Solving problems with the LLL algorithm

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Lattice basis reduction (LLL)

A lattice is a discrete $\mathbb{Z}$-module $\subseteq \mathbb{R}^n$

Example: If $b_1, b_2 \in \mathbb{R}^2$ are $\mathbb{R}$-linearly independent then

$$L = \mathrm{SPAN}_{\mathbb{Z}}(b_1, b_2) = \{n_1 b_1 + n_2 b_2 \mid n_1, n_2 \in \mathbb{Z}\}$$

is a lattice of rank 2 and b_1, b_2 is a basis of L.

Lattice basis reduction

Input: a basis of L.
Output: a good basis of L.

- For rank 2 this is easy ($\approx$ Euclidean algorithm). For a long time it was not known how to handle rank $n > 2$ until:
- [LLL 1982] (Lenstra, Lenstra, Lovász): Efficient algorithm for any rank.
- Has lots of applications!

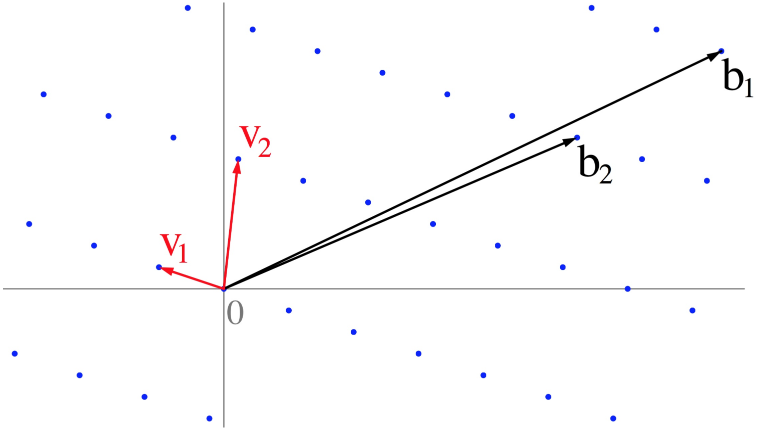
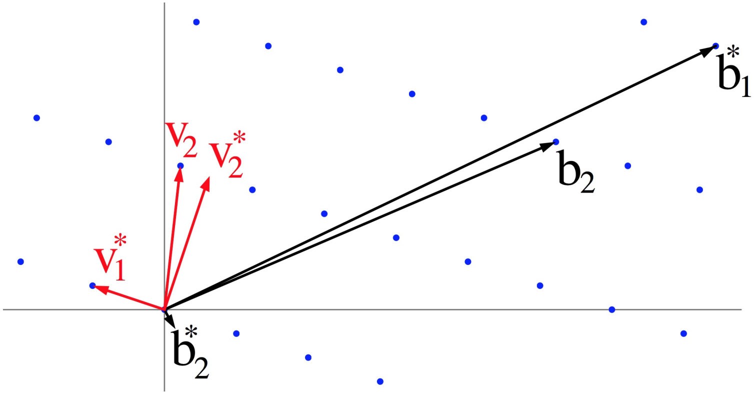


Figure 1: A lattice with a bad basis b_1, b_2 and a good basis v_1, v_2.

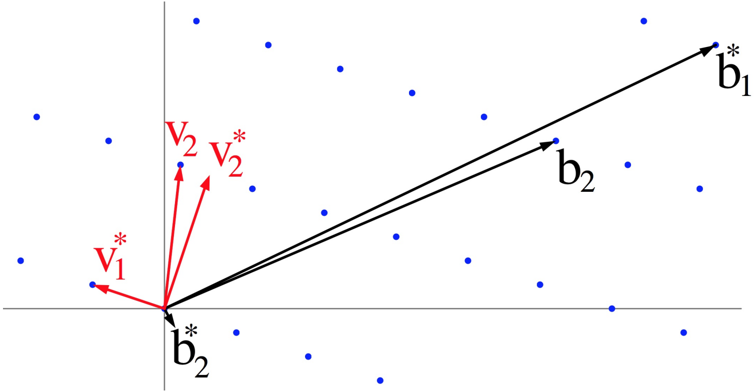
$$L = \{\text{dots in Figure 1}\} = \mathrm{SPAN}_{\mathbb{Z}}(b_1, b_2) = \mathrm{SPAN}_{\mathbb{Z}}(v_1, v_2)$$



Gram-Schmidt process ($n = 2$)

1. $v_1^* = v_1$
2. $v_2^* = v_2 - \mu v_1^*$ Compute $\mu \in \mathbb{R}$ such that $v_1^* \perp v_2^*$.

G.S.-vectors $v_1^*, \ldots, v_n^* \rightsquigarrow$ very useful information on L even though $v_2^*, \ldots, v_n^*$ are generally not in L.

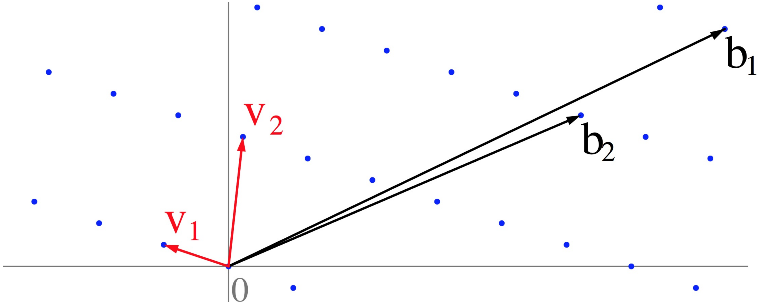


b_1, b_2 is a bad basis because:

1. b_1, b_2 are almost parallel,
2. $||b_2^*|| \ll ||b_2||$ (good basis $\implies$ $||v_i^*|| \approx ||v_i||$)
3. $\min(||b_1^*||, ||b_2^*||)$ is tiny, and thus a poor bound:

Let $b^{\min} := \min(||b_i^*||)$

Shortest-vector-bound: $b^{\min} \leqslant ||\text{shortest } v \neq 0 \text{ in } L||$



Given a bad basis b_1, b_2, how to find a good basis?

1. Subtract an integer-multiple of a one vector from another. (First step in the picture is: replace b_1 with $b_1 - b_2$).
2. Repeat as long as Step 1 can make a vector shorter.

This strategy works well for rank $n = 2$.

Efforts to extend to $n > 2$ failed until the breakthrough [LLL 1982], which uses lengths of G.S.-vectors b_i^* and not the lengths of the b_i themselves!

Application #1: $p = a^2 + b^2$

Theorem (Fermat)

If p prime and $p \equiv 1 \bmod 4$, then $p = a^2 + b^2$ for some $a, b \in \mathbb{Z}$.

Example: $p = 10^{400} + 69 = 10000000000000 \ldots\ldots\ldots 00000000000069$

How to find $a, b \in \mathbb{Z}$ with $a^2 + b^2$ equal to p?

Observation: $a^2 + b^2 \equiv 0 \bmod p$

Hence $a \equiv \alpha b \bmod p$ for some solution of $\alpha^2 + 1 \equiv 0 \bmod p$.

Compute α (e.g. Berlekamp's algorithm). Then

$$\begin{pmatrix} \pm a \\ b \end{pmatrix} \in \mathrm{SPAN}_{\mathbb{Z}}\left(\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \right)$$

(the $\pm$ is irrelevant) $\quad$ ($\alpha^2 + 1 \equiv 0$ has two solutions mod p)

Application #1: $p = a^2 + b^2$

$p = 10^{400} + 69 = 100000000000000000000 \ldots\ldots\ldots 000000000000000000069$

Find: $a, b \in \mathbb{Z}$ with $a^2 + b^2 = p$.

If

$$v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathrm{SPAN}_{\mathbb{Z}}\left(\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1 \end{pmatrix}\right)$$

then

$$||v^2|| = a^2 + b^2 \equiv (\alpha b)^2 + b^2 = (\alpha^2 + 1)b^2 \equiv 0 \mod p.$$

So $||v||^2$ is divisible by p.
So $||v||^2$ is p if v is short enough: $0 < ||v||^2 < 2p$

Such v is easy to find in a good basis.

However, $\left\{\begin{pmatrix} p \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 1 \end{pmatrix}\right\}$ is a bad basis (angle $\approx 10^{-400}$ radians!)

Application #1: $p = a^2 + b^2$

$p = 10^{400} + 69 = 1000000000000000000 \ldots\ldots\ldots 0000000000000000069$

Find: $a, b \in \mathbb{Z}$ with $a^2 + b^2 = p$.

The simple strategy from slide 6 reduces the bad basis to a good basis.

From it we can immediately read off a solution:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8580381359844174226 01\ldots\ldots 0688928009299704710 \\ 5135859783876370541 98\ldots\ldots 0249251426547937937 \end{pmatrix}$$

The computation (finding α and reducing the basis) takes < 0.1 seconds.

Lattice basis reduction for arbitrary rank n

Apply the Gram-Schmidt process to $b_1, \ldots, b_n$

- $b_1^* = b_1$
- $b_2^* = b_2 - \mu_{2,1} b_1^*$ (take $\mu_{ij} \in \mathbb{R}$ s.t. $b_i^* \perp b_j^*$) $(j < i)$
- $b_3^* = b_3 - \mu_{3,1} b_1^* - \mu_{3,2} b_2^*$

  ...

$\mathrm{Det}(L) = ||b_1^*|| \cdots ||b_n^*||$ (the determinant is basis-independent).

Replacing $b_i \leftarrow b_i - k\, b_j$ reduces μ_{ij} to $\mu_{ij} - k$ $(k \in \mathbb{Z})$.

LLL lattice basis reduction

1. Reduce to $|\mu_{ij}| \leqslant 0.51$ ($\leqslant 0.5$ if μ_{ij} known exactly).
2. If swapping $b_{i-1} \leftrightarrow b_i$ increases $||b_i^*||$ at least 10% for some i, then do so and go back to Step 1.

Output: good basis: $||b_{i-1}^*|| \leqslant 1.33 \cdot ||b_i^*||$ and $|\mu_{ij}| \leqslant 0.51$

Properties of LLL reduced basis

If Output(LLL) $= b_1, \ldots, b_n$ then

$$||b_1^*|| \leqslant 1.33 \cdot ||b_2^*|| \leqslant 1.33^2 \cdot ||b_3^*|| \leqslant \cdots \leqslant 1.33^{n-1} \cdot b^{\min}$$

hence

$$||b_1|| \leqslant f_n \cdot ||\text{shortest } v \neq 0 \text{ in } L|| \qquad \text{“fudge factor”} \;\; f_n = 1.33^{n-1}$$

If L has a short non-zero vector then b_1 is not much longer.
If L has short independent $S_1, \ldots, S_k$ then $b_1, \ldots, b_k$ are not much longer.

Many problems P can be solved this way:

1. Construct a lattice $L = \mathrm{SPAN}_{\mathbb{Z}}(b_1, \ldots, b_n)$ for which Solution(P) can be read some solution-vectors $S_1, \ldots, S_k \in L$.
2. Construct L in such a way that vectors in $L - \mathrm{SPAN}_{\mathbb{Z}}(S_1, \ldots, S_k)$ are $\geqslant f_n$ times longer than $S_1, \ldots, S_k$.
3. Replace $b_1, \ldots, b_n$ by an LLL-reduced basis, then:
4. $S_1, \ldots, S_k \in \mathrm{SPAN}_{\mathbb{Z}}(b_1, \ldots, b_k)$. Separates $S_1, \ldots, S_k$ from rest of L

Application #2: Reconstruct algebraic number from an approximation

Suppose β is an algebraic number, a root of an irreducible $P \in \mathbb{Z}[x]$.
Suppose $P = \sum_{i=0}^{n-1} c_i x^i$ with $|c_i| \leqslant 10^b$.

Suppose we have an approximation $\alpha \in \mathbb{R}$ with error $< 10^{-a}$. We need $a \geqslant bn + \epsilon\, n^2$ because P has $\approx bn$ digits of data. (fudge factor $f_n \rightsquigarrow \epsilon\, n^2$)

Problem: Compute exact β (compute P) from the approximation α.

Can read P from solution-vector $S := (c_0, \ldots, c_{n-1}) \in \mathbb{Z}^n$.
Problem: $\mathbb{Z}^n$ contains unwanted vectors as well.

S = Sculpture $\subseteq$ rock.

Use chisel to separate unwanted rock.

Idea:

Add one (or more) entries $\mathbb{Z}^n \to \mathbb{Z}^{n+1}$ that make unwanted vectors at least f_n times longer than S. Use LLL to separate them.

Application #2: Reconstruct algebraic number from an approximation

Define $E : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$

$$(c_0, \ldots, c_{n-1}) \mapsto (c_0, \ldots, c_{n-1}, \ \sum c_i \left[10^a \alpha^i\right] \ )$$

$b_1, \ldots, b_n \ := \ E(\,\text{standard basis of } \mathbb{Z}^n\,)$

$b_1, \ldots, b_n$ spans a lattice $L \subseteq \mathbb{Z}^{n+1}$ of rank n.

$b_1, \ldots, b_n$ is a bad basis. **Typical example**: degree$(P) < 40$ and $|\text{coefficients}| \leqslant 10^{100}$. Angles will be $\approx 10^{-4000}$ radians!

LLL quickly turns this into a good basis.

With suitable precision a, this either leads to the minpoly $P = \sum c_i x^i$ or a proof that no P exists within the chosen bounds.

Application #3: Polynomial-time factorization

Theorem (LLL 1982)

Factoring in $\mathbb{Q}[x]$ can be done in polynomial time.

Proof sketch: Compute a root of f to precision a. Use the previous slide to compute its minpoly. Choose a in such a way that this produces either a non-trivial factor, or an irreducibility proof.

Remarks:

1. [LLL 1982] uses a p-adic root, while [Schönhage 1984] uses a real or complex root. Both work in polynomial time.
2. Neither was used in computer algebra systems; [Zassenhaus 1969] (not polynomial time!) is usually much faster.
3. Faster algorithm [vH 2002]: apply LLL to a much smaller lattice.

Integer solutions of approximate and/or modular equations.

Find: $x_1, \ldots, x_n \in \mathbb{Z}$ when given:

1. Approximate linear equations: $|a_{i,1}x_1 + \cdots + a_{i,n}x_n| < \epsilon_i$ $\qquad (a_{ij} \in \mathbb{R})$
2. or modular linear equations $b_{i,1}x_1 + \cdots + b_{i,n}x_n \equiv 0 \bmod m_i$
3. or a mixture of the above, and other variations

then use LLL.

Remarks:

- Linear equations over $\mathbb{R}$: Ordinary linear algebra gives a basis solutions over $\mathbb{R}$, but this does not help to find solutions over $\mathbb{Z}$.
- Equations (approximate and/or modular etc.) are inserted in a lattice by adding entries (like $E : \mathbb{Z}^n \to \mathbb{Z}^{n+1}$ on p. 13).
- [vH, Novocin 2010]: Efficient method for: "amount(data in equations)" $\gg$ "amount(data in solution)"

Application: Integer relation finding

Given $a_1, \ldots, a_n \in \mathbb{R}$, find $x_1, \ldots, x_n \in \mathbb{Z}$ (say $|x_i| \leqslant 10^{100}$) with

$$a_1 x_1 + \cdots + a_n x_n = 0.$$

Notable algorithms:

- [LLL 1982]
- [PSLQ 1992] ($\equiv$ [HJLS 1986] ?)

Beautiful applications e.g. PSLQ $\rightsquigarrow$ Bailey-Borwein-Plouffe formula for π

Remarks:

- [LLL 1982] is a more complete solution because [PSLQ 1992] gave no bound(precision(a_i)) $\rightsquigarrow$ provable result.
- PSLQ won SIAM Top 10 Algorithms of the Century award.
- The fastest implementations I have seen can handle $n = 500$ (using modern versions of LLL).

Polynomial factorization until 2000.

$f \in \mathbb{Z}[x]$, degree N, square-free and primitive.

Step 1:

Factor $f \equiv f_1 \cdots f_r \bmod p$
and Hensel lift:

$$f \equiv f_1 \cdots f_r \bmod p^a$$

Step 2 in [Zassenhaus 1969]:

- Try $S \subseteq \{f_1, \ldots, f_r\}$ with $1, 2, \ldots \lfloor r/2 \rfloor$ elements, and check if the product (lifted to $\mathbb{Z}[x]$) is a factor of f in $\mathbb{Z}[x]$.
- Up to 2^{r-1} cases $S \subseteq \{f_1, \ldots, f_r\}$ (Combinatorial Problem)

[LLL 1982] Bypasses Combinatorial Problem:

- $L := \{(c_0, \ldots, c_{N-1}) \mid \sum c_i x^i \equiv 0 \bmod (p^a, f_1)\}$ (rank $= N$)
- LLL-reduce, take first vector, and compute $\gcd(f, \sum c_i x^i)$.

Factor f in $\mathbb{Q}[x]$, degree $N = 1000$

[LLL 1982] reduces a lattice of rank N

- Algorithm runs in polynomial time.
- However, lattice reduction for rank 500 is very time consuming.
- rank $N = 1000$ is a problem!

[Zassenhaus 1969] tries $\leqslant 2^{r-1}$ cases

- $r = 12 \rightsquigarrow$ ☺ (Finishes in seconds)
- $r = 80 \rightsquigarrow$ ☹ (Millions of years, even with 10^9 cases per second)

If: f has degree $N = 1000$, few factors in $\mathbb{Q}[x]$ but $r = 80$ factors in $\mathbb{F}_p[x]$
Then: Out of reach for any algorithm in 2000.

However, **80 bits of data** reduces CPU time from **eons** to **seconds!**

[vH 2002]: Use lattice reduction to compute **only those bits!** (**rank** $\approx r$)

Factor f in $\mathbb{Q}[x]$, degree N, with $f \equiv f_1 \cdots f_r \bmod p^a$

[LLL 1982]: (polynomial time)

Reduce a lattice of rank N (and large entries)

[Zassenhaus 1969]: (not poly time, usually faster than [LLL 1982])

Try (exponentially many) subsets $S \subseteq \{f_1, \ldots, f_r\}$ (Combinatorial Problem)

[vH 2002]: (fastest)

- $S \Longleftrightarrow (v_1, \ldots, v_r) \in \{0,1\}^r$
- Insert data: $\{0,1\}^r \subseteq \mathbb{Z}^r \to \mathbb{Z}^{r+\epsilon}$ to construct lattice of rank $r + \epsilon$
- Sequence of lattice reductions leads to $v_1, \ldots, v_r$, and hence S.
- Test (as in Zassenhaus) if $\prod S \bmod p^a \rightsquigarrow$ a factor in $\mathbb{Q}[x]$.
- [vH 2002]: correctness and termination proof, no complexity bound.
- Complexity bound: [vH, Novocin 2010] and [vH 2013].

[vH 2002] factoring

- $S \subseteq \{f_1, \ldots, f_r\} \iff v \in \{0,1\}^r \subseteq \mathbb{Z}^r \to \mathbb{Z}^{r+\epsilon}$
- **If**: we have: approximate/modular linear equations for $v = (v_1, \ldots, v_r)$
  then: lattice reduction $\rightsquigarrow$ v.
- However, the factor $\prod S = \prod f_i^{v_i}$ of f depends non-linearly on v.
- **Idea**: coefficients$(f \cdot f_i'/f_i)$ $\rightsquigarrow$ equations for v
  (f_i'/f_i is the logarithmic derivative; turns products into sums)

Remarks:

- [vH 2002] runs fast; lattice reduction is only used to construct r bits.
- Lots of data in coefficients$(f \cdot f_i'/f_i)$ $N \cdot \log_2(p^a)$ bits $\rightsquigarrow$ r bits.
- How to select from this data? (select all $\rightsquigarrow$ no speedup)
- Arbitrary choice $\rightsquigarrow$ fast in practice but no complexity bound.
- [vH Novocin 2010] and [vH 2013] solve this $\rightsquigarrow$ best complexity bound and practical performance, in the same algorithm.

References, polynomial factorization over $\mathbb{Q}$

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