On the cohomology groups of certain quotients of products of upper half planes and upper half spaces

Amod Agashe* and Lydia Eldredge

Abstract

A theorem of Matsushima-Shimura shows that the the space of harmonic differential forms on the quotient of products of upper half planes under the action of certain groups, when the quotient is compact, is the direct sum of two subspaces called the universal and cuspidal subspaces. We generalize this result to compact quotients of products of upper half planes and upper half spaces (hyperbolic three spaces) under the action of certain groups to obtain a similar decomposition.

1 Introduction and main result

Let n be a positive integer. For $i = 1, \ldots, n$, let $\mathcal{H}_i$ be either the upper half plane $\mathfrak{H}_2 = \{z \in \mathbf{C} : \Im(z) > 0\}$ or upper half space $\mathfrak{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$ (a model for hyperbolic three space), and let $C_i = \mathbf{R}$ if $\mathcal{H}_i$ is the upper half plane and $C_i = \mathbf{C}$ if $\mathcal{H}_i$ is upper half space. The group $SL_2(\mathbf{R})$ acts on $\mathfrak{H}_2$ as usual by fractional linear transformations and the action of $SL_2(\mathbf{C})$ on $\mathfrak{H}_3$ is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} (z,t) = \frac{1}{\delta}((az+b)\overline{(cz+d)} + a\bar{c}t^2, |ad-bc|t), \text{ where } \delta = |cz+d|^2 + |ct|^2 .$$

Let Γ be a subgroup of $\prod_{i=1}^n SL_2(C_i)$, which thus acts on $\prod_{i=1}^n \mathcal{H}_i$.

For $i = 1, \ldots, n$, if $\mathcal{H}_i$ is the upper half plane, then let x_i and y_i denote the x and y coordinates on $\mathcal{H}_i$ (i.e., for which the corresponding point is $x_i + \sqrt{-1}y_i$), and if $\mathcal{H}_i$ is the upper half space, then let x_i, y_i and t_i denote the x, y, and t coordinates on $\mathcal{H}_i$ (i.e., for which the corresponding point is $(x_i + \sqrt{-1}y_i, t_i)$). The standard Riemannian metrics on the upper half plane and upper half space induce a metric (the product metric) on $\prod_{i=1}^n \mathcal{H}_i$. For $i = 1, \ldots, n$, let η_i denote the volume form (also called fundamental form) on $\mathcal{H}_i$, i.e., if $\mathcal{H}_i$ is the upper half plane, then $\eta_i = dx_i/y_i \wedge dy_i/y_i$, and if $\mathcal{H}_i$ is the upper half space, then $\eta_i = dx_i/t_i \wedge dy_i/t_i \wedge dt_i/t_i$. If $C \subseteq \{1, \ldots, n\}$, then let $\eta_C = \wedge_{i \in C} \eta_i$. Note that η_C is invariant under $\prod_{i=1}^n SL_2(C_i)$. For $i = 1, \ldots, n$, let $d_i = 2$ if $\mathcal{H}_i$ is the upper half plane, then let $d_i = 3$ if $\mathcal{H}_i$ is the upper half space. Define

$$\mathcal{H}^m_{\mathrm{univ}} = \begin{cases} \sum_{C \subseteq \{1,\ldots,n\} : \sum_{i \in C} d_i = m} \mathbf{R}\eta_C, & \text{if } m = 2i + 3j \text{ for some integers } i \text{ and } j \\ 0 & \text{otherwise} \end{cases}$$

Note that if $\mathcal{H}_i$ is the upper half plane for all i, then $\mathcal{H}^m_{\mathrm{univ}} \otimes \mathbf{C}$ is isomorphic to the usual universal cohomology group associated to Γ of degree m (e.g., see [Fre90, § III.1]). Let $\mathcal{H}^m(\Gamma)$ denote the group of harmonic differential forms of degree m on $\prod_{i=1}^n \mathcal{H}_i$ that are invariant with respect to Γ. Since the volume form on any $\mathcal{H}_i$ is harmonic, $\mathcal{H}^m_{\mathrm{univ}} \subseteq \mathcal{H}^m(\Gamma)$.

*The first author was partially supported by the Collaboration Grants for Mathematicians from the Simons Foundation (award number 245630).

Now assume that Γ is discrete in $\prod_{i=1}^n SL_2(C_i)$, has no element of finite order other than the identity, and is such that the quotient by its action on $\prod_{i=1}^n \mathcal{H}_i$ is compact. For $i = 1, \ldots, n$, if $\mathcal{H}_i$ is the upper half plane, then let $B_i = \{dx_i/y_i, dy_i/y_i\}$, and if $\mathcal{H}_i$ is the upper half space, then let $B_i = \{dx_i/t_i, dy_i/t_i, dt_i/t_i\}$. For any i, if $A_i \subseteq B_i$, then let $\omega_{A_i} = \wedge_{\omega \in A_i} \omega$. If $a_1, \ldots, a_n$ are integers, then we say that a differential form is of type $a_1, \ldots, a_n$ if it can be written as a linear combination (with coefficients being smooth functions) of some $\omega_{A_1} \wedge \cdots \wedge \omega_{A_n}$, such that for each i, $A_i \subseteq B_i$ and $|A_i| = a_i$. Let $\mathcal{H}^m_{\text{cusp}}$ denote the subgroup of $\mathcal{H}^m(\Gamma)$ generated by elements that are of type $a_1, \ldots, a_n$ such that none of the a_i are equal to 0 or d_i (note that $\sum_i a_i = m$ necessarily).

Assume now that the image of Γ under the projection to any partial product of $\prod_{i=1}^n SL_2(C_i)$ is dense; this is called an "irreducibility condition" in [Fre90, I.2.13] (cf. p. 420–421 of [MS63]), and is often true for arithmetic subgroups if $n > 1$.

Theorem 1.1. *We have $\mathcal{H}^m(\Gamma) = \mathcal{H}^m_{\text{univ}} + \mathcal{H}^m_{\text{cusp}}$ and $\mathcal{H}^m_{\text{univ}} \cap \mathcal{H}^m_{\text{cusp}} = \{0\}$. Hence $H^m(\Gamma, \mathbf{R}) \cong H^m\big(\Gamma\backslash(\prod_{i=1}^n \mathcal{H}_i), \mathbf{R}\big) \cong H^m_{\text{dR}}\big(\Gamma\backslash(\prod_{i=1}^n \mathcal{H}_i), \mathbf{R}\big) \cong \mathcal{H}^m(\Gamma) \cong \mathcal{H}^m_{\text{univ}} \oplus \mathcal{H}^m_{\text{cusp}}$, where the first cohomology is group cohomology, the second one is Betti cohomology, and the third one is deRham cohomology.*

Note that the first three isomorphisms in the second statement are standard, and for this reason, this statement follows from the first. The proof of the first statement is given in the next section. If $\mathcal{H}_i$ is the upper half plane for all i, then the statement of the theorem, when tensored with $\mathbf{C}$, follows from [MS63] (cf. Theorem 1.6 in Chapter III of [Fre90]). The proof given in loc. cit. (see also [Fre90, § III.1] for an exposition) uses crucially the fact that there is a basis of complex differential forms on the upper half plane for which the action of $SL_2(\mathbf{R})$ is diagonal (e.g., one such basis is $\{dz, d\overline{z}\}$, where z denotes the coordinate on the upper half plane); this does not seem to be true for upper half space (as far as we know), and so the proof of [MS63] does not generalize in an obvious way to our situation. While our proof does borrow ideas from loc. cit., even in the case where $\mathcal{H}_i$ is the upper half plane for all i, it differs from the one in loc. cit. and is more elemetary, in the sense that the proof in loc. cit. uses the maximum principle for harmonic functions, while we just use the fact that harmonic forms on compact manifolds are closed. Moreover, we get a result with coefficients in $\mathbf{R}$, as opposed to in $\mathbf{C}$, so our result is more general as well.

The motivation for the authors to generalize the result of Matsushima-Shimura (to include upper half spaces) is for a potential application to the construction of Stark-Heegner points (also called Darmon points) on elliptic curves over arbitrary number fields (the case of totally real fields is considered in [AT]); for an example of the application of the original theorem of Matsushima-Shimura in this context, see the proof of Theorem 8.9 on p. 92 of [Dar04].

Acknowledgement: We are very grateful to E. Aldrovandi for several discussions and advice on work that led to this article.

2 Proof of Theorem 1.1

We continue to use the notation introduced in the previous section. If ω is a differential form, then it can be written uniquely as $\omega = \sum_{\alpha \in S} \omega_\alpha$ where S is some finite indexing set and the ω_α's are non-trivial forms of distinct types. The Laplacian operator and any element of Γ take a form of a given type to a form of the same type. Thus if ω as above is an invariant harmonic form, then each ω_α is also an invariant harmonic form (here and henceforth, when we say that a form is invariant, we shall mean with respect to the full group Γ). Thus it suffices to show that a nonzero harmonic invariant differential form of a given type is either in $\mathcal{H}^m_{\text{univ}}$ or in $\mathcal{H}^m_{\text{cusp}}$, but not in both.

Proposition 2.1. *Let ω be a nonzero invariant harmonic form of type $a_1, \ldots, a_n$ for some $a_1, \ldots, a_n$. Suppose there is an i such that $a_i = 0$ or d_i. Then for all j, $a_j = 0$ or d_j, and ω is a real number times $\wedge_{j=1}^n b_j$, where $b_j = 1$ if $a_j = 0$, and $b_j = \eta_j$ (the volume form on $\mathcal{H}_j$) if $a_j = d_j$.*

We shall prove this proposition next, but note that it immediately implies the theorem, since if a harmonic invariant form is of a type where $a_i = 0$ or d_i for some i, then by the proposition above, it is in $\mathcal{H}^m_{\text{univ}}$, and if not, then it is in $\mathcal{H}^m_{\text{cusp}}$, which proves the first statement of the theorem (as mentioned earlier, the second statement follows from the first).

The rest of this section is devoted to the proof of Proposition 2.1. Without loss of generality, assume that $i = 1$, where i is as in the statement of the proposition. For some finite set I, one can write ω uniquely as $\omega = \sum_{k \in I} f_k \omega_k$ such that $\omega_k = \omega_{A_1} \wedge \cdots \wedge \omega_{A_n}$ for some $A_i \subseteq B_i$ with $|A_i| = a_i$ for all $i = 1, \ldots, n$ and with the requirement that the ω_k's are distinct and f_k's are nonzero smooth functions. Let $\mathbf{x}_i$ denote (x_i, y_i) if $\mathcal{H}_i$ is the upper half plane and $\mathbf{x}_i = (x_i, y_i, t_i)$ if $\mathcal{H}_i$ is the upper half space.

Lemma 2.2. *For each $k \in I$, f_k is independent of $\mathbf{x}_1$ (i.e., independent of the variables in $\mathbf{x}_1$).*

Proof. Let X denote the quotient of $\prod_{i=1}^n \mathcal{H}_i$ under the action of Γ; under our assumptions, it is a compact Riemannian manifold without boundary. If ω' is a form on $\prod_{i=1}^n \mathcal{H}_i$ that is invariant under Γ, then it induces a form on X that we shall denote by $\overline{\omega'}$. We define an inner product on p-forms on $\prod_{i=1}^n \mathcal{H}_i$ that are invariant under Γ by $(\omega_1, \omega_2) = \int_X \overline{\omega_1} \wedge *\overline{\omega_2}$, where the latter integral is well defined since X is compact. With this inner product, the usual proof that a harmonic form on a compact Riemannian manifold is closed generalizes to show that a harmonic form on $\prod_{i=1}^n \mathcal{H}_i$ that is invariant under Γ is closed (as was pointed out to us by E. Klassen, this also follows because it is a local result, which holds on the quotient since it is compact). Since ω is harmonic, we have $d\omega = 0$. We remark that this is the only place in this article where we use the hypothesis that the quotient X is compact.

Consider first the case where $a_1 = 0$. Then note that for each $k \in I$, $\omega_k = \omega_{A_2} \wedge \cdots \wedge \omega_{A_n}$ for some $A_2 \subseteq B_2, \ldots, A_n \subseteq B_n$ (the ω_{A_1} is missing since $a_1 = 0$). For $i = 1, \ldots, n$, if $\mathcal{H}_i$ is the upper half plane, then let $B_i' = \{dx_i, dy_i\}$, and if $\mathcal{H}_i$ is the upper half space, then let $B_i' = \{dx_i, dy_i, dt_i\}$ (thus B_i' is a different basis for the span of B_i). Now, with I as above, one can write ω uniquely as $\omega = \sum_{k \in I} g_k \omega_k'$ such that $\omega_k' = \omega_{A_2'} \wedge \cdots \wedge \omega_{A_n'}$ for some $A_i' \subseteq B_i'$ with $|A_i'| = a_i$ for all i and with the requirement that the ω_k''s are distinct, g_k's are nonzero smooth functions, and $g_k \omega_k' = f_k \omega_k$ (we are just doing a change of basis here).

We have $0 = d\omega = \sum_{k \in I} \sum_{b \in B_1'} \frac{\partial g_k}{\partial b} db \wedge \omega_k + \sum_{k \in I} \sum_{i=2}^n \sum_{b \in B_i'} \frac{\partial g_k}{\partial b} db \wedge \omega_k$. Now each term in the first sum contains db for some $b \in B_1'$, while the terms in the second sum do not contain db for any $b \in B_1'$ (considering the way in which ω_k could be written in the paragraph above), so the first sum must be zero. Secondly, in the first sum, for each $b \in B_1'$, the $db \wedge \omega_k$'s are distinct for the various k's, so we conclude that for all k, $\partial g_k / \partial b = 0$ for all $b \in B_1'$, i.e., g_k is independent of $\mathbf{x}_1$. Now for each k, f_k differs from g_k by $\prod_{i=2}^n c_i$, where c_i is a power of y_i if $\mathcal{H}_i$ is the upper half planes and c_i is a power of t_i if $\mathcal{H}_i$ is the upper half space; this product $\prod_{i=2}^n c_i$ is non-zero and is independent of $\mathbf{x}_1$. Thus for each k, f_k is also independent of $\mathbf{x}_1$.

Now consider the case where $a_1 = d_1$. Then $*\omega$ is also invariant and harmonic, and for it, the corresponding a_1 is 0, while the set of f_k's do not change up to signs. Thus the argument above applied to $*\omega$ shows that the f_k's are independent of $\mathbf{x}_1$ in this case also. □

Now suppose there is a j such that a_j is neither 0 nor d_i. We will show that this leads to a contradiction. Without loss of generality, suppose $j = 2$. Let $M \in SL_2(C_2)$ (recall that $C_2 = \mathbf{R}$ if $\mathcal{H}_2$ is the upper half plane and $C_2 = \mathbf{C}$ if $\mathcal{H}_2$ is the upper half space). Then by the "irreducibility condition", for some matrix $A \in SL_2(C_1)$ that depends on M, there are elements of Γ that are arbitrarily close to $(A, M, I, \ldots, I)$, where I is the identity matrix, as usual. Now ω is invariant with respect to the first component (i.e., with respect to all possible A's as above; here, if $a_1 = d_1$, we use the fact that the volume form is invariant under $SL_2(C_1)$). Since ω is smooth, by pulling back ω to $\mathcal{H}_2$, we get a harmonic differential form ω' on $\mathfrak{H}_2$ or on $\mathfrak{H}_3$ that is invariant under M and is not of degree zero or the top degree (which is 2 for $\mathfrak{H}_2$ and 3 for $\mathfrak{H}_3$). It is perhaps well known that this cannot happen for all such M (and in fact, this may follow from a more general fact), but we prove this anyway for the sake of completeness:

Lemma 2.3. *There is no nonzero harmonic differential form on $\mathfrak{H}_2$ or on $\mathfrak{H}_3$ that is invariant under $SL_2(\mathbf{R})$ or $SL_2(\mathbf{C})$ respectively and is of degree other than zero or the top degree (which is 2 for $\mathfrak{H}_2$ and 3 for $\mathfrak{H}_3$).*

Proof. Assume that a form ω' of the type above actually exists. First consider the case of $\mathfrak{H}_2$. Then ω' is of degree one and so $\omega' = f\frac{dz}{y} + g(-\frac{d\bar{z}}{y})$ for some $f(z)$ and $g(z)$. A direct calculation shows that if $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are points in $\mathfrak{H}_2$, then with $e = x_2 - x_1\frac{y_2}{y_1}$ and $d = \sqrt{\frac{y_1}{y_2}}$, the matrix

$$M = \begin{bmatrix} \frac{1}{d} & ed \\ 0 & d \end{bmatrix} \in SL_2(\mathbb{R}) \tag{1}$$

satisfies $M(P_1) = P_2$. The action of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ on complex differentials in the basis $(\frac{dz}{y}, -\frac{d\bar{z}}{y})$ is given by the matrix

$$\begin{bmatrix} \epsilon^{-2} & 0 \\ 0 & \epsilon^2 \end{bmatrix}, \text{ where } \epsilon = (cz + d)/|cz + d| \ . \tag{2}$$

Using this, one sees that the matrix in (1) leaves $\frac{dz}{y}$ and $-\frac{d\bar{z}}{y}$ invariant (keeping in mind that $d > 0$). Since ω' is invariant under $SL_2(\mathbb{R})$, we conclude that $f(z)$ and $g(z)$ are constants.

Now consider the following matrix

$$\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL_2(\mathbb{R})$$

Since $f(z)$ and $g(z)$ are constants, we shall write them as just f and g respectively. Then

$$f\Big(\frac{dz}{y}\Big) + g\Big(-\frac{d\bar{z}}{y}\Big) = \gamma^*\Big(f\frac{dz}{y} + g\Big(-\frac{d\bar{z}}{y}\Big)\Big) = f\gamma^*\Big(\frac{dz}{y}\Big) + g\gamma^*\Big(-\frac{d\bar{z}}{y}\Big) = f\frac{|z|^2}{z^2}\frac{dz}{y} + g\frac{z^2}{|z|^2}\Big(-\frac{d\bar{z}}{y}\Big) \ ,$$

where we used formula (2) above in the last step. Putting $z = i$ (for example) in the equation above, we see that $f = g = 0$ and so $\omega' = 0$, which is a contradiction.

Now consider the case of $\mathfrak{H}_3$. First consider the case where the degree of ω' is one. Let $\beta_0 = -dz/t$, $\beta_1 = dt/t$, and $\beta_2 = d\bar{z}/t$. Then $\omega' = f_0\beta_0 + f_1\beta_1 + f_2\beta_2$ for some functions f_0, f_1, and f_2. A direct calculation shows that if $P_1 = (x_1, y_1, t_1)$ and $P_2 = (x_2, y_2, t_2)$ are points in $\mathfrak{H}_3$, then with $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $e = z_2 - z_1\frac{t_2}{t_1}$, and $d = \sqrt{\frac{t_1}{t_2}}$, the matrix

$$M = \begin{bmatrix} \frac{1}{d} & ed \\ 0 & d \end{bmatrix} \in SL_2(\mathbf{C}) \tag{3}$$

satisfies $M(P_1) = P_2$. The action of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbf{C})$ on complex differentials in the basis $(\beta_0, \beta_1, \beta_2)$ is given by the matrix

$$\frac{1}{|r|^2+|s|^2}\begin{bmatrix} r^2\Delta & -2rs\Delta & s^2\Delta \\ \frac{r\overline{s}}{s^2\Delta} & \frac{|r|^2-|s|^2}{2rs\Delta} & \frac{-\overline{r}s}{r^2\Delta} \end{bmatrix}, \text{ where } \Delta = ad-bc = 1, r = \overline{cz+d}, \text{ and } s = \overline{c}t \ . \tag{4}$$

Using this, one sees that the matrix in (3) leaves each of β_0, β_1, and β_2 invariant (considering that d is real). Since ω' is invariant under $SL_2(\mathbf{C})$, we conclude that f_0, f_1, and f_2 are constants.

It is easy to check that $*\beta_0 = i\beta_1 \wedge \beta_2$, $*\beta_1 = \frac{i}{2}\beta_2 \wedge \beta_0$, and $*\beta_2 = i\beta_0 \wedge \beta_1$, and that $d\beta_0 = \beta_0 \wedge \beta_1$, $d\beta_1 = 0$, and $d\beta_2 = \beta_2 \wedge \beta_1$ (see, e.g., the proof of Lemma 61 in §4.2.5 of [Byg99]). Using these formulas, and the fact that ω' is harmonic, we have

$$\begin{aligned}
0 &= \Delta\omega = (d\delta + \delta d)\omega = d\delta(f_0\beta_0 + f_1\beta_1 + f_2\beta_2) + \delta d(f_0\beta_0 + f_1\beta_1 + f_2\beta_2) \\
&= d((-1)^{3(1+1)+1} * d * (f_0\beta_0 + f_1\beta_1 + f_2\beta_2)) + \delta(f_0\beta_0 \wedge \beta_1 + 0 + f_2\beta_2 \wedge \beta_1) \\
&= -d * d(i\overline{f_0}\beta_1 \wedge \beta_2 + \frac{i}{2}\overline{f_1}\beta_2 \wedge \beta_0 + i\overline{f_2}\beta_0 \wedge \beta_1) + (-1)^{3(2+1)+1} * d * (f_0\beta_0 \wedge \beta_1 + f_2\beta_2 \wedge \beta_1) \\
&= -d * (0 + \frac{i}{2}\overline{f_1}(-2\beta_0 \wedge \beta_1 \wedge \beta_2) + 0) + *d(\overline{f_0}(i\beta_2) + \overline{f_2}(-i\beta_0)) \\
&= d * (i\overline{f_1}\beta_0 \wedge \beta_1 \wedge \beta_2) + *d(\overline{f_0}i\beta_2 - \overline{f_2}i\beta_0) = d(-if_1(2i)) + *(\overline{f_0}i\beta_2 \wedge \beta_1 - \overline{f_2}i\beta_0 \wedge \beta_1) \\
&= d(2f_1) - f_0 i(-i\beta_0) + f_2 i(i\beta_2) = 0 - f_0 i\beta_0 - f_2\beta_2
\end{aligned}$$

It follows that $f_0 = f_2 = 0$. Recall that f_1 is a constant. Let

$$\gamma = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SL_2(\mathbb{C}) \ .$$

Then

$$f_1\beta_1 = \omega' = \gamma^*(\omega') = \gamma^*(f_1\beta_1) = f_1\gamma^*(\beta_1) = f_1\frac{1}{|z|^2+t^2}(-2\overline{z}t\beta_0 + (|z|^2 - t^2)\beta_1 + 2zt\beta_2) \ ,$$

where we used formula (4) above in the last step. Putting $z = 0, t = 1$ in the equation above, we see that $f_1 = 0$ and so $\omega' = 0$, which is a contradiction.

If the degree of ω' is 2, then $*\omega'$ is a non-zero harmonic form of degree one that is invariant under $SL_2(\mathbf{C})$. Thus from the argument above, we again get a contradiction. □

Thus, for all j, $a_j = 0$ or d_j. Then one sees that I consists of only one element k (since the dimension of the space of forms of degree 0 or d_j is one for each j), and applying Lemma 2.2 above for any j instead of for $i = 1$, we see that f_k is indepedent of $\mathbf{x}_j$ for all j. Thus f_k is a real number. This finishes the proof of Proposition 2.1.

Remark 2.4. We take the opportunity to point out a correction to the literature (this remark and the rest of this article are independent of each other). Recall that $\mathfrak{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$ is the upper half space (a model for hyperbolic three space). Let $f_0, f_1, \text{and} f_2$ be smooth functions on $\mathfrak{H}_3$. It has been wrongly stated in the literature that $\omega = f_0(-dz/t) + f_1 dt/t + f_2 d\overline{z}/t$ is harmonic if and only if

$$\begin{aligned}\frac{\partial f_0}{\partial \overline{z}} + \frac{\partial f_2}{\partial z} &= 0,\\ \frac{\partial f_1}{\partial z} + \frac{\partial f_0}{\partial t} - t^{-1} f_0 &= 0,\\ \frac{\partial f_1}{\partial \overline{z}} - \frac{\partial f_2}{\partial t} + t^{-1} f_2 &= 0,\\ \frac{t}{2}\frac{\partial f_1}{\partial t} - f_1 - 2t\frac{\partial f_0}{\partial \overline{z}} &= 0 \end{aligned} \tag{5}$$

The system of equations above is equivalent to saying that ω is closed and coclosed (e.g., see the proof of Lemma 61 in §4.2.5 of [Byg99]). On a compact manifold, a differential form is harmonic if and only if it is closed and coclosed. However $\mathfrak{H}_3$ is not compact. Now

$$\begin{aligned} \delta\Big(\frac{dt}{t}\Big) &= (-1)^{3(1+1)+1} * d * \frac{dt}{t} = - * d\Big(\frac{dx}{t} \wedge \frac{dy}{t}\Big) = - * d\Big(\frac{1}{t^2} dx \wedge dy\Big)\\ &= - * \Big(- 2\frac{1}{t^3} dt \wedge dx \wedge dy\Big) = 2 * \Big(\frac{dt}{t} \wedge \frac{dx}{t} \wedge \frac{dy}{t}\Big) = 2 \cdot 1 = 2, \text{and}\\ \Delta\Big(\frac{dt}{t}\Big) &= (\delta d + d\delta)\Big(\frac{dt}{t}\Big) = \delta d\Big(\frac{dt}{t}\Big) + d\delta\Big(\frac{dt}{t}\Big) = \delta\Big(\frac{-1}{t^2} dt \wedge dt\Big) + d(2) = 0 + 0 = 0\ . \end{aligned}$$

Thus the differential dt/t is harmonic, but not coclosed. It does not satisfy equation (5) above (in our case, $f_1 = 1$ and $f_0 = 0$). It also follows that $*(\frac{dt}{t})$ is harmonic, but not closed.

References

[AT] Amod Agashe and Mak Trifkovic, *Darmon points for totally real fields*, submitted, available at arXiv:1310.4237.

[Byg99] J. Bygott, *Modular forms and modular symbols over imaginary quadratic fields*, Ph.D. thesis, Exeter (1999), available at http://hdl.handle.net/10871/8322.

[Dar04] Henri Darmon, *Rational points on modular elliptic curves*, CBMS Regional Conference Series in Mathematics, vol. 101, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004. MR 2020572 (2004k:11103)

[Fre90] Eberhard Freitag, *Hilbert modular forms*, Springer-Verlag, Berlin, 1990. MR 1050763 (91c:11025)

[MS63] Yozô Matsushima and Goro Shimura, *On the cohomology groups attached to certain vector valued differential forms on the product of the upper half planes*, Ann. of Math. (2) **78** (1963), 417–449. MR 0155340 (27 #5274)