The special L-value of the winding quotient of level a product of two distinct primes

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Abstract

Let p and q be two distinct primes and let J_e denote the winding quotient at level pq. We give an explicit formula that expresses the special L-value of J_e as a rational number, and interpret it in terms of the Birch-Swinnerton-Dyer conjecture.

1 Introduction and results

Let N be a positive integer and let $X_0(N)$ denote the usual modular curve of level N and $J_0(N)$ its Jacobian. Let $\{0, i\infty\}$ denote the projection of the path from 0 to $i\infty$ in $\mathcal{H} \cup \mathbf{P}^1(\mathbf{Q})$ to $X_0(N)(\mathbf{C})$, where $\mathcal{H}$ is the complex upper half plane. We have an isomorphism $H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{R} \xrightarrow{\simeq} \mathrm{Hom}_{\mathbf{C}}(H^0(X_0(N), \Omega^1), \mathbf{C})$, obtained by integrating differentials along cycles. Let $e \in H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{R}$ correspond to the map $\omega \mapsto -\int_{\{0,i\infty\}} \omega$ under this isomorphism. It is called the *winding element*. Let $\mathbf{T}$ denote the Hecke algebra, i.e., the sub-ring of endomorphisms of $J_0(N)$ generated by the Hecke-operators T_l for primes $l|N$ and by U_p for primes $p \not| N$. We have an action of $\mathbf{T}$ on $H_1(X_0(N), \mathbf{Z}) \otimes \mathbf{R}$. Let I_e be the annihilator of e with respect to this action; it is an ideal of $\mathbf{T}$. We consider the quotient abelian variety $J_e = J_0(N)/I_e J_0(N)$ over $\mathbf{Q}$. It is called the *winding quotient* of $J_0(N)$. It is the largest quotient of $J_0(N)$ whose L-function does not vanish at $s = 1$, and every optimal quotient of $J_0(N)$ whose L-function does not vanish at $s = 1$ factors through the winding quotient. Thus the winding quotient is especially interesting from the point of view of the second part of the Birch and Swinnerton-Dyer conjecture. The winding quotient also played a crucial role in the proof of the uniform boundedness conjecture for elliptic curves [Mer96a].

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The goal of this article is to give a formula that expresses the algebraic part of the special L-value of J_e as a rational number in the situation where N is a product of two distinct primes (see Theorem 1.1 below). Such a formula is given implicitly for the case where N is prime in [Aga99]. Also an analogous formula is given for newform quotients in [AS05] (see also [Aga10]). The last two quotients mentioned above factor through the new part of $J_0(N)$. The winding quotient considered in this article for N a product of two distinct primes need not factor through the new quotient of $J_0(N)$. To our knowledge, this article is the first occasion where an explicit formula has been given that expresses the algebraic part of the special L-value as a rational number for a quotient of $J_0(N)$ of arbitrary dimension that does *not* factor through the new quotient of $J_0(N)$. This is especially interesting from the point of view of the Birch and Swinnerton-Dyer conjecture, and we see some peculiarities that do not arise when the quotient factors through the new quotient of $J_0(N)$. For example, we are led to a conjecture regarding the Manin constant of J_e (when N is a product of two distinct primes).

We now prepare to state the formula alluded to above. Let W denote the $\mathbf{Z}$-module of invariant differentials on the Néron model of J_e. Then $\mathrm{rank}(W) = d$ where $d = \dim(J_e)$ and $\wedge^d W$ is a free $\mathbf{Z}$-module of rank 1 contained in $H^0(J_e, \Omega^d_{J_e/\mathbf{Q}})$. Let D be a generator of $\wedge^d W$. Let $\{\omega_1, ..., \omega_d\}$ be any $\mathbf{Q}$-basis of $H^0(J_e, \Omega_{J_e/\mathbf{Q}})$. Then $D = c \cdot \wedge_j \omega_j$ for some $c \in \mathbf{Q}$. Let $\{\gamma_1, ..., \gamma_d\}$ be a basis of $H_1(J_e, \mathbf{Z})^+$, where the superscript + always denotes the group of elements invariant under the action of complex conjugation. Let $c_\infty(J_e)$ denote the number of connected components of $J_e(\mathbf{R})$. Then define the real volume of J_e as $\Omega(J_e) = c_\infty(J_e) \cdot c \cdot \det(\int_{\gamma_i} \omega_j)$. Note that $\Omega(J_e)$ is independent of the choice of the basis $\{\omega_j\}$; it is the volume of $J_e(\mathbf{R})$ computed using the measure given by the Néron differentials. It is known that $L(J_e, 1)/\Omega(J_e)$ is a rational number (we shall show this below when N is the product of two distinct primes); this is what we meant by the algebraic part of the special L-value above.

Let $S_2(\Gamma_0(N), \mathbf{Z})$ denote the $\mathbf{Z}$-module of cusp forms over $\Gamma_0(N)$ with coefficients in $\mathbf{Z}$. Pulling back differentials along $X_0(N) \to J_0(N)$, one gets an injection of $H^0(J_0(N), \Omega_{J_0(N)/\mathbf{Z}})$ into $S_2(\Gamma_0(N), \mathbf{Z})$, where if $f \in S_2(\Gamma_0(N), \mathbf{Z})$, then the corresponding differential on $X_0(N)$ is given by $\omega_f = 2\pi i f(z) dz$. One can show that a $\mathbf{Q}$-basis for $H^0(J_e, \Omega_{J_e/\mathbf{Q}})$ is given by the differentials corresponding the set of generators of $S_e = \{f \in S_2(\Gamma_0(N), \mathbf{Z}) : I_e f = 0\}$. If we use this for the basis $\{\omega_j\}$ in the paragraph above, the constant c in the paragraph above will be denoted $c_M(J_e)$. It is the Manin constant of J_e, as defined in [ARS06]. For future use, we note that from the discussion above, we have

$$\Omega(J_e) = c_\infty(J_e) \cdot c_M(J_e) \cdot \mathrm{disc}(H_1(J_e, \mathbf{Z})^+ \times S_e \to \mathbf{C}), \tag{1}$$

where "disc" always denotes the discriminant of a pairing of $\mathbf{Z}$-modules.

Let $H = H_1(X_0(N), \mathbf{Z})$, $H_e = H[I_e]$, $\hat{I}_e = \mathrm{Ann}_{\mathbf{T}} I_e$, $\hat{H}_e = H[\hat{I}_e]$ and $\Im = \mathrm{Ann}_{\mathbf{T}}((0) - (\infty))$. If M is a positive integer, then let S_M denote the set of newforms f of level M with $L(f, 1) \neq 0$. If f is a newform of level M, then as usual, by $a_n(f)$ we mean the n-th Fourier coefficient of f.

Theorem 1.1. *Let N be a product of two distinct primes p and q. Then*

$$\frac{L(J_e,1)}{\Omega(J_e)} = \frac{\left|\frac{(H/\hat{H}_e)^+}{H^+/\hat{H}_e{}^+}\right|}{c_\infty(J_e)} \cdot \frac{\left|\frac{H^+}{\hat{H}_e{}^+ + H_e^+}\right| \cdot \left|\frac{H_e^+}{\Im e}\right|}{\prod_{f \in S_p}(1 + q - a_q(f)) \cdot \prod_{g \in S_q}(1 + p - a_p(g)) \cdot |\mathbf{T}e/\Im e|} \cdot \frac{q^{|S_p|} \cdot p^{|S_q|}}{c_M(J_e)}. \tag{2}$$

Note that the product $\prod_{f \in S_p}(1 + q - a_q(f)) \cdot \prod_{g \in S_q}(1 + p - a_p(g))$ in the denominator on the right side is non-zero because of the Weil bound. It should be possible to compute the special L-value using the formula above, considering that a similar calculation was done when N is a prime in [Aga00] and [Aga99].

We shall prove the theorem above in Section 2, but in the rest of this section, we shall discuss its implications. It is instructive to first consider the corresponding situation when N is prime. In that case, it follows from equations (2) and (3) of [Aga99] (noting the correction just before the Acknowledgments on page 374 of loc. cit., where "torsion-full" should really be "torsion-free") that

$$\frac{L(J_e,1)}{\Omega(J_e)} = \frac{\left|\frac{H^+}{\hat{H}_e{}^+ + H_e^+}\right| \cdot \left|\frac{H_e^+}{\Im e}\right|}{|\mathbf{T}e/\Im e|} \cdot \frac{1}{c_M(J_e)}. \tag{3}$$

We now compare the formulas above to the conjectured value given for the left hand sides by the second part of the Birch and Swinnerton-Dyer conjecture. Let $Ш_{J_e}$ denote the Shafarevich-Tate group of J_e. If p is a prime that divides N, then let $c_p(J_e)$ denote the Tamagawa number at p. Finally, let $\hat{J}_e$ denote the dual abelian variety of J_e. For any N, by the proof of Theorem 3.9 of [Par99], $L(J_e, 1) \neq 0$ (this follows for N prime and a product of two distinct primes by the formulas above), and hence by [KL89], $J_e(\mathbf{Q})$ and the Shafarevich-Tate group of J_e are both finite. Thus the second part of the Birch-Swinnerton-Dyer conjecture (as generalized by Tate and Gross) gives the formula (see [Lan91, III, §5]):

$$\frac{L(J_e,1)}{\Omega(J_e)} = \frac{|\text{Ш}_{J_e}| \cdot \prod_{p|N} c_p(J_e)}{|J_e(\mathbf{Q})_{\rm tor}| \cdot |\hat{J}_e(\mathbf{Q})_{\rm tor}|} \tag{4}$$

Supose now that N is prime, and denote it by p. It follows from [Maz77, II.9.7] and Props. 3.4.1 and 3.4.2 of [Aga00] that $|J_e(\mathbf{Q})_{\rm tor}|=|\hat{J}_e(\mathbf{Q})_{\rm tor}|= |\mathbf{T}e/\Im e|$. Also, by [Eme03, Thm. 4.13], $c_p(J_e) =| J_e(\mathbf{Q})_{\rm tor}|$. In view of this, comparing formulas (3) and (4), the second part of the Birch and Swinnerton-Dyer conjecture predicts that

$$|\text{Ш}_{J_e}|= |\frac{H^+}{\hat{H_e}^+ + H_e^+}| \cdot |\frac{H_e^+}{\Im e}| \cdot \frac{1}{c_M(J_e)}.$$

In analogy with formula (14) in [Aga10], we suspect that

$$|\text{Ш}_{J_e}|= |\frac{H^+}{\hat{H_e}^+ + H_e^+}| \cdot |\frac{H_e^+}{\Im e}|,$$

and thus we are led to the following:

Conjecture 1.2. *If N is prime, then $c_M(J_e) = 1$.*

Note that it is conjectured in [ARS06] that the Manin constant is one for quotients of $J_0(N)$ associated to newforms, which lends some credence to the conjecture above, considering that all cuspforms on $\Gamma_0(N)$ are new when N is prime. At the same time, Adam Joyce found an example of a quotient of the new part of $J_0(N)$ whose Manin constant is not one.

Now let us consider the case where N is a product of two distinct primes p and q, where we now compare formulas (2) and (4). First off, the first term $\frac{|\frac{(H/\hat{H}_e)^+}{H^+/\hat{H_e}^+}|}{c_\infty(J_e)}$ on the right side of (2) is a power of 2. Also, by Lemma 3.3, if N were prime, then this term is actually one, and so one would suspect that it is one in our case also. Anyhow, For both reasons, we shall not focus much on it, and instead prefer to work away from the prime 2.

The primes dividing the level (in this case, p and q) rarely divide any of the terms on the right side of formula (4), while the terms of the sort $(1+q-a_q(f))$ on the right side of formula (2) do have something to do with torsion subgroups (e.g., if f has integer Fourier coefficients, then this term is the number of points on the associated elliptic curve modulo p, to which the torsion subgroup of the elliptic curve maps). These considerations and the analogy with the case where N is prime (as discussed above) leads us to suspect that (at least up to a power of 2)

$$|\text{Ш}_{J_e}| = |\frac{H^+}{\hat{H_e}^+ + H_e^+}| \cdot |\frac{H_e^+}{\Im e}|, \text{ and}$$

$$\frac{|J_e(\mathbf{Q})_{\mathrm{tor}}| \cdot |\hat{J}_e(\mathbf{Q})_{\mathrm{tor}}|}{c_p(J_e) \cdot c_q(J_e)} = \prod_{f \in S_p} (1 + q - a_q(f)) \cdot \prod_{g \in S_q} (1 + p - a_p(g)) \cdot |\mathbf{T}e/\Im e|,$$

and to make the following conjecture:

Conjecture 1.3. *When N is a product of two distinct primes p and q, then $c_M(J_e) = q^{|S_p|} \cdot p^{|S_q|}$.*

We remark that the terms $|\frac{H^+}{\hat{H_e}^+ + H_e^+}|$ and $|\frac{H_e^+}{\Im e}|$ likely have interpretations similar to the analogous terms in [Aga10].

The plan for the rest of this article is as follows: in Section 2, we give the proof of Theorem 1.1. In Section 3, which serves as an appendix, we prove a result mentioned in the introduction and also state and prove some results about the number of components in the real locus of an abelian variety that may be of independent interest.

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2 Proof of Theorem 1.1

The proof is similar to the proof of the analogous result when N is prime, treated in [Aga99, Prop 2.1], with some added complications since N is not prime in our case.

Let T_M denote the set of Galois orbits of newforms f of level M with $L(f, 1) \neq 0$. If f is a newform of level M, then let J_f denote the quotient of $J_0(M)$ associated to f by Shimura. By the proof of Theorem 3.9 of [Par99], it follows that

$$J_e \sim \prod_{f \in T_p} \cdot J_f^2 \cdot \prod_{g \in T_q} J_g^2 \cdot \prod_{h \in T_{pq}} J_h. \tag{5}$$

We have the pairing $(H^+ \otimes \mathbf{C}) \times S_2(\Gamma_0(N), \mathbf{C}) \to \mathbf{C}$ given by $(\gamma, f) \mapsto \langle \gamma, f \rangle = \int_\gamma \omega_f$. In the following, at various points, we will be considering pairings between two $\mathbf{Z}$-modules; each such pairing is obtained in a natural way from the pairing in the previous sentence. Then by equation (5), we have

$$\begin{aligned} L(J_e,1) &= \prod_{f\in T_p} L(J_f,1)^2 \cdot \prod_{g\in T_q} L(J_g,1)^2 \cdot \prod_{h\in T_{pq}} L(J_h,1) \\ &= \prod_{f\in S_p} \langle e,f\rangle^2 \cdot \prod_{g\in S_q} \langle e,g\rangle^2 \cdot \prod_{h\in S_{pq}} \langle e,h\rangle. \end{aligned} \tag{6}$$

Note that in the formula above, for example, we should really be taking $\langle e,f\rangle$ "at level p", but it is the same as taking it "at level pq" by the functoriality of the de Rham pairing.

Using the fact that $J_e = J_0(N)/I_eJ_0(N)$, one sees that $H_1(J_e,\mathbf{Z})$ is isomorphic to the quotient of H by the saturation of I_eH in H, i.e., by $\hat{H}_e$. Thus, by formula (1),

$$\Omega(J_e) = c_{\scriptscriptstyle M} \cdot c_\infty(J_e) \cdot \mathrm{disc}((H/\hat{H}_e)^+ \times S_e \to \mathbf{C}). \tag{7}$$

We are going to replace $(H/\hat{H}_e)^+$ by another lattice. Consider the homomorphism

$$H^+ \to (H/\hat{H}_e)^+/H_e^+. \tag{8}$$

Its kernel is $(\hat{H}_e + H_e)^+$ and the map is not necessarily surjective. Consider the following map to the cokernel:

$$(H/\hat{H}_e)^+ \to \frac{(H/\hat{H}_e)^+/H_e^+}{H^+/(\hat{H}_e+H_e)^+}.$$

It is surjective with kernel $\frac{H^+}{(\hat{H}_e)^+}$. Hence the cokernel of the map in (8) is $\frac{(H/\hat{H}_e)^+}{H^+/(\hat{H}_e)^+}$. Thus we have

$$\frac{H^+}{(\hat{H}_e+H_e)^+} \cong \frac{\frac{(H/\hat{H}_e)^+}{H_e^+}}{\frac{(H/\hat{H}_e)^+}{H^+/(\hat{H}_e)^+}}.$$

Using the equation above, we perform some change of lattices:

$$
\begin{aligned}
\frac{1}{\mathrm{disc}((H/\hat{H}_e)^+ \times S_e \to \mathbf{C})} &= \frac{1}{\mathrm{disc}(H_e^+ \times S_e \to \mathbf{C})} \cdot \left|\frac{(H/\hat{H}_e)^+}{H^+/\hat{H}_e^{\ +}}\right| \cdot \left|\frac{H^+}{\hat{H}_e^{\ +} + H_e^+}\right| \\
&= \frac{1}{\mathrm{disc}(\Im e \times S_e \to \mathbf{C})} \cdot \left|\frac{(H/\hat{H}_e)^+}{H^+/\hat{H}_e^{\ +}}\right| \cdot \left|\frac{H^+}{\hat{H}_e^{\ +} + H_e^+}\right| \cdot \left|\frac{H_e^+}{\Im e}\right| \\
&= \frac{1}{\mathrm{disc}(\mathbf{T}e \times S_e \to \mathbf{C})} \cdot \left|\frac{(H/\hat{H}_e)^+}{H^+/\hat{H}_e^{\ +}}\right| \cdot \\
&\quad \cdot \left|\frac{H^+}{\hat{H}_e^{\ +} + H_e^+}\right| \cdot \left|\frac{H_e^+}{\Im e}\right| \Big/ \left|\frac{\mathbf{T}e}{\Im e}\right| . \qquad (9)
\end{aligned}
$$

Thus from formulas (6), (7), and (9), we have

$$
\begin{aligned}
\frac{L(J_e,1)}{\Omega(J_e)} &= \frac{\prod_{f\in S_p}\langle e,f\rangle^2 \cdot \prod_{g\in S_q}\langle e,g\rangle^2 \cdot \prod_{h\in S_{pq}}\langle e,h\rangle}{\mathrm{disc}(\mathbf{T}e \times S_e \to \mathbf{C})} \cdot \\
&\quad \cdot \frac{\left|\frac{(H/\hat{H}_e)^+}{H^+/\hat{H}_e^{\ +}}\right|}{c_\infty(J_e)} \cdot \frac{\left|\frac{H^+}{\hat{H}_e^{\ +}+H_e^+}\right|\left|\frac{H_e^+}{\Im e}\right|}{\left|\frac{\mathbf{T}e}{\Im e}\right|} \cdot \frac{1}{c_M(J_e)}. \qquad (10)
\end{aligned}
$$

We now focus on the term

$$
\frac{\prod_{f\in S_p}\langle e,f\rangle^2 \cdot \prod_{g\in S_q}\langle e,g\rangle^2 \cdot \prod_{h\in S_{pq}}\langle e,h\rangle}{\mathrm{disc}(\mathbf{T}e \times S_e \to \mathbf{C})} \qquad (11)
$$

on the right side of equation (10). There is a perfect pairing $\mathbf{T}/I_e \times S_e \to \mathbf{Z}$ which associates to (t,f) the first Fourier coefficient $a_1(tf)$ of the modular form tf. This defines $t_e \in \mathbf{T}/I_e \otimes \mathbf{C}$ characterized by $\langle e,f\rangle = a_1(t_e f)$ $(f \in S_e)$. The discriminant of the pairing $\mathbf{T}e \times S_e \to \mathbf{C}$ which associates to (te,f) the complex number $\langle te,f\rangle$ coincides, via the identification above and the canonical isomorphism $\mathbf{T}e \cong \mathbf{T}/I_e$, with the discriminant of the pairing $\mathbf{T}/I_e \times \mathrm{Hom}(\mathbf{T}/I_e,\mathbf{Z}) \to \mathbf{C}$ which associates to (t,ψ) the complex number $\psi(t_e t)$ (obtained by extending ψ by $\mathbf{C}$-linearity). The latter discriminant is equal to $\det_{\mathbf{T}/I_e\otimes\mathbf{C}} t_e = \det_{S_e\otimes\mathbf{C}} t_e$. So we need to compute the action of t_e on $S_e\otimes\mathbf{C}$. Note that a basis for $S_e\otimes\mathbf{C}$ is given by $\{h : h \in S_{pq}\} \cup \{f(z), f(qz) : f \in S_p\} \cup \{g(z), g(pz) : g \in S_q\}$.

If f' is an eigenform for all the Hecke operators (including U_p and U_q), then it is easy to see that $t_e f' = \langle e,f'\rangle f'$ (compare the first Fourier coefficients of either side). Now $h \in S_{pq}$ is an eigenform for all the Hecke operators. So the contribution from h to $\det_{S_e \otimes \mathbf{C}} t_e$ is $\langle e, h\rangle$, cancelling with the corresponding contribution in the numerator of (11).

If $g \in S_q$, then $g(z)$ and $g(pz)$ are eigenvectors for all the Hecke operators T_l for $l \neq p, q$ and for U_q, but not for U_p. However U_p preserves the subspace spanned by $g(z)$ and $g(pz)$. Let $B_p g = g(pz)$. Then on q-expansions (see [Par99, 3.1]),

$$B_p(\sum_{n\geq 1} a_n q^n) = \sum_{n\geq 1} a_n q^{np},$$
$$U_p(\sum_{n\geq 1} a_n q^n) = \sum_{n\geq 1} a_{np} q^n, \text{ and}$$
$$T_l(\sum_{n\geq 1} a_n q^n) = \sum_{n\geq 1} a_{np} q^n + \sum_{n\geq 1} a_n q^{np}.$$

Since g is an eigenform for T_p with eigenvalue $a_p = a_p(g)$, from the above formulas, it is easy to see that $U_p(g) = a_p g - pB_p g$ and $U_p(B_p g) = g$. Thus the matrix of U_p on the subspace spanned by $g(z)$ and $g(pz)$ is given by

$$\begin{pmatrix} a_p & 1 \\ -p & 0 \end{pmatrix}.$$

Hence the action of U_p on this subspace is diagonalizable. The characteristic polynomial of U_p on this subspace is $U_p^2 - a_p U_p + p$. If α_1 and α_2 are the eigenvalues, then an easy check shows that $g - \alpha_2 B_p g$ and $g - \alpha_1 B_p g$ are eigenvectors (with eigenvalues α_1 and α_2 respectively). Thus we can use this eigenbasis to compute $\det t_e$ on this subspace. Using the fact that $\langle e, B_p g\rangle = \langle e, g\rangle/p$ (see the proof of Lemma 3.10 in [Par99]), we find that this determinant is

$$\begin{aligned} & \langle e, g - \alpha_1 B_p g\rangle \cdot \langle e, g - \alpha_2 B_p g\rangle \\ = \quad & \langle e, g\rangle^2 (1 - \alpha_1/p)(1 - \alpha_2/p) \\ = \quad & \langle e, g\rangle^2 (1 - (\alpha_1 + \alpha_2)/p + \alpha_1\alpha_2/p^2) \\ = \quad & \langle e, g\rangle^2 (1 - a_p/p + p/p^2) \\ = \quad & \langle e, g\rangle^2 (1 + p - a_p)/p. \end{aligned}$$

But the contribution to the numerator of (11) corresponding to the subspace spanned by $g(z)$ and $g(pz)$ is $\langle e, g\rangle^2$. Thus the contribution to (11) coming from the the subspace spanned by $g(z)$ and $g(pz)$ is a factor of $p/(1 + p - a_p(g))$.

Similarly, if $f \in S_p$, then the contribution to (11) coming from the the subspace spanned by $f(z)$ and $f(qz)$ is a factor of $q/(1+q-a_q(f))$. Thus we find that

$$\frac{\prod_{f\in S_p}\langle e,f\rangle^2\cdot\prod_{g\in S_q}\langle e,g\rangle^2\cdot\prod_{h\in S_{pq}}\langle e,h\rangle}{\operatorname{disc}(\mathbf{T}e\times S_e\to\mathbf{C})}$$
$$=\prod_{f\in S_p}\frac{q}{1+q-a_q(f)}\cdot\prod_{g\in S_q}\frac{p}{1+p-a_p(g)}.$$

The theorem now follows from the equation above and equation (10).

3 Appendix: number of components in real locus

The main goal of the appendix from the point of view of this article is to prove Lemma 3.3 below. In the process, we state some results about the number of components in the real locus of an abelian variety.

The following lemma is probably well known, but we could not find a suitable reference. The proof was provided to us by H. Lenstra.

Lemma 3.1. *Let A be an abelian variety over $\mathbf{Q}$, and let c denote the action of complex conjugation on $A(\mathbf{C})$ as well as the induced action on $H_1(A,\mathbf{Z})$. Then the number of components in $A(\mathbf{R})$ is the order of the 2-group $H_1(A,\mathbf{Z})^+/(1+c)H_1(A,\mathbf{Z})$.*

Proof. Write L for the lattice $H_1(A,\mathbf{Z})$ and V for its tensor product with $\mathbf{R}$; so $V = H_1(A,\mathbf{R})$. Now consider the exact sequence

$$0\to L\to V\to A(\mathbf{C})\to 0$$

of $\langle c\rangle$-modules, and take its Tate cohomology sequence. The group V is uniquely divisible and so has trivial cohomology (note first that the cohomology groups are 2-groups, by [AW67, §6, Cor. 1], and multiplication by any integer is an isomorphism on them). So the long exact sequence gives us:

$$0\to\widehat{H}^0(\langle c\rangle, A(\mathbf{C}))\to H^1(\langle c\rangle, L)\to 0.$$

But $\widehat{H}^0(\langle c\rangle, A(\mathbf{C}))$ is (by definition) equal to $A(\mathbf{C})^+/(1+c)A(\mathbf{C})$, so we get an isomorphism $A(\mathbf{R})/(1+c)A(\mathbf{C}) \xrightarrow{\simeq} H^1(\langle c\rangle, L)$. Now $A(\mathbf{C})$ is compact and connected, so its continuous image $(1+c)A(\mathbf{C})$ is compact (hence closed) and connected as well. Hence $A(\mathbf{R})/(1+c)A(\mathbf{C})$ is a Hausdorff group (for example, using [Bou66, §III.2.6, Prop. 18]) with the same group of components as $A(\mathbf{R})$ itself. But it is also finite, since $H^1(\langle c\rangle, L)$ is finite (using [AW67, §6, Cor 2] and the fact that L is finitely generated). So $A(\mathbf{R})/(1+c)A(\mathbf{C})$, being Hausdorff and finite, is discrete and equal to its own component group. Thus, the group of components of $A(\mathbf{R})$ is canonically isomorphic to $H^1(\langle c\rangle, L)$.

To deduce that the number of components equals the order of $L^+/(1+c)L$, observe that the latter group is $\widehat{H}^0(\langle c\rangle, L)$, so that all that remains to be proved is that L has Herbrand quotient equal to 1. Now by the semilinearity of c on V (with respect to the complex structure on V), it follows that c has equally many eigenvalues $+1$ as -1 on V. Now use [AW67, §8, Prop. 12], with $L' = \mathbf{Z}\langle c\rangle$, and the fact that $L\otimes\mathbf{Q}\cong\mathbf{Z}\langle c\rangle\otimes\mathbf{Q}$ as $\langle c\rangle$-representation spaces (which can be checked by looking at traces), to conclude that the Herbrand quotient of L is the same as that of $\mathbf{Z}\langle c\rangle$. But the latter lattice has Herbrand quotient 1. Hence so does L, and that finishes the proof. □

The following corollary was pointed out to us by L. Merel:

Corollary 3.2. *If p is a prime, then $J_0(p)(\mathbf{R})$ is connected.*

Proof. This follows from the Proposition above, in view of the fact that $H_1(J_0(p),\mathbf{Z})^+ = (1+c)H_1(J_0(p),\mathbf{Z})$ by [Mer96b, Prop. 5]. □

The following result was stated without proof in [Aga99]:

Lemma 3.3. *We use the notation introduced in Section 1. Suppose N is prime. Then* $\left|\frac{(H/\hat{H}_e)^+}{H^+/(\hat{H}_e)^+}\right| = c_\infty(J_e)$.

Proof. Since the level is prime, by [Mer96b, Prop. 5], $H^+ = (1+c)H$. Now, $H_1(J_e,\mathbf{Z})\cong H/\hat{H}_e$, and so $(1+c)H_1(J_e,\mathbf{Z})\cong(1+c)H/(1+c)\hat{H}_e\cong H^+/\hat{H}_e{}^+$. Thus

$$\left|\frac{(H/\hat{H}_e)^+}{H^+/(\hat{H}_e)^+}\right| = \left|\frac{H_1(J_e,\mathbf{Z})^+}{(1+c)H_1(J_e,\mathbf{Z})}\right| = c_\infty(J_e),$$

by the proposition above. □

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