A generalization of Kronecker's first limit formula

Amod Agashe*

Abstract

Kronecker's first limit formula gives the polar and constant terms of the Laurent series expansion of the Eisenstein series for $\mathrm{SL}(2, \mathbf{Z})$ at $s = 1$, which in turn can be used to find expressions for the polar and constant terms of partial or Dedekind zeta functions of quadratic fields. In this article, we generalize the formula to certain maximal parabolic Eisenstein series associated to $\mathrm{SL}(n, \mathbf{Z})$ for $n \geq 2$. We also show how the generalized formula can be used to find expressions for the polar and constant terms of partial or Dedekind zeta functions of arbitrary number fields at $s = 1$.

Keywords: Kronecker's first limit formula, Zeta functions of number fields, Eisenstein series.
Mathematics Subject Classification codes: 11R42, 11M36.

1 Introduction

Let $n \geq 2$ be an integer. Let τ be in the generalized upper half-plane $\mathfrak{H}^n$, which consists of $n \times n$ matrices with real number entries that are the product of an upper triangular matrix with 1's along the diagonal and a diagonal matrix with positive diagonal entries such that the lowermost diagonal entry is 1. When $n = 2$, one can identify $\mathfrak{H}^n$ with the usual complex upper half plane (for details, see, e.g., [Gol06, §1.2]).

In the following, $m_1, \ldots, m_n$ denote integers with perhaps some added restrictions as noted; in particular, we follow the convention that in any sum over a subset of $m_1, \ldots, m_n$, if a term has denominator zero for some values of $m_1, \ldots, m_n$, then that term is to be skipped in the sum.

Consider the maximal parabolic Eisenstein series

$$E_n(\tau, s) = \sum_{(m_1, \ldots, m_n) = 1} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n)\tau \|^{ns/2}} \ ,$$

where $\| (m_1 \ldots m_n)\tau \|$ denotes the norm of the row vector that is the product of the row vector $(m_1 \ldots m_n)$ and the matrix τ; this series converges when $\mathrm{Re}(s) > 1$, and is known to have a meromorphic continuation to all of $\mathbf{C}$.

Let

$$\begin{aligned} E_n^*(\tau, s) &= \pi^{-ns/2}\Gamma(ns/2)\zeta(ns)E(\tau, s) \\ &= \pi^{-ns/2}\Gamma(ns/2) \sum_{m_1, \ldots, m_n} \frac{(\det \tau)^s}{\| (m_1 \ldots m_n)\tau \|^{ns/2}} \ . \end{aligned} \tag{1}$$

*The author was partially supported by the Collaboration Grants for Mathematicians from the Simons Foundation.

Note that in the case where $n = 2$, if τ corresponds to the point $z = x + iy$ in the complex upper half plane, then

$$E_2^*(\tau, s) = \pi^{-s}\Gamma(s) \sum_{m_1,m_2} \frac{y^s}{|m_1 z + m_2|^{2s}} \ .$$

The classical Kronecker's first limit formula gives the first two terms of the Laurent expansion of $E_2^*(\tau, s)$ at $s = 1$:

$$E_2^*(\tau, s) = \frac{1}{s-1} + \big(\gamma - \log 4\pi - \log y - 4 \log |\eta(\tau)|\big) + O(s-1) \ , \tag{2}$$

where γ is the Euler-Mascheroni constant and $\eta(z)$ is the Dedekind eta-function.

Recall that if K is a number field and A is an ideal class of K, then the partial zeta function associated to A is given by

$$\zeta_K(s, A) = \sum_{\mathfrak{a} \in A} \frac{1}{N\mathfrak{a}^s}.$$

The Dedekind zeta function is the sum of the partial zeta functions over all ideal classes. For a number field K, we denote by w_K the number of roots of unity in K and by d_K the discriminant of K.

Using (2), Kronecker showed that if K is a quadratic imaginary field, then

$$\zeta_K(s, A) = \frac{1}{w_K} \frac{2\pi}{\sqrt{d_K}} \left(\frac{1}{s-1} + 2\gamma - \log 2 - \log y - 4 \log(|\eta(\tau)|) \right) + O(s-1) \ ,$$

where τ is an element of the upper half plane such that $\{1, \tau\}$ is a basis for an ideal in the inverse class of A and y is the imaginary part of τ.

Later, Hecke used Kronecker's first limit formula (2) to show that if K is a real quadratic field and ϵ is the fundamental unit of K, then

$$\begin{aligned} \frac{1}{2}(\pi^{-1} d_K^{1/2})^s \Gamma(s/2)^2 \zeta_K(s, A) &= \frac{\log \epsilon}{s-1} + (\gamma - \log 4\pi) \log \epsilon - \int_1^\epsilon \log y(t) \frac{dt}{t} \\ &\quad -4 \int_1^\epsilon \log |\eta(\tau(t))| \frac{dt}{t} + O(s-1) \ , \end{aligned} \tag{3}$$

where $\tau(t)$ is a certain curve in the upper half plane (which depends on A), and $y(t)$ denotes its y-coordinate.

Kronecker's first limit formula (2) and Hecke's formula (3) were generalized to the case $n = 3$ in [Efr92] (see also [BG84] for a slightly different generalization in this case). A generalization of Kronecker's first limit formula to arbitrary $n \geq 2$ for Epstein zeta functions is given in [Ter73]. In this article, we generalize Kronecker's first limit formula (2) and Hecke's formula (3) to arbitrary $n \geq 2$ (Theorem 1.1 below and Theorem 2.1 in Section 2 respectively).

Theorem 1.1. *For a given $\tau \in \mathfrak{H}^n$, let $y_1, \ldots, y_{n-1}$ denote the unique positive real numbers such that for $i \geq 1$, we have $\tau_{n-i,n-i} = \prod_{j=1}^i y_j$. If $m_1, \ldots, m_n$ are integers, then for $j = 2, \ldots, n$, let $b_j = \sum_{i=1,\ldots,j-1} m_i \tau_{i,j}$ and $c_j = \tau_{j,j}$; also let m be the nonnegative real number such that $m^2 = m_n^2/c_n^2 + m_{n-1}^2/c_{n-1}^2 + \cdots + m_2^2/c_2^2$ and $d = b_n m_n/c_n + b_{n-1}m_{n-1}/c_{n-1} + \cdots b_2 m_2/c_2$. Let τ' be the submatrix of τ obtained by removing the topmost row and leftmost column and let*

$$g(\tau) = \exp\left\{ -\frac{1}{4}\left(\left(\prod_{i=1}^{n-1} y_i^{(\frac{i}{n-1})} \right) E_{n-1}^*\left(\tau', \frac{n}{n-1}\right) + \sum_{m_1 \neq 0} \frac{1}{|m_1|} \sum_{\substack{(m_2,\ldots,m_n) \\ \neq (0,\ldots,0)}} \exp\left(2\pi i d - 2\pi |m_1| m \prod_{i=1}^{n-1} y_i\right) \right) \right\}.$$

Then

$$E_n^*(\tau,s) \;=\; \frac{2/n}{s-1} + \left(\gamma - \log 4\pi - \frac{2}{n}\log\left(\prod_{i=1}^{n-1} y_i^i\right) - 4\log g(\tau)\right) + O(s-1)\;. \tag{4}$$

The theorem is proved in Section 3. The classical Kronecker's first limit formula has several applications (see, e.g., [Sie80]); many of these generalize to give applications of our generalization of the limit formula. In this article, we shall limit ourselves to one such application: in Section 2, we a formula analogous to (3) for the polar and constant terms of the Laurent series of the partial or Dedekind zeta function for arbitrary number fields K: see Theorem 2.1 (as mentioned earlier, this was already done for cubic fields in [Efr92]). As pointed out in [Zag75, §1], the interest in the constant terms mentioned above is that they can be used to compute the values at $s=1$ of the L-functions associated to non-trivial characters of the ideal class group of K, and this in turn leads to an evaluation of the residue of the zeta function (and hence of the class number) of unramified abelian *extensions* of K, since these zeta functions are the product of the zeta function of K and the L-functions mentioned above.

Our proof of Theorem 1.1 is a generalization of the proof of the classical Kronecker's first limit formula given in [Lan87, §20.4], and the key observation is to break the sum in (1) over $m_1,\ldots,m_n$ into two parts conveniently (we break it as a sum over $\{m_1\}$ and a sum over $\{m_2,\ldots,m_n\}$; when $n=2$, there is not much of a choice) and to apply the Poisson summation formula in the second sum in reverse order (over m_n first, followed by m_{n-1}, and so on, up to m_2). Even in the case $n=3$, our proof differs in some key steps with that in [Efr92] (who introduces complex coordinates on $\mathfrak{H}^3$, while we don't) and that in [BG84] (who use minimal parabolic Eisenstein series). Our proof techniques are similar to those used in [Ter73] (about which we learned only after a first draft of this article was written), but are more direct and elementary (the main goal of [Ter73] was to prove the functional equation of the Epstein zeta function using generalizations of the Selberg-Chowla formula). After this article was first submitted, the author came to know of the paper of Liu-Masri [LM15], where the authors prove Theorem 1.1. [1] However, their proof relies crucially on [Ter73], while ours is independent of loc. cit. (or any other articles, for that matter). Apart from our article being self-contained, another difference between our article and [LM15] is that in loc. cit., the formula analogous to (3) for the polar and constant terms of the Laurent series of the partial zeta function (Theorem 1 in loc. cit.) is given only for totally real fields K, while we give the formula for all number fields (it seems possible that the techniques of [LM15] might work if the number field K has at least one real embedding, but when that is not the case, i.e., if K is totally complex, then one seems to need a new trick, which is formula (12) below). We remark that [LM15] also gives some other applications of the generalization of Kronecker's first limit formula (Theorem 1.1, which is also their theorem).

Finally, we make a comment that is historical in nature. As mentioned in [Zag75, §1], Hecke considered the problem of determining the constant terms of the partial zeta functions of arbitrary number fields and claimed to have a formula analogous to (3), based on Epstein's generalization of the Kronecker limit formula, in the general case; the details never appeared. While Theorem 2.1 below achieves what Hecke seems to have claimed, our proof does not use Epstein's generalization, which involves Epstein zeta functions. At the same time, Epstein zeta functions can be easily related to the maximal parabolic Eisenstein series considered in this article (see, e.g., [Efr92, §1] in the case $n=3$), and we could have proved a formula analogous to Theorem 2.1 by adjusting our proof by using Epstein zeta functions and Epstein's generalization of Kronecker's limit formula (for

[1] A first version of our paper that included Theorem 1.1 and its present proof was posted on arxiv.org before [LM15] was published.

Epstein zeta functions rather than Eisenstein series)[2] instead of our generalization of Kronecker's limit formula (for Eisenstein series); this is perhaps what Hecke had in mind (especially considering that the key to the proof of Theorem 2.1 is a generalization of the trick of Hecke that he used to prove (3)).

2 Zeta functions of number fields

In this section, we give a formula for the Laurent series expansion of the partial and Dedekind zeta function of a number field times some explicit functions; the main idea is to use a generalization of a trick of Hecke to express the partial zeta function as an integral of an Eisenstein series over a suitable region and then use our limit formula for the Eisenstein series. The procedure is described for $n = 3$ in [Efr92, §4], and the discussion below is its generalization. However, we do need a new input, which is formula (12), the analog of which was not required in loc. cit.

Let K be a number field of degree n over $\mathbf{Q}$. Let r denote the number of real embeddings and c denote the number of complex conjugate embeddings of K. We assume that $r + c > 1$ since the cases where $r + c = 1$ are classical (the field of rational numbers and quadratic imaginary fields).

Let A be an ideal class of K. Fix $B \in A^{-1}$. Let U denote the unit group of K and as before, let w_K denote the number of roots of unity in K. Let $m = r + c - 1$, and let $\epsilon_1, \ldots, \epsilon_m$ denote a fundamental set of units. Then

$$\zeta_K(s, A) = \sum_{\mathfrak{a} \in A} \frac{1}{N\mathfrak{a}^s} = NB^s \sum_{\lambda \in B/U} \frac{1}{|N\lambda|^s} = \frac{NB^s}{w_K} \sum_{\lambda \in B/\langle \epsilon_1, \ldots, \epsilon_m \rangle} \frac{1}{|N\lambda|^s}. \tag{5}$$

Order the embeddings of K so that the first c are complex and the remaining r are real. If $x \in K$, then for $i = 1, \ldots, r + c$, let $|x|_i$ denote the absolute value of the image of x under the i-th embedding. For $i = 1, \ldots, m + 1 = r + c$, let $\delta_i = 1$ if the i-th embedding is real and $\delta_i = 2$ otherwise.

We first deal with the case where K has at least one real embedding. Then the $(m + 1)$-st embedding is real, and thus $\delta_{m+1} = 1$. A change of variables shows that for any positive real numbers $a_1, \ldots, a_{m+1}$,

$$\begin{aligned} &\int_0^\infty \cdots \int_0^\infty (a_1^2 t_1^2 + \ldots + a_m^2 t_m^2 + a_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\ &\quad = (a_1^{\delta_1} \cdots a_{m+1}^{\delta_{m+1}})^{-s} \int_0^\infty \cdots \int_0^\infty (t_1^2 + \ldots + t_m^2 + (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \ . \end{aligned} \tag{6}$$

Now for $i = 1, \ldots, r + c$, let $a_i = |\lambda|_i$. Then $a_1^{\delta_1} \cdots a_{m+1}^{\delta_{m+1}} = |N\lambda|$. Let

$$d(s) = \int_0^\infty \cdots \int_0^\infty (t_1^2 + \ldots + t_m^2 + (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} = \frac{\Gamma(\delta_1 s/2) \cdots \Gamma(\delta_{m+1} s/2)}{2^m (1 + \delta_1 + \cdots + \delta_m) \Gamma(ns/2)} \ . \tag{7}$$

Putting all this in formula (6), we see that

$$d(s) \frac{1}{|N\lambda|^s} = \int_0^\infty \cdots \int_0^\infty (|\lambda|_1^2 t_1^2 + \ldots + |\lambda|_m^2 t_m^2 + |\lambda|_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m},$$

[2] We were not aware of Epstein's generalization (reproved in [Ter73]) when we first wrote this article.

and so

$$\begin{aligned} & d(s) \sum_{\lambda \in B/\langle \epsilon_1, \ldots, \epsilon_m \rangle} \frac{1}{|N\lambda|^s} \\ = & \sum_{\lambda \in B/\langle \epsilon_1, \ldots, \epsilon_m \rangle} \int_0^\infty \cdots \int_0^\infty (|\lambda|_1^2 t_1^2 + \ldots + |\lambda|_m^2 t_m^2 + |\lambda|_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \end{aligned} \tag{8}$$

Now $1 = |N\epsilon_i| = \prod_{j=1}^{m+1} |\epsilon_i|_j^{\delta_j}$, and so

$$|\epsilon_i|_{m+1} = (|\epsilon_i|_1^{\delta_1} \cdots |\epsilon_i|_m^{\delta_m})^{-1}, \tag{9}$$

considering that $\delta_{m+1} = 1$. For $i = 1, \ldots, m$, let ϵ_i act on $(\mathbf{R}_+^\times)^m$ by multiplying the j-th coordinate by $|\epsilon_i|_j$. Letting D be a fundamental domain under the action of $\langle \epsilon_1, \ldots, \epsilon_m \rangle$ on $(\mathbf{R}_+^\times)^m$, we get

$$\begin{aligned} & \sum_{\lambda \in B/\langle \epsilon_1, \ldots, \epsilon_m \rangle} \int_0^\infty \cdots \int_0^\infty (|\lambda|_1^2 t_1^2 + \ldots + |\lambda|_m^2 t_m^2 + |\lambda|_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\ = & \sum_{\lambda \in B} \int_D (|\lambda|_1^2 t_1^2 + \ldots + |\lambda|_m^2 t_m^2 + |\lambda|_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \end{aligned} \tag{10}$$

Let $\alpha_1, \ldots, \alpha_n \in K$ be a $\mathbf{Z}$-basis of B. Let $\lambda \in B$. Then $\lambda = m_1\alpha_1 + \cdots m_n\alpha_n$ for some $m_1, \ldots, m_n \in \mathbf{Z}$. For $i = 1, \ldots, r + c$, let λ_i denote the image of λ under the i-th embedding. We denote $(\alpha_j)_i$ by $\alpha_{j,i}$. Recalling the way we ordered the embeddings of K, we see that for $i = 1, \ldots, c$, $|\lambda|_i^2 = \Re(\lambda_i)^2 + \Im(\lambda_i)^2$, while for $i = c + 1, \ldots, c + r$, $|\lambda|_i^2 = \lambda_i^2$ (if $c = 0$, then any expression below containing $\Re$ or $\Im$ should be ignored). Thus

$$\begin{aligned} & |\lambda|_1^2 t_1^2 + \ldots + |\lambda|_m^2 t_m^2 + |\lambda|_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2} \\ = & (m_1\Re(\alpha_{1,1}) + \ldots + m_n\Re(\alpha_{n,1}))^2 t_1^2 + (m_1\Im(\alpha_{1,1}) + \ldots + m_n\Im(\alpha_{n,1}))^2 t_1^2 + \ldots \\ + & (m_1\Re(\alpha_{1,c}) + \ldots + m_n\Re(\alpha_{n,c}))^2 t_1^2 + (m_1\Im(\alpha_{1,c}) + \ldots + m_n\Im(\alpha_{n,c}))^2 t_c^2 \\ + & (m_1\alpha_{1,c+1} + \ldots + m_n\alpha_{n,c+1})^2 t_{c+1}^2 + \ldots \\ + & (m_1\alpha_{1,m} + \ldots + m_n\alpha_{n,m})^2 t_m^2 \\ + & (m_1\alpha_{1,m+1} + \ldots + m_n\alpha_{n,m+1})^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2} \\ = & \mathbf{m} M M^T \mathbf{m}^T, \end{aligned}$$

where $\mathbf{m}$ is the row vector with entries $m_1, \ldots, m_n$ and M is the $n \times n$ matrix whose i-th row has entries

$$\Re(\alpha_{i,1})t_1,\ \Im(\alpha_{i,1})t_1,\ \ldots,\ \Re(\alpha_{i,c})t_c,\ \Im(\alpha_{i,c})t_c,\ \alpha_{i,c+1}t_{c+1},\ \ldots,\ \alpha_{i,m}t_m,\ \alpha_{i,m+1}(t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2}\ .$$

Let $Q = MM^T$. Then Q is an $n \times n$ positive definite symmetric matrix, and thus can be written as $Q = (\det Q)^{1/n} (\det \tau(t_1, \ldots, t_m))^{-2/n} \tau(t_1, \ldots, t_m) \tau(t_1, \ldots, t_m)^T$ for some uniquely defined τ :

$\mathbf{R}^m \to \mathfrak{H}^n$ (note that $\det Q$ is independent of $t_1, \ldots, t_m$). Thus

$$\begin{aligned}
&\sum_{\lambda \in B} \int_D (|\lambda|_1^2 t_1^2 + \ldots + |\lambda|_m^2 t_m^2 + |\lambda|_{m+1}^2 (t_1^{\delta_1} \cdots t_m^{\delta_m})^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\
= &\sum_{m_1, \ldots, m_n} \int_D (\mathbf{m} M M^T \mathbf{m}^T)^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\
= &\sum_{m_1, \ldots, m_n} \int_D (\mathbf{m} Q \mathbf{m}^T)^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\
= &\sum_{m_1, \ldots, m_n} \int_D (\mathbf{m} (\det Q)^{1/n} (\det \tau(t_1, \ldots, t_m))^{-2/n} \tau(t_1, \ldots, t_m) \tau(t_1, \ldots, t_m)^T \mathbf{m}^T)^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\
= &(\det Q)^{-s/2} \int_D (\det \tau(t_1, \ldots, t_m))^s \sum_{m_1, \ldots, m_n} (\mathbf{m} \tau(t_1, \ldots, t_m) \tau(t_1, \ldots, t_m)^T \mathbf{m}^T)^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\
= &(\det Q)^{-s/2} \pi^{ns/2} \Gamma(ns/2)^{-1} \int_D E_n^*(\tau(t_1, \ldots, t_m), s) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m},
\end{aligned}$$

by definition of $E_n^*(\tau, s)$ (see formula (1)). Thus from the equation above and equations (5), (8), and (10), we see that

$$d(s)\zeta_K(s, A) = \frac{NB^s}{w_K} (\det Q)^{-s/2} \pi^{ns/2} \Gamma(ns/2)^{-1} \int_D E_n^*(\tau(t_1, \ldots, t_m), s) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}.$$

Considering that for a complex number z, $\Re(z) = (z + \overline{z})/2$ and $\Im(z) = (z - \overline{z})/2$, we see that $4^c \det Q = \mathrm{disc} B = (NB)^2 d_K$, where recall that d_K denotes the discriminant of K. So

$$d(s)\zeta_K(s, A) = \frac{(2^c d_K^{-1/2})^s}{w_K} \pi^{ns/2} \Gamma(ns/2)^{-1} \int_D E_n^*(\tau(t_1, \ldots, t_m), s) \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}. \tag{11}$$

Now consider the case where K does not have a real embedding, i.e., K is totally complex. In that case, formula (6) does not work, and instead, we use the following formula:

$$\begin{aligned}
&\int_0^\infty \cdots \int_0^\infty (a_1^2 t_1^2 + \ldots + a_m^2 t_m^2 + a_{m+1}^2 (t_1 \cdots t_m)^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} \\
&\quad = (a_1^2 \cdots a_{m+1}^2)^{-s} \int_0^\infty \cdots \int_0^\infty (t_1^2 + \ldots + t_m^2 + (t_1 \cdots t_m)^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}\ ,
\end{aligned} \tag{12}$$

The argument above starting right after formula (6) goes through, with the following changes: take

$$d(s) = \int_0^\infty \cdots \int_0^\infty (t_1^2 + \ldots + t_m^2 + (t_1 \cdots t_m)^{-2})^{-ns/2} \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m} = \frac{\Gamma(s/2)^{m+1}}{2^m (m+1) \Gamma(ns/2)}\ , \tag{13}$$

replace throughout the term $(t_1^{\delta_1} \cdots t_m^{\delta_m})$ by $(t_1 \cdots t_m)$, replace the line containing equation (9) by "Now $1 = |N\epsilon_i| = \prod_{j=1}^{m+1} |\epsilon_i|_j^2$, and so $|\epsilon_i|_{m+1} = (|\epsilon_i|_1^{\delta_1} \cdots |\epsilon_i|_m^{\delta_m})^{-1}$", and replace the last entry in the i-th row of M by the two entries $\Re(\alpha_{i,m+1})(t_1 \cdots t_m)^{-2}$ and $\Im(\alpha_{i,m+1})(t_1 \cdots t_m)^{-2}$. So equation (11) is still valid, but with $d(s)$ given by formula (13), and with the definition of $\tau(t_1, \ldots, t_m)$ modified as per the change in M mentioned above.

Let $V = \int_D \frac{dt_1}{t_1} \cdots \frac{dt_m}{t_m}$. Then from equation (11) and our generalization of Kronecker's limit formula (formula (4)), we get

Theorem 2.1. *Recall that K is a number field that is not the rational numbers or a quadratic imaginary field. With notation as above, in paricular, taking $d(s)$ to be given by formula (7) if K has a real embedding, and by formula (13) if not, we have*

$$\begin{aligned} w_K\big(2^{-c}\pi^{-n/2}d_K^{1/2}\big)^s d(s)\Gamma(ns/2)\zeta_K(s,A) &= \frac{2V/n}{s-1}+(\gamma-\log 4\pi)V \\ &\quad -\frac{2}{n}\int_D \log\left(\prod_{i=1}^{n-1} y_i(t_1,\ldots,t_m)^i\right)\frac{dt_1}{t_1}\cdots\frac{dt_m}{t_m} \\ &\quad -4\int_D \log g(\tau(t_1,\ldots,t_m))\frac{dt_1}{t_1}\cdots\frac{dt_m}{t_m}+O(s-1). \end{aligned}$$

When $n=2$, we recover Hecke's formula (3); when $n=3$ and K has a complex embedding, we get Theorem 3 of [Efr92]; and if K is a totally real field, we recover Theorem 1 of [LM15]. Finally, note that the Dedekind zeta function is the sum of the partial zeta functions over all ideal classes, so from the formula above, we get a corresponding formula involving the Dedekind zeta function.

3 Proof of Theorem 1.1

We first prove the formula for $E_n^*(\tau,s)$ and deduce from it the formula for $E_n^\dagger(\tau,s)$. In formula (1), the term corresponding to $m_1=0$ is

$$\begin{aligned} S_1 &= \pi^{-ns/2}\Gamma(ns/2)\sum_{m_2,\ldots,m_n}\frac{(\prod_{i=1}^{n-1} y_i^{n-i})^s}{\|\ (m_2\ldots m_n)\tau'\ \|^{ns/2}} \\ &= \pi^{-ns/2}\Gamma(ns/2)\cdot\left(\prod_{i=1}^{n-1} y_i^{(\frac{i}{n-1})}\right)^s\cdot\sum_{m_2,\ldots,m_n}\frac{(\prod_{i=1}^{n-2} y_i^{n-1-i})^{(\frac{n}{n-1}s)}}{\|\ (m_2\ldots m_n)\tau'\ \|^{(n-1)(\frac{n}{n-1}s)/2}} \\ &= \left(\prod_{i=1}^{n-1} y_i^{(\frac{i}{n-1})}\right)^s\cdot\pi^{-ns/2}\Gamma(ns/2)\cdot E_{n-1}\Big(\tau',\frac{n}{n-1}s\Big) \\ &= \left(\prod_{i=1}^{n-1} y_i^{(\frac{i}{n-1})}\right)^s\cdot\pi^{-(n-1)(\frac{n}{n-1}s)/2}\Gamma\Big((n-1)\Big(\frac{n}{n-1}s\Big)/2\Big)\cdot E_{n-1}\Big(\tau',\frac{n}{n-1}s\Big) \\ &= \left(\prod_{i=1}^{n-1} y_i^{(\frac{i}{n-1})}\right)^s\cdot E^*_{n-1}\Big(\tau',\frac{n}{n-1}s\Big). \end{aligned} \tag{14}$$

Let

$$S_2=\sum_{m_1\neq 0}\sum_{m_2,\ldots,m_n}\frac{\pi^{-ns/2}\Gamma(ns/2)}{\|\ (m_1\ldots m_n)\tau\ \|^{ns/2}},$$

so that

$$E_n^*(\tau,s)=S_1+\left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^s\cdot S_2. \tag{15}$$

Our next goal is to find a suitable expression for S_2, which will not be achieved till equation (26) below. We use the formula

$$\frac{\pi^{-s}\Gamma(s)}{a^s}=\int_0^\infty \exp(-\pi at)t^s\frac{dt}{t} \tag{16}$$

with $a = \| (m_1 \ldots m_n)\tau \|$, and s replaced by $ns/2$ to get

$$S_2 = \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \int_0^\infty \exp(-\pi t \| (m_1 \ldots m_n)\tau \|) t^{ns/2} \frac{dt}{t} \ . \tag{17}$$

For $j = 2, \ldots, n$, let

$$a_j = (m_1\tau_{1,1})^2 + \cdots + \left(\sum_{i=1,\ldots,j-1} m_i \tau_{i,j-1} \right)^2, \tag{18}$$

and recall that $b_j = \sum_{i=1,\ldots,j-1} m_i \tau_{i,j}$, and $c_j = \tau_{j,j}$. Then

$$\begin{aligned} \| (m_1 \ldots m_n)\tau \| &= (m_1\tau_{1,1})^2 + \cdots + \left(\sum_{i=1,\ldots,n-1} m_i \tau_{i,n-1} \right)^2 + \left(\left(\sum_{i=1,\ldots,n-1} m_i \tau_{i,n} \right) + m_n \tau_{n,n} \right)^2 \\ &= a_n + (b_n + c_n m_n)^2, \end{aligned} \tag{19}$$

Putting (19) in (17), we get

$$\begin{aligned} S_2 &= \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \int_0^\infty \exp(-\pi t a_n) \exp(-\pi t (b_n + c_n m_n)^2) t^{ns/2} \frac{dt}{t} \\ &= \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_{n-1}} \int_0^\infty \exp(-\pi t a_n) \sum_{m_n} \exp(-\pi t (b_n + c_n m_n)^2) t^{ns/2} \frac{dt}{t} \end{aligned} \tag{20}$$

The Poisson summation formula says that for real numbers t, b, c, with $c \neq 0$,

$$\sum_{m \in \mathbf{Z}} \exp(-\pi t (b + cm)^2) = \frac{1}{c\sqrt{t}} \sum_{m \in \mathbf{Z}} \exp(2\pi i b m / c) \exp(-\pi m^2 / t c^2). \tag{21}$$

Using this with $b = b_n$ and $c = c_n$, replacing m by m_n, and noting that c_n, being a diagonal entry of τ, is always positive, we get

$$\sum_{m_n} \exp(-\pi t (b_n + c_n m_n)^2) = \frac{1}{c_n \sqrt{t}} \exp(-(\pi t a_n)) \sum_{m_n} \exp(2\pi i b_n m_n / c_n) \exp(-\pi m_n^2 / t c_n^2).$$

Putting this in (20), we get

$$\begin{aligned} S_2 &= \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_{n-1}} \int_0^\infty \exp(-\pi t a_n) \sum_{m_n} \exp(2\pi i b_n m_n / c_n) \exp(-\pi m_n^2 / t c_n^2) t^{ns/2 - 1/2} \frac{dt}{t} \\ &= \frac{1}{c_n} \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \exp(2\pi i b_n m_n / c_n) \int_0^\infty \exp(-(\pi a_n t + \pi m_n^2 / t c_n^2)) t^{ns/2 - 1/2} \frac{dt}{t} \\ &= \frac{1}{c_n} \sum_{m_1 \neq 0, m_n} \exp(2\pi i b_n m_n / c_n) \\ &\quad \cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2) / t) \sum_{m_2,\ldots,m_{n-2}} \sum_{m_{n-1}} \exp(-(\pi a_n t)) t^{ns/2 - 1/2} \frac{dt}{t} \end{aligned} \tag{22}$$

Now from equation (18),

$$\begin{aligned} a_n &= (m_1\tau_{1,1})^2 + \cdots + \left(\sum_{i=1,\ldots,n-2} m_i \tau_{i,n-2} \right)^2 + \left(\left(\sum_{i=1,\ldots,n-2} m_i \tau_{i,n-1} \right) + m_{n-1} \tau_{n-1,n-1} \right)^2 \\ &= a_{n-1} + (b_{n-1} + c_{n-1} m_{n-1})^2 \ . \end{aligned}$$

So using formula (21) again as above, we have

$$\begin{aligned}
&\sum_{m_{n-1}} \exp(-(\pi a_n t)) \\
= \quad & \exp(-(\pi t a_{n-1})) \sum_{m_{n-1}} \exp(-\pi t (b_{n-1} + c_{n-1} m_{n-1})^2) \\
= \quad & \frac{1}{c_{n-1}\sqrt{t}} \exp(-(\pi t a_{n-1})) \sum_{m_{n-1}} \exp(2\pi i b_n m_{n-1}/c_{n-1}) \exp(-\pi m_{n-1}^2 / t c_{n-1}^2).
\end{aligned}$$

Putting this in (22),

$$\begin{aligned}
S_2 \quad = \quad & \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0, m_n} \exp(2\pi i b_n m_n / c_n) \int_0^\infty \exp(-\pi m_n^2 / t c_n^2) \sum_{m_2,\ldots,m_{n-2}} \frac{1}{\sqrt{t}} \exp(-(\pi t a_{n-1})) \\
& \cdot \sum_{m_{n-1}} \exp(2\pi i b_{n-1} m_{n-1}/c_{n-1}) \exp(-\pi m_{n-1}^2 / t c_{n-1}^2) t^{ns/2 - 2/2} \frac{dt}{t} \\
= \quad & \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0, m_n} \sum_{m_{n-1}} \exp(2\pi i (b_n m_n / c_n + b_{n-1} m_{n-1} / c_{n-1})) \\
& \cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2 + m_{n-1}^2 / c_{n-1}^2)/t) \sum_{m_2,\ldots,m_{n-2}} \exp(-(\pi t a_{n-1})) t^{ns/2 - 2/2} \frac{dt}{t} \\
= \quad & \frac{1}{c_{n-1}c_n} \sum_{m_1 \neq 0, m_n, m_{n-1}} \exp(2\pi i (b_n m_n / c_n + b_{n-1} m_{n-1} / c_{n-1})) \\
& \cdot \int_0^\infty \exp(-\pi (m_n^2 / c_n^2 + m_{n-1}^2 / c_{n-1}^2)/t) \sum_{m_2,\ldots,m_{n-3}} \sum_{m_{n-2}} \exp(-(\pi t a_{n-1})) t^{ns/2 - 2/2} \frac{dt}{t} \qquad (23)
\end{aligned}$$

Looking at equations (22) and (23), we see that repeated use of Poisson summation gives

$$S_2 = \frac{1}{\prod_{i=2}^n c_i} \sum_{m_1 \neq 0, m_n, m_{n-1}, \ldots, m_2} \exp(2\pi i d) \int_0^\infty \exp(-\pi m^2 / t) \exp(-(\pi t a_1)) t^{ns/2 - (n-1)/2} \frac{dt}{t},$$

where recall that m and d were defined in Theorem 1.1. Now $a_1 = m_1^2 y^2$ where $y = \tau_{1,1} = y_1 y_2 \cdots y_{n-1}$. So

$$\left(\prod_{i=2}^n c_i \right) S_2 = \sum_{m_1 \neq 0} \sum_{m_2,\ldots,m_n} \exp(2\pi i d) \int_0^\infty \exp(-(\pi (m_1 y)^2 t + \pi m^2 / t)) t^{n(s-1)/2 + 1/2} \frac{dt}{t}.$$

Denote the term corresponding $m_2 = \cdots = m_n = 0$ by S_2', i.e.,

$$S_2' = \sum_{m_1 \neq 0} \int_0^\infty \exp(-\pi (m_1 y)^2 t) t^{n(s-1)/2 + 1/2} \frac{dt}{t}.$$

Using formula (16), with s replaced by $n(s-1)/2 + 1/2$, we get

$$\begin{aligned}
S_2' \quad = \quad & \sum_{m_1 \neq 0} \frac{\pi^{-(n(s-1)/2 + 1/2)} \Gamma(n(s-1)/2 + 1/2)}{(m_1 y)^{2(n(s-1)/2 + 1/2)}} \\
= \quad & y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2 + 1/2)} \Gamma(n(s-1)/2 + 1/2) \cdot 2 \sum_{m_1 \geq 0} \frac{1}{m_1^{n(s-1)+1}} \\
= \quad & 2 y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2 + 1/2)} \Gamma(n(s-1)/2 + 1/2) \zeta(n(s-1)+1) \qquad (24)
\end{aligned}$$

Let

$$\begin{aligned} S_2'' &= \left(\prod_{i=2}^n c_i\right) S_2 - S_2' \qquad (25) \\ &= \sum_{m_1\neq 0} \sum_{(m_2,\ldots,m_n)\neq(0,\ldots,0)} \exp(2\pi i d) \int_0^\infty \exp(-(\pi(m_1 y)^2 t + \pi m^2/t)) t^{n(s-1)/2+1/2} \frac{dt}{t}. \end{aligned}$$

For a and b positive real numbers, recall the function

$$K_s(a,b) = \int_0^\infty \exp(-(a^2 t + b^2/t)) t^s \frac{dt}{t}$$

Noting that $m \neq 0$ if not all $m_2, \ldots, m_n$ are zero,

$$S_2'' = \sum_{m_1\neq 0} \sum_{(m_2,\ldots,m_n)\neq(0,\ldots,0)} \exp(2\pi i d) K_{n(s-1)/2+1/2}(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|) \qquad (26)$$

From equations (14), (15), (24), (25), and (26), we finally get an expression for $E_n^*(\tau, s)$:

$$\begin{aligned} E_n^*(\tau,s) &= \left(\prod_{i=1}^{n-1} y_i^{(\frac{i}{n-1})}\right)^s \cdot E_{n-1}^*\left(\tau', \frac{n}{n-1}s\right) \\ &+2\left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^s \frac{1}{\prod_{i=2}^n c_i} y^{-(n(s-1)+1)} \pi^{-(n(s-1)/2+1/2)} \Gamma(n(s-1)/2+1/2)\zeta(n(s-1)+1) \\ &+\left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^s \frac{1}{\prod_{i=2}^n c_i} \sum_{m_1\neq 0} \sum_{\substack{(m_2,\ldots,m_n)\\ \neq(0,\ldots,0)}} \exp(2\pi i d) K_{n(s-1)/2+1/2}(\sqrt{\pi}|m_1|y, \sqrt{\pi}|m|) \qquad (27) \end{aligned}$$

The good thing about the formula above is that it is easy to read off the polar part and the constant term in each of the summands above, which is what we do now. It is known that K_s is an entire function of s, and so all the functions appearing in the expression above are holomorphic at $s=1$ except $\zeta(2s-1)$, which has a simple pole at $s=1$, and perhaps $E_{n-1}^*\left(\tau', \frac{n}{n-1}s\right)$. By induction, $E_{n-1}^*\left(\tau', \frac{n}{n-1}s\right)$ is also holomorphic except perhaps when $(\frac{n}{n-1})s = 1$, and in particular is homlomorphic at $s=1$. So the first and last summands on the right side of equation (27) are holomorphic at $s=1$; using the fact that $K_{1/2}(a,b) = \frac{\sqrt{\pi}}{a}\exp(-2ab)$, their sum is

$$\left(\prod_{i=1}^{n-1} y_i^{(\frac{i}{n-1})}\right) E_{n-1}^*\left(\tau', \frac{n}{n-1}\right) + \sum_{m_1\neq 0} \sum_{(m_2,\ldots,m_n)\neq(0,\ldots,0)} \exp(2\pi i d) \frac{1}{|m_1|} \exp(-2\pi|m_1||m|y) + O(s-1),$$

which is $-4\log g(\tau) + O(s-1)$.

In order to deal with the second summand on the right side of equation (27), note that

$$\zeta(n(s-1)+1) = \frac{1}{n(s-1)} + \gamma + O(s-1),$$

$$\Gamma(n(s-1)/2+1/2) = \sqrt{\pi}(1 + \frac{n}{2}(\gamma - \log 4)(s-1) + O(s-1)^2,$$

$$\pi^{-(n(s-1)/2+1/2)} = \frac{1}{\sqrt{\pi}}(1 - \frac{n}{2}\log \pi (s-1) + O(s-1)^2,$$

$$y^{-(n(s-1)+1)} = y^{-1}(1 - n\log y(s-1) + O(s-1)^2,$$

and

$$\left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^s = \left(\prod_{i=1}^{n-1} y_i^{n-i}\right)^{(s-1)+1} = \left(\prod_{i=1}^{n-1} y_i^{n-i}\right)(1 + \log\left(\prod_{i=1}^{n-1} y_i^{n-i}\right)(s-1) + O(s-1)^2).$$

Using the formulas above, the second summand in on the right side of equation (27) becomes

$$\frac{2/n}{s-1} + \left(\gamma - \log 4\pi - \frac{2}{n}\log\left(\prod_{i=1}^{n-1} y_i^i\right)\right) + O(s-1)$$

Using the formulas obtained above for the three summands on the right side of equation (27), we get the formula for $E_n^*(\tau, s)$ given in Theorem 1.1.

References

[BG84] Daniel Bump and Dorian Goldfeld, *A Kronecker limit formula for cubic fields*, Modular forms (Durham, 1983), Ellis Horwood Ser. Math. Appl.: Statist. Oper. Res., Horwood, Chichester, 1984, pp. 43–49. MR 803361

[Efr92] Isaac Efrat, *On a* GL(3) *analog of* $|\eta(z)|$, J. Number Theory **40** (1992), no. 2, 174–186. MR 1149736 (93a:11040)

[Gol06] Dorian Goldfeld, *Automorphic forms and L-functions for the group* $\mathrm{GL}(n, \mathbf{R})$, Cambridge Studies in Advanced Mathematics, vol. 99, Cambridge University Press, Cambridge, 2006, With an appendix by Kevin A. Broughan. MR 2254662 (2008d:11046)

[Lan87] Serge Lang, *Elliptic functions*, second ed., Graduate Texts in Mathematics, vol. 112, Springer-Verlag, New York, 1987, With an appendix by J. Tate. MR 890960 (88c:11028)

[LM15] Sheng-Chi Liu and Riad Masri, *A Kronecker limit formula for totally real fields and arithmetic applications*, Res. Number Theory **1** (2015), Art. 8, 20. MR 3500992

[Sie80] Carl Ludwig Siegel, *Advanced analytic number theory*, second ed., Tata Institute of Fundamental Research Studies in Mathematics, vol. 9, Tata Institute of Fundamental Research, Bombay, 1980. MR 659851 (83m:10001)

[Ter73] Audrey A. Terras, *Bessel series expansions of the Epstein zeta function and the functional equation*, Trans. Amer. Math. Soc. **183** (1973), 477–486. MR 0323735 (48 #2091)

[Zag75] Don Zagier, *A Kronecker limit formula for real quadratic fields*, Math. Ann. **213** (1975), 153–184. MR 0366877 (51 #3123)