Lecture 27

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1 Proj S

Definition 1.1. If *(*a) is a homogeneous ideal of S then we define V(*(*a)) $= \{P \in Proj(S) | (a) \supseteq P\}$

Lemma 1.2. *(a) If* (a) *and* (b) *are homogeneous ideals of* S*, then* $V((a) \cdot (b)) = V((a)) \cup V((b))$
(b) If $\{a_i\}_{i \in I}$ *is a family of homogeneous ideals of* S *then* $V(\sum a_i) = \cap V(a_i)$

Proof. Same as for spectra, using the fact that a homogeneous ideal P is prime iff whenever $a, s \in S$ are homogeneous elements such that $ab \in P$ then $a \in P$ or $b \in P$. □

Remark 1.3. $Proj(S) = V(0)$ and $\phi = V(S)$

Definition 1.4. The Zariski topology on $Proj(S)$ is the one with closed subsets being V(*(*a)) for homogeneous ideals *(*a).

Example 1.5. If $S = k[x_1, ..., x_n]$ with k algebraically closed, then the set of closed points of $Proj(S)$ (with the induced topology) is homeomorphic to P_k^n

Example 1.6. n=2 $(x - z, y - 2z)$ is a homogeneous prime ideal and corresponds to [1:2:1] in P_k^2. Also $k[x, y, z]/(x - y, y - 2z) \cong k[z]$ with points in P_k^2 are lines through the origin in A_k^3.

Next: Define a sheaf of rings on $Proj(S)$ If $P \in Proj(S)$; let T_P be the set of all homogeneous elements $S \backslash P$ and let $S_(P)$ denote the degree zero element in $T_P^- 1S$ and $S_{(P)}$ will be the local ring at P.

Example 1.7. $S = k[x, z]$, the point P=(0,1) $\longleftrightarrow$ P=(homogeneous polynomials that do not vanish at [0:1]) then $x^2 + y^2 \in T_P$ $x^2 + xz = x(x + z) \in P$ and $\frac{xz}{x^2+y^2} \in S_(P)$

Definition 1.8. If $v \in Proj(S)$ is open, then
$O(U) = \{$functions :U$\rightarrow \coprod S_{(P)} | \forall p \in U$, $\exists$ a nbhd. V of P and homogeneous elements $a, f \in S$ of the same degree such that $\forall a \in V$, $f \notin ($a) and $S((a)) = \frac{a}{f}$ in $S_{(y)}$.

Example 1.9. This defines a sheaf, $O((a), O_{Proj(S)}$

Proposition 1.10. *Let S be a graded ring.*
(a) $\forall p \in Proj(S)$, $O_p \cong S_{(P)}$ *(stalk), a local ring*
(b) $\forall$ *homogeneous* $f \in S_t$ *let* $D_t(f) = \{p \in Proj(S) | f \notin P\} = Proj(S) \backslash V((f))$.

Proof. See book. □

[e.g. If $S = k[x, y, z]$ and $f = z$, then the closed points of $D_t(z)$ correspond to points [x,y,z] in P_k^2 with $z \neq 0$
i.e. these are the points of the type [x:y:1], a copy of A^2]
Then $D_t(f)$ is open in $Proj(S)$ and the $D_t(f)$'s cover $Proj(S)$, and there is an isomorphism of locally ringed spaces $(D_t(f), O|_{D_t(f)}) \cong Spec(S_{(f)}$ where $S_{(f)}$ is the subring of degree zero elements in S_f.

Example 1.11. Again if $S = k[x, y, z]$ and $f = z$ then $\frac{x^2+yz}{z^k} \in S_{(z)}$; think about putting $z = 1$ to get the function $x^2 + y \in A^2$

So $Proj(S)$ is a scheme and is not Affine. This is an example of a scheme "created intrinscally"' i.e. not by gluing. Same as with the sphere. Motivates: If A is a ring, we define projective n-space over A to be the scheme $P_A^n := Proj(A[x_0, \cdots, x_n])$

Remark 1.12. If k is algebraically closed field, then P_k^n may denote the corresponding scheme of variety.

For pictures of P_k^n see Mumford or Eisenbud-Harris.

Remark 1.13. The scheme P_k^n is the right place to do intersection theory and there is a Bezout's theorem(See E-H).

Remark 1.14. Varieties first defined A_k^n and P_k^n and we considered closed subsets $Z(a)$
but for Affine Schemes: $Spec(k[x_1, \cdots, x_n])$ and $Spec(k[x_1, \cdots, x_n])/(a)$
for Projective Schemes: $Proj(k[x_1, \cdots, x_n]) \longleftrightarrow P_k^n$ and
$Proj(k[x_1, \cdots, x_n]/(a)) \longleftrightarrow Z((a))$