Algebraic Geometry I
Lectures 3, 4, and 5

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1 Varieties

Proposition 1.1. *(a) If X and Y are algebraic sets in $\mathbb{A}^n$, then so is $X \cup Y$*

(b) If I is a set and $\forall i \in I$, X_i is an algebraic set in $\mathbb{A}^n$, then $\bigcap_{i \in I} X_i$ is an algebraic set.

(c) ϕ and $\mathbb{A}^n$ are algebraic sets.

Proof. (a) Let $X = Z(T_1)$ and $Y = Z(T_2)$, where T_1 and T_2 are subsets in $k[x_1, \ldots, x_n]$. Then $X \cup Y = Z(T_1 T_2) = \{f_1 f_2 | f_1 \in T_1, f_2 \in T_2\}$ whence $X \cup Y$ is an algebraic set.

(b) Let $X_i = Z(T_i)$ $\forall i \in I$. Then $\bigcap_{i \in I} X_i = Z(\bigcup_{i \in I} T_i)$.

(c) Note that $\phi = Z(1)$ and $\mathbb{A}^n = Z(0)$.

□

So the algebraic sets are the closed sets of a topology, called the *Zariski topology.* In other words, in the Zariski topology on $\mathbb{A}^n$, the closed sets are precisely the algebraic subsets.

Example 1.2. The complement of $y = x^2$ in $\mathbb{A}^n$ is open. Thus we see that open sets are big!

Example 1.3. Consider the Zariski topology on $\mathbb{A}^1$. Note that $k[x_1] = k[x]$ is a PID. So the algebraic sets are of the form $Z(f)$ for some $f(x) \in k[x]$. If $f(x) = c(x - a_1)\cdots(x - a_n)$, then $Z(f) = \{(a_1, \cdots, (a_n)\}$ which is a finite set of points unless $n = 0$. If $n = 0$, then

$$Z(f) = \begin{cases} \phi & \text{if } c \neq 0 \\ \mathbb{A}^n & \text{if } c = 0 \end{cases}$$

Conversely, if $a_1, \ldots, a_n \in k$ with $n \geq 1$ then $\{(a_1), \ldots, (a_n)\} = Z\big((x - a_1), \ldots, (x - a_n)\big)$. Also, as shown earlier, $\phi = Z(1)$ and $\mathbb{A}^n = Z(0)$. Thus the closed sets of $\mathbb{A}^1$ are either ϕ, $\mathbb{A}^n$ or a finite se of points. In particular, the topology is not Hausdorff.

Now let $Y \subseteq \mathbb{A}^n$ be a subset (not necessarily algebraic). If $f(\vec{x}) \in k[\vec{x}]$ such that $f(\vec{a}) = 0 \; \forall \; \vec{a} \in Y$ then $fg = 0$ on Y for any $g \in k[\vec{x}]$. Also if $f = 0$ and $g = 0$ on Y then $f + g = 0$ on Y. Thus $\{f \in k[\vec{x}] | f = 0 \text{ on } Y\}$ is an ideal of $k[\vec{x}]$. It is denoted by $I(Y)$ and is called the ideal of Y.

Proposition 1.4. *(a) If $T_1 \subseteq T_2$ are subsets of $k[\vec{x}]$ then $Z(T_1) \supseteq Z(T_2)$.*

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$ then $I(Y_1) \supseteq I(Y_2)$.

Proof. Homework. □

Now if $\mathfrak{a}$ is an ideal of $k[\vec{x}]$, then a natural question to ask is : how is $I(Z(\mathfrak{a}))$ related to $\mathfrak{a}$? Certainly $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$. Does the reverse inclusion hold as well?

Example 1.5. Let $\mathfrak{a} = (x^n) \subseteq k[x]$ for some $n > 1$. Then $Z(\mathfrak{a}) = \{(0)\}$. Now observe that $x \in I(Z(\mathfrak{a})$ but $x \notin \mathfrak{a}$.

Thus $I(Z(\mathfrak{a})) \nsubseteq \Omega$.

Definition 1.6. If $\mathfrak{a}$ is an ideal in a ring A then the radical of $\mathfrak{a}$ is defined as

$$\sqrt{\mathfrak{a}} = \{a \in A | a^n \in \mathfrak{a}; n \geq 1\}$$

An ideal $\mathfrak{a}$ is said to be a radical ideal if $\mathfrak{a} = \sqrt{\mathfrak{a}}$.

(HW): Show that for any ideal $\mathfrak{a}$, $\sqrt{\mathfrak{a}}$ is an ideal and is a radical ideal.

Remark 1.7. $\forall$ ideals $\mathfrak{a}$, $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$.
If $\mathfrak{a}$ is an ideal in $k[\vec{x}]$, then $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$.

Theorem 1.8. (Hilbert's Nullstellensatz) *Assume k is algebraically closed. Let $\mathfrak{a}$ be an ideal of $k[\vec{x}]$ and let $f \in k[\vec{x}]$ such that $f = 0$ on $Z(\mathfrak{a})$. Then $f^r \in \mathfrak{a}$ for some r.*

Proof. See textbook. □

Corollary 1.9. *If $\mathfrak{a}$ is an ideal of $k[\vec{x}]$ then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$*

Remark 1.10. The conclusion of Theorem 1.8 need not hold if k is not algebraically closed.

Example 1.11. Consider $k = \mathbb{R}$, $n = 2$ and $\mathfrak{a} = (x^2 + y^2 + 1) \subseteq k[x, y]$. Then

$$Z(\mathfrak{a}) = \phi$$

and

$$I(Z(\mathfrak{a})) = I(\phi) = k[x, y] \not\supseteq \mathfrak{a}$$

Example 1.12. Consider $k = \mathbb{R}$, $n = 1$, $\mathfrak{a} = (x^3 - 1) \subseteq k[x]$. Then

$$Z(\mathfrak{a}) = \{(1)\} \subseteq \mathbb{A}^1_{\mathbb{R}}$$

and

$$I(Z(\mathfrak{a})) = (x - 1) \not\supseteq \sqrt{(x^3 - 1)} = \sqrt{\mathfrak{a}}$$

Proposition 1.13. *If Y is any subset of $\mathbb{A}^n$ then $Z(I(Y)) = \overline{Y}$, where $\overline{Y}$ is the closure of Y in the Zariski topology.*

Corollary 1.14. *There is a one-to-one inclusion reversing correspondence:*

$$\{\textit{radical ideals of } k[x_1, \ldots, x_n]\} \longleftrightarrow \{\textit{closed subsets of } \mathbb{A}^n\}$$

given by

$$\mathfrak{a} \longmapsto Z(\mathfrak{a})$$

and

$$Y \longmapsto I(Y)$$

We now introduce the concept of irreducible sets in $\mathbb{A}^n$.

Consider $f(x,y) = x^2 - y^2 \in k[x,y]$ and $Z(f) \subseteq \mathbb{A}^2$. We see that

$$Z(f) = Z(x+y) \cup Z(x-y)$$

so that $Z(f)$ is the union two proper algebraic (i.e closed) subsets.

Definition 1.15. A non-empty subset Y of the topological space X is said to be *irreducible* if it cannot be expressed as a union of two proper closed subsets (they need not be disjoint).

Note that the empty set is considered to be *not irreducible* .

Example 1.16. $Z(x^2 - y^2) \subseteq \mathbb{A}^2$ is not irreducible.

Recall that a topological space X is said to be *connected* if it cannot be written as a union of disjoint non-empty open subsets.

Remark 1.17. Consider $Y = Z(x^2 - y^2)$. We claim that Y is connected in the Zariski topology.

Lemma 1.18. *Irreducible* $\Longrightarrow$ *connected*

Proof. Suppose Y is not connected i.e $\exists$ non-empty open subsets U_1, U_2 such that $U_1 \cup U_2 = Y$ and $U_1 \cap U_2 = \phi$. We can then write

$$Y = (Y \backslash U_1) \cup (Y \backslash U_2)$$

□

Irreducibility is not as relevant in the "usual topology" as it is in algebraic geometry.

Corollary 1.19. *Every algebraic set in $\mathbb{A}^n$ can be expressed uniquely as a union of irreducible subsets with not one containing any other.*

Proof. See textbook. □

Definition 1.20. An *affine algebraic variety* is an irreducible closed subset of $\mathbb{A}^n$ with the induced (Zariski) topology.

Proposition 1.21. *If Y_1, Y_2 are subsets of $\mathbb{A}^n$, then*

$$I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$$

Corollary 1.22. *An algebraic set Y in $\mathbb{A}^n$ is irreducible $\Longleftrightarrow$ $I(Y)$ is a prime ideal.*

Proof. ($\Longrightarrow$) Let Y be irreducible and suppose that $f,\ g \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$. Thus $Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$, both being closed subsets of Y. However, Y is irreducible so that we must have either $Y = Y \cap Z(f)$ whence $Y \subseteq Z(f)$, or $Y = Y \cap Z(g)$ whence $Y \subseteq Z(g)$. Thus either $f \in I(Y)$ or $g \in I(Y)$.
($\Longleftarrow$) Suppose $\mathfrak{p} = I(Y)$ is prime. Then $Y = Z(\mathfrak{p}) = Z(I(Y))$. Suppose $Z(\mathfrak{p}) = Y_1 \cup Y_2$ with Y_1, Y_2 closed subsets. Then $\mathfrak{p} = I(Z(\mathfrak{p})) = I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$ whence $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$. Thus $Z(\mathfrak{p}) = Y_1$ or Y_2, hence it is irreducible. □

Example 1.23. $\mathbb{A}^n$ is irreducible since $I(\mathbb{A}^n) = (0)$ which is prime

Example 1.24. If $f \in k[x_1, \ldots, x_n]$ is irreducible then (f) is a prime ideal and so $Z(f)$ is irreducible. For $n = 2$, $Z(f)$ is called a *curve* . For $n = 3$, $Z(f)$ is called a *surface* . For $n > 3$, $Z(f)$ is called a *hypersurface* .

HW* A non-empty open subset of an irreducible space is irreducible and dense.

Non-empty open subsets of an irreducible space carry *a lot* of information about the topology.

Definition 1.25. A non-empty open subset of an affine variety is called a *quasi-affine variety* .

Lemma 1.26. *If $Y = Y_1 \cup Y_2$ with Y_1, Y_2 connected and $Y_1 \cap Y_2 \neq \phi$, then Y is connected.*

Proof. Suppose Y is not connected so that $\exists$ open subsets U_1, U_2 with $Y = U_1 \cup U_2$ and $U_1 \cap U_2 = \phi$. Consider a point $P \in (Y_1 \cap Y_2)$. Then either $P \in Y_1$ or $P \in Y_2$. WLOG, suppose that $P \in Y_1$. Now note that

$$Y_1 = (Y_1 \cap U_1) \cup (Y_1 \cap U_2)$$

Since Y_1 is connected, one of $Y_1 \cap U_1$ and $Y_1 \cap U_2$ must be empty. But we know that $P \in Y_1 \cap U_1$ and so $Y_1 \cap U_2$ is empty. Thus $Y_1 \subseteq U_1$. Similarly, $Y_2 \subseteq U_1$ and thus $Y \subseteq U_1$ whence $U_2 = \phi$. ($\Rightarrow\Leftarrow$) □

Example 1.27. It immediately follows from the above lemma that $Z(x^2 - y^2)$ is connected in $\mathbb{A}^2$ since both $Z(x - y)$ and $Z(x + y)$ are connected (they are both irreducible).