In practice, the functions whose limits we have to find are often of the form

$$q(x) \quad = \quad \frac{f(x)}{g(x)}$$

where f and g are both continuous functions (on their respective domains, whose intersection is the domain of q). Then, broadly speaking, there are only four different cases that may arise, as follows:*

1. The first case arises when $g(a) \neq 0$. Then, because f and g are continuous,

$$\lim_{x \to a} q(x) \quad = \quad \frac{f(a)}{g(a)}.$$

For example, $\lim_{x \to 4} \frac{\sqrt{x-3}}{\sqrt{x+3}} \quad = \quad \frac{\sqrt{4-3}}{\sqrt{4+3}} \quad = \quad \frac{1}{\sqrt{7}}$. So the first case is really rather trivial.†

2. The second case arises when $f(a) \neq 0$, $g(a) = 0$ *and* the sign of g does not change at $x = a$. Then

$$\lim_{x \to a} q(x) \quad = \quad \pm\infty,$$

where the positive or negative sign is taken according to whether $g(x)$ has the same sign as $f(a)$ or the opposite one for x close to a (but $\neq a$). For example, $\lim_{x \to 4} \frac{1-x}{(5-x)(4-x)^2} \quad = \quad -\infty$, because $1 - 4 < 0$ while $(5-x)(4-x)^2 > 0$ near $x = 4$.‡

3. The third case arises when $f(a) \neq 0$, $g(a) = 0$ and the sign of g *does* change at $x = a$. Then only one-sided limits exist. For example,

$$\lim_{x \to 4-} \frac{1-x}{(5-x)(4-x)} \quad = \quad -\infty \quad \text{and} \quad \lim_{x \to 4+} \frac{1-x}{(5-x)(4-x)} \quad = \quad \infty \quad \text{but} \quad \lim_{x \to 4} \frac{1-x}{(5-x)(4-x)} \nexists$$

because $(5-x)(4-x)$ changes sign from positive to negative as you move through $x = 4$ from left to right.

4. The fourth case arises when $q(a)$ is undefined because $f(a)$ and $g(a)$ are both either zero or infinite (both $\frac{0}{0}$ and $\frac{\infty}{\infty}$ being meaningless). We must then find $p(x)$ such that

$$q(x) \quad = \quad p(x) \qquad \forall x \neq a$$

implying

$$\lim_{x \to a} q(x) \quad = \quad \lim_{x \to a} p(x)$$

(because, you will recall, the value of q at $x = a$ is completely irrelevant to the limit of q as $x \to a$ and so, in particular, it will not matter in the least if the value of q at a isn't even defined). For example (recalling the first day of classes), because

$$\frac{3-\sqrt{(1-h)(9+h)}}{h} \quad = \quad \frac{8+h}{3+\sqrt{(1-h)(9+h)}}$$

for all $h \neq 0$, we have

$$\lim_{h \to 0} \frac{3-\sqrt{(1-h)(9+h)}}{h} \quad = \quad \lim_{h \to 0} \frac{8+h}{3+\sqrt{(1-h)(9+h)}} \quad = \quad \frac{8+0}{3+\sqrt{(1-0)(9+0)}} \quad = \quad \frac{4}{3}.$$

Similarly, because $\frac{x+1}{x-1} = \frac{1+1/x}{1-1/x}$ for all finite x, we have

$$\lim_{x \to \infty} \frac{x+1}{x-1} \quad = \quad \lim_{x \to \infty} \frac{1+\frac{1}{x}}{1-\frac{1}{x}} \quad = \quad \frac{1+0}{1-0} = 1.$$

---

*To keep things simple, we assume throughout that a can be approached from both the right and the left.

†Note that f has domain $[3, \infty)$ and g has domain $[-4, \infty)$. So q has domain $[3, \infty) \cap [-4, \infty) = [3, \infty)$.

‡Note that $(5-x)(4-x)^2$ is not positive everywhere—but only its behavior near $x = 4$ is relevant.