Algebraic Geometry I
Lectures 9, 10, and 11

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Housekeeping Note: Dr. Agashe's office hours are now MW 2:15-3:15.

1 Covering Projective Space by Affine Spaces

In this section, we will develop a natural covering of $\mathbb{P}^n_k$ by n-dimensional affine spaces over k. As we will see, this yields homeomorphisms between projective algebraic sets in $\mathbb{P}^n$ and affine algebraic sets in $\mathbb{A}^n$.

Notation 1.1. We use the abbreviation $\{$*[some condition]*$\}$ to denote the set of all points that satisfy *[some condition]*. For example, $\{x_0 \neq 0\}$ is an abbreviation for $\{[x_0 : \ldots : x_n] \in \mathbb{P}^n \mid x_0 \neq 0\}$, and if $f : X \longrightarrow Y$ and $c \in Y$, then $\{f = c\}$ is an abbreviation for $\{x \in X \mid f(x) = c\}$.

We now define our covering. For $i = 0, 1 \ldots, n$, let $U_i = \{x_i \neq 0\} \subseteq \mathbb{P}^n$, and define $\phi_i : U_i \longrightarrow \mathbb{A}^n$ as

$$\phi_i([a_0 : \ldots : a_n]) = \left(\frac{a0}{ai}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$

Also define $\phi_i^{-1} : \mathbb{A}^n \longrightarrow \mathbb{P}^n$ as

$$\phi_i^{-1}(a_1, \ldots, a_n) = [a_1 : \ldots : a_i : 1 : a_{i+1} : \ldots : a_n]$$

Note ϕ_i is well defined, since if $(a_0, \ldots, a_n)$ and $(b_0, \ldots, b_n)$ are two sets of homogeneous coordinates for a point in U_i, then $(b_0, \ldots, b_n) = \lambda(a_0, \ldots, a_n)$ for some $\lambda \neq 0 \in k$, and we have

$$\phi_i([b_0 : \ldots : b_n]) = \phi_i([\lambda a_0 : \ldots : \lambda a_n])$$
$$= \left(\frac{\lambda a_0}{\lambda a_i}, \ldots, \frac{\lambda a_{i-1}}{\lambda a_i}, \frac{\lambda a_{i+1}}{\lambda a_i}, \ldots, \frac{\lambda a_n}{\lambda a_i}\right) = \left(\frac{a_0}{a_i}, \ldots, \frac{a_{i-1}}{a_i}, \frac{a_{i+1}}{a_i}, \ldots, \frac{a_n}{a_i}\right)$$
$$= \phi_i([a_0 : \ldots : a_n])$$

It is easily verified that $\phi_i^{-1}(\mathbb{A}^n) \subseteq U_i$, and that ϕ_i and ϕ_i^{-1} are (two-sided) inverses; hence, ϕ_i is a bijection.

Example 1.2. Recall that when discussing $\mathbb{P}^2$, we considered points $[x_0 : y_0 : z_0]$ with $z_0 \neq 0$, and mapped them to $\mathbb{A}^2$ as $[x_0 : y_0 : z_0] \mapsto (\frac{x_0}{z_0}, \frac{y_0}{z_0})$. This is $\phi_2 : U_2 \longrightarrow \mathbb{A}^2$.

We now state two results from [**?**], Section I.2.

Proposition 1.3. *([**?**], Proposition I.2.2) The map ϕ_i is a homeomorphism of U_i with its induced topology to $\mathbb{A}^n$ with the Zariski topology.*

Corollary 1.4. *([**?**], Corollary I.2.3) If Y is a projective (respectively, quasi-projective) algebraic set, then Y is covered by the open sets $Y \cap U_i$ for $i = 0, 1, \ldots, n$, and these sets are homeomorphic to affine (respectively, quasi-affine) algebraic sets via ϕ_i.*

Remark 1.5. "So we can do local geometry using an affine cover."

Example 1.6. Find the tangent to $y^2 z = x^3 - xz^2$ at the point $p = [\,0 : 1 : 0\,]$.
Solution: Note that $p \in \{y \neq 0\} = U_1$. Set $y = 1$. Then we have the affine algebraic set

$$\{f(x,z) \ := \ z - x^3 - xz^2 \ = \ 0\} \subseteq \mathbb{A}^2$$

p becomes $(0,0)$, and the tangent to the curve at p is given by

$$\frac{\delta f}{\delta x} \mid_{(0,0)} (x - 0) + \frac{\delta f}{\delta z} \mid_{(0,0)} (z - 0) = 0$$

$$(-3 \cdot 0^2 - 0^2)x + (1 - 2 \cdot 0 \cdot 0)z = 0$$

$$z = 0$$

2 The Category of Algebraic Varieties

Definition 2.1. An *algebraic set* is an algebraic subset of affine *or* projective space. A *variety* is an irreducible algebraic set or an open subset thereof.

These are the objects that we want to study. We would like to define a category of varieties. In order to do so, we must decide what the morphisms in this category are to be.

2.1 Regular Functions

As a first guess, we might try continuous maps (with respect to the Zariski topology). The problem with this approach is that there are too many continuous maps. As an example, consider maps $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$. Since the closed sets in $\mathbb{A}^1$ are $\emptyset$, $\mathbb{A}^1$, and finite sets, *any* bijection on $\mathbb{A}^1$ is a homeomorphism. Since this would include maps that have nothing to do with the structure of varieties, we need more restrictions on the types of maps we take as morphisms in our category.

Remark 2.2. Here are two considerations:

1. If $\phi : \mathbb{A}^n \longrightarrow \mathbb{A}^m$, and π_i is the i^{th} coordinate function on $\mathbb{A}^m$, then the composite map $\pi_i \circ \phi$ is just a function $\mathbb{A}^n \longrightarrow k$. What functions $\mathbb{A}^n \longrightarrow k$ should we allow? Certainly, we want to allow the coordinate functions and functions built from them using pointwise addition and multiplication.

2. Recall the parameterization for the Pythagorean triples:

$$t \longmapsto \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$$

   where our base field k is $\mathbf{C}$. This suggests that we should allow rational functions of coordinates. The only problem is that, if the denominator vanishes at some point, then the function is not well-defined there.

Example 2.3. Let $Y = Z(xy - zw) - Z(wy) \subseteq \mathbb{A}^4$. The function $(x, y, z, w) \mapsto \frac{x}{w}$ is well-defined provided $w \neq 0$. Similarly, the function $(x, y, z, w) \mapsto \frac{z}{y}$ is well-defined when $y \neq 0$. If $w, y \neq 0$, then $\frac{x}{w} = \frac{z}{y}$ on $Z(xy - zw)$. Hence, the map $\phi : Y \longrightarrow k$ defined as

$$\phi(x, y, z, w) = \begin{cases} \frac{x}{w} & \text{if } w \neq 0 \\ \frac{z}{y} & \text{if } y \neq 0 \end{cases}$$

is well-defined by different rational functions at different points.

This motivates the following definition.

Definition 2.4. Let $Y \subseteq \mathbb{A}^n$ be a quasi-affine algebraic set. A function $f : Y \longrightarrow k$ is *regular* at $p \in Y$ if there exists an open neighborhood U of p with $U \subseteq Y$ and polynomials $g, h \in k[x_1, \ldots, x_n]$ such that h is nowhere zero on U and $f = \frac{g}{h}$ on U. f is *regular on* Y if f is regular at every point in Y.

Remark 2.5. We will use the same definition for quasi-projective algebraic sets. However, for that case, we will insist that g and h be homogeneous of the same degree, so that $\frac{g}{h}$ is well-defined at p.

We now state a result from [**?**], Section I.3; a proof is provided in the text.

Lemma 2.6. *([**?**], Lemma I.3.1) A regular function $\mathbb{A}^n \longrightarrow k$ is continuous when k is identified with $\mathbb{A}^1$ under the Zariski topology.*

Remark 2.7. If $Y \subseteq \mathbb{A}^n$ is irreducible and $f, g : Y \longrightarrow k$ are continuous functions (with respect to the Zariski topology) such that $f = g$ on some non-empty open subset $U \subseteq Y$, then $f = g$ on Y.

Proof. Recall that when k is identified with $\mathbb{A}^1$ under the Zariski topology, any finite subset of k is closed.

Now, note that the set $\{ f = g \}$ is closed, since it is the preimage of the closed set $\{0\}$ under the continuous map $f - g$. By assumption, $U \subseteq \{ f = g \}$, so $\{ f = g \}$ is non-empty. If f and g are not equal on all of Y, then $\{ f = g \}$ and $Y - U$ are non-empty proper closed subsets of Y, with

$$Y = U \cup (Y - U) \subseteq \{ f = g \} \cup (Y - U) \subseteq Y$$

so

$$Y = \{ f = g \} \cup (Y - U)$$

contradicting our assumption that Y is irreducible. Thus $f = g$ on all of Y. □

2.2 Morphisms

We can now define morphisms of algebraic sets. In the theory of differentiable manifolds, there are two ways of defining morphisms between two manifolds M and N:

1. Cover M and N by charts, and insist that the induced maps from subsets of $\mathbf{R}^n$ to subsets of $\mathbf{R}^m$ are differentiable. The problem with this approach is that this definition depends on the choice of charts, and is not intrinsic.

2. Locally, a pullback of differentiable functions is differentiable.

Our definition of morphisms of algebraic sets will mirror the second approach.

Definition 2.8. Let X and Y be algebraic sets. A *morphism* $\phi : X \longrightarrow Y$ is a continuous map (with respect to the Zariski topology) such that for all open subsets $V \subseteq Y$ and all regular functions $f : V \longrightarrow k$, the function $f \circ \phi : \phi^{-1}(V) \longrightarrow k$ is regular. ϕ is an *isomorphism* if there exists a morphism $\psi : Y \longrightarrow X$ such that $\phi \circ \psi = id_Y$ and $\psi \circ \phi = id_X$.

Lemma 2.9. *([**?**], Lemma I.3.6)*
Let X be an algebraic set, let $Y \subseteq \mathbb{A}^n$ be a quasi-affine algebraic set, and let $x_1, x_2, \ldots, x_n$ denote the coordinate functions on $\mathbb{A}^n$. Then a map $\psi : X \longrightarrow Y$ is a morphism if and only if $x_i \circ \psi$ is regular on X for $i = 1, 2, \ldots, n$.

Proof. See the text. $\square$

Example 2.10. The function $\phi : \mathbb{A}^1_{\mathbf{C}} - \{ i, -i \} \longrightarrow \mathbb{A}^2_{\mathbf{C}}$ defined as

$$\phi(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$$

is a morphism.

Homework 2.11. ([**?**], Ex. I.3.2(a))
Define $\phi : \mathbb{A}^1 \longrightarrow \{ y^2 = x^3 \} \subseteq \mathbb{A}^2$ as $\phi(t) = (t^2, t^3)$. Show that

1. ϕ is a morphism
2. ϕ is a bijection
3. (*) ϕ^{-1} is continuous (with respect to the Zariski topology)
4. (*) ϕ^{-1} is not a morphism

2.3 Rings of Regular Functions

Let's go back to regular functions on an algebraic set X.

Definition 2.12. For an open subset $U \subseteq X$, we define $\mathcal{O}(U)$ as the ring of functions that are regular on U.

Remark 2.13. If f is regular on U, then it is regular on any subset of U. Consequently, if $V \subseteq U$ are open subsets of X, then $\mathcal{O}(U) \subseteq \mathcal{O}(V)$. However, if a function is regular on V, it may not be regular on all of U.

Definition 2.14. Let X be a topological space. A *presheaf* $\mathcal{F}$ consists of the following data:

1. for every open subset $U \subseteq X$, a set $\mathcal{F}(U)$
2. for every inclusion $V \subseteq U$ of open subsets of X, a map $\rho_{UV} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$

subject to the following constraints:

1. $\mathcal{F}(\emptyset) = \emptyset$
2. for all open subsets $U \subseteq X$, $\rho_{UU} = id_U$
3. if $W \subseteq V \subseteq U$ are open subsets of X, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

Example 2.15. Let X be an algebraic set, let $\mathcal{F} = \mathcal{O}$, and for all open subsets $U \subseteq V$ of X, define ρ_{UV} as

$$\rho_{UV}(s) = s\,|_V, \forall\, s \in \mathcal{O}\,(\,U\,)$$

This is an example of a presheaf.

We can place extra structure on the sets $\mathcal{F}(U)$ in the presheaf. For example, they may be groups or rings. In this case, we insist that the maps ρ_{UV} be morphisms of the corresponding structures (e.g., group or ring homomorphisms).

For example, $\mathcal{O}$ is actually a presheaf of rings, and is denoted $\mathcal{O}_X$.

Remark 2.16.
1. The notion of presheaves is related to that of covering spaces (see exercise II.1.13 in [**?**])
2. We call the set $\mathcal{F}(U)$ a *section* of $\mathcal{F}$ over U. $\mathcal{F}(U)$ is sometimes denoted by $\Gamma(U, \mathcal{F})$. We also call the maps ρ_{UV} *restriction maps*, and if $s \in \mathcal{F}(U)$, then we denote $\rho_{UV}(s)$ by $s\,|_V$. Elements of $\mathcal{F}(X)$ are called *global sections*.

Example 2.17. $\mathcal{O}_X(X)$ is the set of functions that are regular on all of X.

2.4 Affine Coordinate Rings and Local Rings

Consider an affine set $Y \subseteq \mathbb{A}^n$. Any polynomial in $k[x_1, \cdots, x_n]$ is clearly regular on $\mathbb{A}^n$. If $f, g \in k[x_1, \cdots, x_n]$ then f and g give the same function on Y if and only if $f - g \in I(Y)$. Therefore, if we want to identify functions that agree on an affine algebraic set, we can mod out the ideal of that set and work in the resulting quotient ring. This motivates the following definition.

Definition 2.18. The *affine coordinate ring* of $Y \subseteq \mathbb{A}^n$, denoted $A(Y)$, is defined as $A(Y) = k[x_1, \cdots, x_n] \;/\; I(Y)$.

Remark 2.19. $A(Y) \subseteq \mathcal{O}(Y)$. Is there anything else in $\mathcal{O}(Y)$? This question will be answered soon.

Next, we turn to the question of which functions are regular at a point $p \in Y$. We can (and will) assume Y is irreducible. Any function f that is regular at p is defined in a neighborhood U of p. If g is another function regular on a neighborhood V of p such that $f = g$ on $U \cap V$, then we will identify f and g. The objects we consider are thus pairs $< U, f >$, where $U \subseteq Y$ is open, $p \in U$, and f is regular on U, under the equivalence relation

$$< U, f > \sim < V, g > \iff f = g \text{ on } U \cap V$$

Homework 2.20. (*) Prove that $\sim$ is an equivalence relation. (Hint: if X is irreducible, $U \subseteq X$ is open, and f and g are continuous functions on X such that $f = g$ on U, then $f = g$ on all of X since U is dense in X).

Fact 2.21. The equivalence classes of the pairs $< U, f >$ under $\sim$, with the operations

$$< U, f > + < V, g > \;=\; < U \cap V, f + g >$$

and

$$< U, f > \cdot < V, g > \;=\; < U \cap V, f \cdot g >$$

form a ring, called the *local ring of p on Y*, and denoted $\mathcal{O}_{p,Y}$ (or $\mathcal{O}_p$ for simplicity). Elements of $\mathcal{O}_{p,Y}$ are called *germs* of functions regular near p.

Example 2.22. Consider the affine algebraic set $Y : y = x^2 \subseteq \mathbb{A}^2$, and let $p = (1,1) \in \mathbb{A}^2$. The function $f(x,y) = \frac{x^2+y}{x+1}$ is regular on $U = Y - \{x + 1 = 0\}$, so $< U, f > \in \mathcal{O}_p$.

Example 2.23. On $Y = \{xy = zw\} - \{z = y = 0\} \subseteq \mathbb{A}^4$, $< Y - \{z = 0\}, \frac{x}{z} >, < Y - \{y = 0\}, \frac{w}{y} > \in \mathcal{O}_{(1,1,1,1)}$, with $< Y - \{z = 0\}, \frac{x}{z} > \sim < Y - \{y = 0\}, \frac{w}{y} >$

Remark 2.24.

1. A ring is said to be *local* if it has only one maximal ideal. The ring $\mathcal{O}_p$ is local, since the equivalence classes $< U, f >$ with $f(p) = 0$ form a maximal ideal that contains all other proper ideals.

   Proof. (Sketch) Suppose I is an ideal of $\mathcal{O}_p$ that properly includes the ideal in question. That is, suppose I contains some element $< V, g >$, where $g(p) \neq 0$. Then $\frac{1}{g}$ is regular on some neighborhood W of p. Hence, $< V,\ g >< W,\ \frac{1}{g} >=< V\ \cap\ W,\ 1 >$ is a unit in I, so $I = \mathcal{O}_p$. $\square$

2. Suppose $Y \subseteq \mathbb{A}^n$. If $f, g \in A(Y)$ such that $g(p) \neq 0$, then $\frac{f}{g}$ is regular in some neighborhood of p. Let $\mathfrak{m}_p = \{g \in A(Y) \mid g(p) = 0\}$, and note that $\mathfrak{m}_p$ is a maximal ideal of $A(Y)$, so that we may invert elements outside of $\mathfrak{m}_p$.

   If A is an integral domain and S is a multiplicative subset of A (i.e., $0 \notin S$, and $\forall x, y \in S$, $xy \in S$), define the relation $\sim$ on $A \times S$ as

   $$(f, g) \sim (f', g') \iff fg' = gf'$$

   (i.e., $(f, g) \sim (f', g') \iff \frac{f}{g} = \frac{f'}{g'}$). It is easy to check that $\sim$ is an equivalence relation, and that the set of equivalence classes under the operations

   $$(f, g) + (f', g') = (fg' + gf', gg')$$

   and

   $$(f, g) \cdot (f', g') = (ff', gg')$$

   forms a ring. This ring is called the *localization of A at S*, and is denoted $S^{-1}A$.

   If A is an integral domain and $\mathfrak{p}$ is a prime ideal of A, then $S = A - \mathfrak{p}$ is a multiplicative set, and $S^{-1}A$ is denoted $A_{\mathfrak{p}}$, and called the *localization of A at $\mathfrak{p}$*. So $A(Y)_{\mathfrak{m}_p} \subseteq \mathcal{O}_p$. Conversely, if $< U, f' >\in \mathcal{O}_p$, then for some open $V \subseteq U$, $f'|_V = \frac{f}{g}$, with $g(p) \neq 0$, so $< U, f' >=< V, \frac{f}{g} >\in A(Y)_{\mathfrak{m}_p}$. Thus, $A(Y)_{\mathfrak{m}_p} = \mathcal{O}_p$.

3. For readers familiar with direct limits,

   $$\mathcal{O}_p \ = \ \varinjlim_{p \in U} \mathcal{O}(U)$$

   We may take this as the definition of $\varinjlim_{p \in U} \mathcal{O}(U)$, or we may characterize it by its universal property. It is a ring R with a map $r_U : \mathcal{O}(U) \longrightarrow R$ for every U, such that the "usual compatibility" condition is satisfied. That is, for open sets $V \subseteq U$, $r_U = r_V \circ \rho_{UV}$.

   Now, if R' is another ring with maps $r'_U : \mathcal{O}(U) \longrightarrow R'$ satisfying compatibility as above, there exists a unique $\psi_{R'} : R \longrightarrow R'$ such that for all U, the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{O}(U) \arrow[r, "r_U"] \arrow[dr, "r'_U"'] & R \arrow[d, "\psi_{R'}"] \\
& R'
\end{tikzcd}
$$

Mimicking the above construction, we have the following.

Definition 2.25. If $\mathcal{F}$ is a presheaf on a topological space X and $p \in X$, then the *stalk of* $\mathcal{F}$ *at* p, denoted $\mathcal{F}_p$, is defined as

$$
\mathcal{F}_p \;=\; \varinjlim_{p \in U} \mathcal{F}(U)
$$