Agebraic Geomerty I
Lectures 6 and 7

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1 Projective Space

In Euclidean geometry, two distinct lines in the plane intersect in exactly in one point except if they are parallel; this exception is a bit of a nuisance. So we add one extra point to the Euclidean plane for each direction (these are called points at infinity) and declare that parallel lines meet at the point at infinity corresponding to their common direction. The corresponding geometry is called projective geometry. Just like Euclidean geometry has an axiomatic definition, so does projective geometry. The former has an analytic description in terms of the "usual" coordinate geometry (Descarte). An analytic description of projective geomerty was introduced by Plücker, Mobius, and others.

1.1 General Idea

Put the xy-plane in 3-dimensions and shift it up by 1. Each point on the xy-plane is in correspondence with a line in through the originwhich is not in the xy-plane. Each line in the xy-plane corresponds to a direction. So we can think of the projective plane as the set of lines through the origin; classical points and points at infinity. Now a line through the origin is an [equivalence class] of points under scaling and can be specified by a point in 3-dimensions, except for zero.

1.2 Definition of $\mathbb{P}^n$ Projective n-space

Assume the field k is algebraically closed.

Let (x_0, y_0, z_0) be a triple of elements in k. Let $\sim$ be the equivalence relation given by

$$(x_0, y_0, z_0) \sim (x_1, y_1, z_1)$$

if and only if $\exists\lambda \in k \setminus \{0\}$ such that $x_0 = \lambda x_1$, $y_0 = \lambda y_1$, and $z_0 = \lambda z_1$.

Definition 1.1. *Projective n-space* over k is

$$\mathbb{P}^n := \{(a_0, \ldots, a_n) \in k^{n+1} \setminus \{0\}\}/\sim .$$

Definition 1.2. An element of $\mathbb{P}^n$ is called a point. Note that points in $\mathbb{P}^n$ are (n+1)-tuples. If $P \in \mathbb{P}^n$, then $(a_0, \ldots, a_n)$ any (n+1)-tuple in the equivalence class of P is called a *set of homogeneous coordinates.* The equivalence class of $(a_0, \ldots, a_n)$ is denoted $[a_0\colon \ldots : a_n]$.

We can classify points as follows: $P = [a_0\colon \ldots : a_n]$ with $a_n \neq 0$. Note that this condition is independent of the representative. One can easily see this in affine space by

$$[a_0\colon \ldots : a_n] = [\frac{a_0}{a_n}\colon \ldots : \frac{a_{n-1}}{a_n} : 1] \rightsquigarrow (\frac{a_0}{a_n}, \ldots, \frac{a_{n-1}}{a_n}) \in \mathbb{A}^n.$$

Remember, in $\mathbb{A}^n$ every coordinate may take value of 0 simultaneously.

Conversely, a tuple $(b_0, \ldots, b_n) \in \mathbb{A}^n$ gives a point $[b_0\colon \ldots : b_{n-1}\colon h]$, $h \neq 0$. This corresponds to "lifting" the n-plane up by h.

Remark 1.3. The set of points $[a_0\colon \ldots : a_n] \in \mathbb{P}^n$ with $a_n \neq 0$ is a copy of $\mathbb{A}^n$ inside $\mathbb{P}^n$.

Remark 1.4. Points $[a_0\colon \ldots : a_n]$ with $a_n = 0$ correspond up to scalars to points in $\mathbb{A}^n - (0)$ which are called points at infinity. The collection of such points is called a *hyperplane* at infinity.

Example 1.5. Take n=2. Then points in $\mathbb{P}^2$ correspond to lines in the xy-plane through the origin. Points that are further from the origin in the "lifted" plane $\rightsquigarrow$ lines in the actual xy-plane; eventually the points at infinity.

Remark 1.6. In the classification above we chose the (n+1)th coordinate, but any index would do. The former leads itself to the intuitively seeing the first n points as a cpoy of $\mathbb{A}^n$ together with points added at infinity.

2 How to do geometry in $\mathbb{P}^n$

The equations of lines, conics, etc. are defined as the zeros of certian polynomials. However, it does not make sense to evaluate an arbitrary $f \in k[x_o. \ldots, x_n]$ at a point P because it may have different values depending upon the representaions used.

Example 2.1. $f(x, y, z) = x + y^2$ and $P = [1\colon 1\colon 1\colon 1]$. Then $f(1, 1, 1) = 2$ but $f(2, 2, 2) = 6$ and $f(-1, -1, -1) = 0$.

If f satisfies $f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n)$ for d the degree of f and all $\lambda \in k \setminus \{0\}$, then it makes sense to ask whether f is zero at a point P or not.

Example 2.2. $f(x, y, z) = x^2 + y^2$ satisfies this condition.

Definition 2.3. A polynomial $f \in k[x_0, \ldots, x_n]$ is *homogeneous of degree d* if each of its terms has degree d. This allows one to think about zero sets of homogeneous polynomials since in this case

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n).$$

Definition 2.4. If T is a set of homogeneous polynomials in $k[x_0, \ldots, x_n]$, the zero set of T in $\mathbb{P}^n$ is $Z(T) = \{P \in \mathbb{P}^n | f(P) = 0, \forall f \in T\}$. $Y \subseteq \mathbb{P}^n$ is *algebraic* if there exists a subset $T \subset k[x_0, \ldots, x_n]$ of homogeneous elements such that $Z(T) = Y$.

Consider $y^2 = x^3 - x \in \mathbb{A}^2$. What is its analogue in $\mathbb{P}^2$? Multiply through by a monomial to make $f = y^2 - x^3 + x$ homogeneous; $y^2 z = x^3 - xz^2$ in $\mathbb{P}^n$. This technique is used in the general case as well:

Definition 2.5. To any polynomial $f \in k[x_1, \ldots, x_n]$ of degree d we may associate a homogeneous polynomial

$$f(x_1, \ldots, x_n) \rightsquigarrow x_0^d \cdot f(x_1/x_0, \ldots, x_n/x_0).$$

This process is called *homogenization*.

Let $f \in k[x_0, \ldots, x_{n-1}]$ and $F \in k[x_0, \ldots, x_n]$ be obtained by homogenozing f. Now points in $Z(F) \subset \mathbb{P}^n$ with $a_n \neq 0$ are in one-to-one correspondence with $Z(f) \subset \mathbb{A}^n$.

Example 2.6.
Consider $Z(y^2 z - x^3 + xz^2) \subset \mathbb{P}^2$ and let $P = [x_0\colon y_0\colon z_0] \in Z(y^2 z - x^3 + xz^2)$. We have the following cases:

<u>case 1</u>: $z_0 \neq 0$. Then $y_0^2 z_0 - x_0^3 + x_0 z_0^2 = 0$ and $\left(\frac{y_0}{z_0}\right)^2 - \left(\frac{x_0}{z_0}\right)^3 + \left(\frac{x_0}{z_0}\right) = 0$. So $\left(\frac{y_0}{z_0}, \frac{x_0}{z_0}\right) \in Z(y^2 - x^3 + x^2) \subset \mathbb{A}^2$.
If $(a_0, b_0) \in Z(y^2 - x^3 + x^2) \subset \mathbb{A}^2$, then $[a_0\colon b_0\colon 1] \in Z(y^2 z - x^3 + xz^2)$

because $F(x, y, 1) = f(x, y) = y^2 - x^3 + x^2$.
case 2: $z_0 = 0$. Then $y_0^2 \cdot 0 - x_0^3 - x_0 \cdot 0^2 = 0 \implies x_0 = 0$. Therefore y_0 is, up to nonzero scaling, arbitrary. These points are all in the class $[0\colon 1\colon 0]$, so there is only one such point.

Find the points at infinity of the tangent line to $E : y^2 z = x^3 - xz^2$ at $[-1\colon 0\colon 1] \in \mathbb{P}^2$ corresponding to $(-1, 0) \in \mathbb{A}^2$.

Solution: The tangent line to a homogeneous polynomial ,F, at a point $P = [x_0\colon y_0\colon z_0]$ is given by

$$\frac{\partial F}{\partial x}|_P(x - x_0) + \frac{\partial F}{\partial y}|_P(y - y_0) + \frac{\partial F}{\partial z}|_P(z - z_0) = 0.$$

This gives $-3x^2 + z^2|_P(x - (-1)) + 2xz|_P(z - 1) = 0$. That is $x = -z$. The points on this line with $z \neq 0$ correspond to $x = -1 \in \mathbb{A}^2$. Put this into the initial equation to get $y^2 z = -z^3 + z^3 = 0 \implies y = 0$ or $z = 0$ but not both since $[0\colon 0\colon 0] \notin \mathbb{P}^2$.
subcase 1: $y = 0$. Then z is arbitrary (but nonzero) and the solutions are $[-z\colon 0\colon z] \sim [-1\colon 0\colon 1]$.
subcase 2: $z = 0$. Now y is arbitrary (but nonzero) and the solutions are $[0\colon y\colon 0] \sim [0\colon 1\colon 0]$.

Remark 2.7. Counting multiplicities we get three points of intersection, as Bezout's Theorem predicts. In $\mathbb{P}^2$, the group law on an elliptc curve can be specified by saying that the points of intersection of any line with the curve E add up to zero, taking multiplicities into account. The identity element, O, should satisfy $2O + O = O$. That is O should be an inflection point on E.

3 More on Algebraic Sets in $\mathbb{P}^n$

Proposition 3.1. *The union of two algebraic sets is algebraic. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

Definition 3.2. We define the *Zariski topology* on $\mathbb{P}^n$ by taking the open sets to be the complements of algebraic sets.

Let H_n be the hyperplane given by $x_n = 0$. Let $U_n = \mathbb{P}^n \setminus H_n$. Define

$$\varphi_n : U_n \to \mathbb{A}^n$$

as follows: if $P = (a_0, \ldots, a_n) \in U_n$ then $\varphi(P) = Q$ where Q is the point with affine coordinates $\left(\frac{a_0}{a_n}, \cdots, \frac{a_{n-1}}{a_n}\right)$ with $\left(\frac{a_n}{a_n}\right)$ omitted.

Proposition 3.3. *The map φ_n defined above is a homeomorphism of U_n with its topology to $\mathbb{A}^n$ with its Zariski topology.*

Corollary 3.4. *If y is a projective (respectively, quasi-projective) variety, then Y is covered by the open sets $Y \cap U_i$ which are homeomorphic to affine (respectively, quasi-affine) varieties via the mapping varphi$_i$ defined above.*

Remark 3.5. If $T \subset k[x_0, \ldots, x_n]$ consists of homogeneous elements and $P \in Z(T)$, then for all $g \in (T) \subset k[x_0, \ldots, x_n]$, $g(a_0, \ldots, a_n) = 0$ for any representative, $(a_0, \ldots, a_n)$, of P.