A generalization of Kronecker's first limit formula with application to zeta functions of number fields

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Kronecker's first limit formula has many other applications; we mention one such next.

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where w_K denotes the number of roots of unity in K, d_K denotes the discriminant of K, τ is an element of the upper half plane such that $\{1, \tau\}$ is a basis for an ideal in the inverse class of A, and y is the imaginary part of τ.

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The preceding formula was generalized by Bump-Goldfeld to real cubic fields, Efrat to all cubic fields, and Liu-Masri to all totally real fields.

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We generalize it to all number fields.

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Our proof is self contained; the proof of Liu-Masri relies on work of Terras (all use the Poisson summation formula).

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where $\tau : \mathbf{R}^m \rightarrow \mathfrak{H}^n$ is an explicit function, r denotes the number of real embeddings of K, c denotes the number of complex conjugate embeddings, $m = r + c - 1$, $\epsilon_1, \ldots, \epsilon_m$ denotes a fundamental set of units of K, D is a fundamental domain under the action of $\langle \epsilon_1, \ldots, \epsilon_m \rangle$ on $(\mathbf{R}_{>0})^m$, $V = \int_D \frac{dt_1}{t_1}\cdots\frac{dt_m}{t_m}$, and $d(s)$ is an explicit function.

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Was proved for cubic fields by Efrat and for totally real fields by Liu-Masri. All proofs use generalizations of a trick of Hecke.

Thank you!