -f(t)$, and so if inflow is represented by the function v, i.e., $I = v(t)$, then $v(t) = -f(t)$ throughout the domain. We write $v = -f$ and call $-f$ the **negative** of f. Figure 4 shows the graph of $-f$, with that of f below it for comparison. We see that any minimum of f (regardless of whether it is local or global) is a maximum of $-f$, that any maximum of f is a minimum of $-f$, and vice versa. This result is general: it applies to any function. A consequence in practice is that mathematical software packages may have a routine for finding local maxima or minima, but never both—as you will discover for yourself if you use such a package on Exercises 4-9.

References

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Exercises 1

1.1 If f is an increasing function on [a, b] and g denotes its inverse, what are (i) the range of f, (ii) the domain of g and (iii) the range of g?

1.2 If f is a decreasing function on [a, b] and g denotes its inverse, what are (i) the range of f, (ii) the domain of g and (iii) the range of g?

1.3 Find all subdomains on which f in Figure 3 is invertible. In each case, sketch the graph of the inverse function.

1.4* In February, 1919, the U.S. Coast and Geodetic Survey used a tide staff to record the height of the tide at Morro, California. Schureman (1994, p. 105) gives heights y in feet above the zero of the tide staff at hourly intervals. In the table below, t = 0 and t = 24 correspond to midnight on February 12 and February 13, respectively.

t	y	t	y	t	y	t	y	t	y
0	3.9	5	3.6	10	5.6	15	1.9	20	3.2
1	3.4	6	4.4	11	4.8	16	1.2	21	4.0
2	3.0	7	5.1	12	3.9	17	1.0	22	4.3
3	2.8	8	5.7	13	3.4	18	1.3	23	4.5
4	3.0	9	6.0	14	2.6	19	2.3	24	4.2

A function f with domain [0, 24] is defined by f(t) = height of the tide at time t. Use a graphical method to find both the global maximum and global minimum of f.

1.5* In February, 1919, the U.S. Coast and Geodetic Survey used a tide staff to record the height of the tide at Morro, California. Schureman (1994, p. 105) gives heights y in feet above the zero of the tide staff at hourly intervals. In the table below, t = 24 and t = 48 correspond to midnight on February 13 and February 14, respectively.

t	y	t	y	t	y	t	y	t	y
24	4.2	29	3.1	34	6.2	39	2.6	44	3.1
25	3.8	30	3.6	35	5.8	40	2.0	45	3.9
26	3.3	31	4.5	36	5.1	41	1.6	46	4.5
27	3.0	32	5.3	37	4.3	42	1.6	47	4.7
28	2.8	33	6.0	38	3.4	43	2.2	48	4.6

A function f with domain [24, 48] is defined by f(t) = height of the tide at time t. Use a graphical method to find both the global maximum and global minimum of f.

1.6* In February, 1919, the U.S. Coast and Geodetic Survey used a tide staff to record the height of the tide at Morro, California. Schureman (1994, p. 105) gives heights y in feet above the zero of the tide staff at hourly intervals. In the table below, t = 48 and t = 72 correspond to midnight on February 14 and February 15, respectively.

t	y	t	y	t	y	t	y	t	y
48	4.6	53	2.5	58	5.4	63	2.8	68	2.6
49	4.2	54	2.8	59	5.5	64	2.2	69	3.4
50	3.5	55	3.5	60	5.1	65	1.7	70	4.1
51	3.0	56	4.3	61	4.4	66	1.5	71	4.5
52	2.6	57	4.9	62	3.5	67	1.8	72	4.5

A function f with domain [48, 72] is defined by f(t) = height of the tide at time t. Use a graphical method to find both the global maximum and global minimum of f.

1.7 In February, 1919, the U.S. Coast and Geodetic Survey used a tide staff to record the height of the tide at Morro, California. Schureman (1994, p. 105) gives heights y in feet above the zero of the tide staff at hourly intervals. In the table below, t = 72 and t = 96 correspond to midnight on February 15 and February 16, respectively.

t	y	t	y	t	y	t	y	t	y
72	4.5	77	2.2	82	4.6	87	2.9	92	2.0
73	4.2	78	2.2	83	4.9	88	2.2	93	2.8
74	3.7	79	2.6	84	4.8	89	1.6	94	3.6
75	3.1	80	3.3	85	4.3	90	1.3	95	4.1
76	2.5	81	4.1	86	3.6	91	1.4	96	4.4

A function f with domain [72, 96] is defined by f(t) = height of the tide at time t. Use a graphical method to find both the global maximum and global minimum of f.

1.8 In February, 1919, the U.S. Coast and Geodetic Survey used a tide staff to record the height of the tide at Morro, California. Schureman (1994, p. 105) gives heights y in feet above the zero of the tide staff at hourly intervals. In the table below, t = 96 and t = 120 correspond to midnight on February 16 and February 17, respectively.

t	y	t	y	t	y	t	y	t	y
96	4.4	101	2.2	106	3.9	111	3.1	116	2.3
97	4.2	102	1.9	107	4.3	112	2.6	117	3.0
98	3.8	103	2.0	108	4.4	113	2.1	118	3.8
99	3.3	104	2.4	109	4.2	114	1.9	119	4.4
100	2.7	105	3.1	110	3.7	115	1.9	120	4.7

A function f with domain [96, 120] is defined by f(t) = height of the tide at time t. Use a graphical method to find both the global maximum and global minimum of f.

1.9 In February, 1919, the U.S. Coast and Geodetic Survey used a tide staff to record the height of the tide at Morro, California. Schureman (1994, p. 105) gives heights y in feet above the zero of the tide staff at hourly intervals. In the table below, t = 120 and t = 144 correspond to midnight on February 17 and February 18, respectively.

t	y	t	y	t	y	t	y	t	y
120	4.7	125	3.0	130	3.6	135	3.8	140	2.5
121	4.9	126	2.6	131	4.1	136	3.2	141	3.0
122	4.6	127	2.5	132	4.5	137	2.7	142	3.6
123	4.1	128	2.7	133	4.5	138	2.4	143	4.2
124	3.5	129	3.1	134	4.3	139	2.3	144	4.6

A function f with domain [120, 144] is defined by f(t) = height of the tide at time t. Use a graphical method to find both the global maximum and global minimum of f.

Answers and Hints for Selected Exercises

1.1 (i) $[f(a), f(b)]$ (ii) $[f(a), f(b)]$ (iii) $[a, b]$

1.2 (i) $[f(b), f(a)]$ (ii) $[f(b), f(a)]$ (iii) $[a, b]$

1.3 Quoting answers to an accuracy of 2 s.f., f is invertible on subdomains [0.05, 0.14], [0.14, 0.33], [0.33, 0.35], [0.4, 0.52], [0.52, 0.75], [0.75, 0.8] and [0.8, 0.9]; but f is not invertible on subdomain [0, 0.05] or [0.35, 0.4].

1.4 The global maximum is $f(9.0) = 6.0$
The global minimum is $f(16.9) = 1.00$

1.5 The global maximum is $f(33.9) = 6.21$
The global minimum is $f(41.5) = 1.54$

1.6 The global maximum is $f(58.7) = 5.52$
The global minimum is $f(66.0) = 1.50$