Agebraic Geomerty I
Lectures and

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1 Some Motivation and Perspective

Definition 1.1. Let M be a topological n-manifold. If (U, φ), (V, ψ) are two charts such that $U \cap V \neq \emptyset$ the composite map,

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called a transition map from φ to ψ. Since this is a composition of homeomorphisms, it is itself a homeomorphism. Two cahrts, (U, φ) and (V, ψ), are said to be *smoothly compatible* if either $U \cap V = \emptyset$ or the transition map $\psi \circ \varphi^{-1}$ is a diffeomorphism (bijective smooth map with smooth inverse). An *atlas* for M is a collection of charts that cover M. An atlas A is called a smooth atlas if any two charts of A are smoothly compatible with each other. A smooth atlas A on M is called *maximal* if it is not strictly contained in any other smooth atlas on M (i.e. any chart that is smoothly compatible with every other chart in A is already in A). A *smooth manifold* is a pair (M, A) where M is a topological n-manifold and A is a smooth structure on M, that is a maximal smooth atlas on M.

An equivalent definition of a differentiable manifold:

Definition 1.2. Let X be a topological space and $\mathcal{O}_X$ the sheaf of $\mathbb{R}$-valued functions such that for all $p \in X$, there exists a niegborhood $p \in U \subset X$ such that $(U, \mathcal{O}_X|_U) \cong (B_n, \mathcal{O}_{B_n})$ as lrs. Here B_n is the open unit ball in $\mathbb{R}^n$ and $\mathcal{O}_{B_n}$ the structure sheaf of differentialble functions on B_n. Similarly, morphisms of diffferentiable manifolds are morphisms of lrs.

Remark 1.3. We can replace the B_n with affine varieties to get a notion of an abstract algebraic set.

Remark 1.4. Replacing $(B_n, \mathcal{O}_{B_n})$ by $(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R})$ for a ring R gives a scheme.

Definition 1.5. A scheme that is isomorphic as a lrs to the spectrum of a ring with its structure sheaf, for some ring R, is called an *Affine Scheme*.

Remark 1.6. Some times one gets a topological manifold by starting with a topological space and showing it is a manifold: S^2 withU_1, U_2 overlapping hemispheres, each of which is homeomorphic to the disc. Other times one consturcts topological manifolds by glueing open subsets of $\mathbb{R}^n$ for some n, and using continuous functions for transition functions.

More generally, we can glue manifolds together as follows:

Let X_1 and X_2 be manifolds with homeomorphisms $\varphi : U_1 \to U_2$. We can glue U_1, U_2 using φ to get a manifold $X = (X_1 \backslash U_1) \coprod U_2 \coprod (X_2 \backslash U_2)$. Define $\pi : X_1 \coprod X_2 \to X$ given by if $p \in X_1 \backslash U_1$, then $\pi(p) = p$, if $p \in U_1$, then $\pi(p) = \varphi(p) \in U_2 \subset X$, and if $p \in X_2$, then $\pi(p) = p \in X$. Give the topology defined by $U \subset X$ is open if and only if $\varphi^{-1}(U) \subset X_1 \coprod X_2$ is open.

Example 1.7. $\mathbb{P}^n$ can be obtained by gluing copies of $\mathbb{A}^n$ because $\mathbb{P}^n$ can be covered by $U_i := \{x_i \neq= 0\}$ when $\phi_i : U_i \to \mathbb{A}^n$.

Remark 1.8. If X is a scheme, $U \subset X$ open, then $(U, \mathcal{O}_X|_U)$ is a scheme.

Example 1.9. Let X_1, X_2 be schemes and $U_i \subset X_i$ be open. Let $(\phi, \phi^\sharp)$ be an isomorphism of lrs. The gluing of X_1 and X_2 using $(\phi, \phi^\sharp)$ gives the scheme $(X, \mathcal{O}_X)$ where X is as in the previous remark and the structure sheaf is defined as follows:
Let $\pi : X_1 \amalg X_2 \to X_1$. Then for all open $V \subset X$ define

$$\mathcal{O}_X(V) := \{(s_1, s_2) | s_j \in \mathcal{O}_{X_j}(i_j^{-1}(V)) \text{ and } \phi^\sharp(s_2|_{i_2^{-1}(v) \cap U_2}) = s_1|_{i_2^{-1}(v) \cap U_2}\}.$$

So $(X, \mathcal{O}_X)$ is a lrs since X_1, and x_2 are schemes, any $p \in X$ has an open neigborhood isomorphic to the spectrum of a ring.

Remark 1.10. Similarly, one can glue schemes indexed by any set (cf. exercise II 2.12 [H]).

2 Are Varieties Schemes?

Remark 2.1. Let be an algebraic set over a field k (not necessarily algebraically closed) associated to an ideal $\wp \subset k[x_1, \ldots, x_n]$, i.e. $V = \mathrm{Z}(\wp) \subset \mathbb{A}^n$. For example $\mathrm{E} : y^2 = x^3 - x$ is defined over $\mathbb{Q}$ but we do geometry over an algebraically closed field. We have a map $\psi : k \to k[x_1, \ldots, x_n]/\wp = A(v)$. This induces a map $\mathrm{Spec}A \to \mathrm{Spec}A(V)$. This holds for any algebraic set V over k, so $\mathrm{Spec}k$ is a common base for all varieties over K. Notice that $\mathrm{Spec}k = \{(0)\}$ a singleton set. So as a map of sets ψ is trivial.

Definition 2.2. Let S be a fixed scheme. A *scheme over* S is a scheme X, together with a morphism $X \to S$. A morphism of X to Y as schemes over S is a morphism that is compatible with the given morphisms to S.

Example 2.3. $X = [x_1, \ldots, x_n]/\wp)$, $Y = \mathrm{Spec}(k[y_1, \ldots, y_m]/\wp'))$, and $S = \mathrm{Spec}k$ with the natural maps $X \to S \leftarrow Y$. Then $(X \to Y \to S) = (X \to S)$ if and only if $(k \to B \to A) = (k \to A)$. That is $f^\sharp$ is a $k-$ algebra homomorphism. So to give $f^\sharp$, it suffices to give $f^\sharp(x_i)$, each of which is a polynomial (mod $\wp$) with coefficients in k.

Remark 2.4. Even though $\mathrm{Spec}k \to \mathrm{Spec}(V(\wp))$ does not give much information, the corresponding map on structure sheaves does!

Example 2.5. E:$y^2 = x^3 - x$ defined over $\mathbb{Q}$ or any extension there of. Consider the map on $\bar{\mathbb{Q}}$ points: $(x, y) \mapsto (-x, iy)$. Check that $f^\sharp(x) = -x$ and $f^\sharp(y) = iy$. So $f^\sharp$ is not a $\mathbb{Q}$-algebra map, but it is a $\mathbb{Q}(i)$-algebra map. So f is defined over $\mathbb{Q}(i)$ but not over $\mathbb{Q}$!

Definition 2.6. Let S be a scheme. Define $\mathrm{Sch}(S)$ to be the category of schemes over S with morphisms, morphisms of schemes over S. Similarly, let k be a field. Let $\mathrm{Var}(k)$ be the category of varieties over k.

Remark 2.7. If k is a field, one often writes k for $\mathrm{Spec}(k)$. Assume $k = \bar{k}$. If V is a variety, we want to see if it is a scheme. A good candidate is $\mathrm{Spec}A(V)$. This has more points than V. Its points are all non-empty irreducible closed subsets of V. This motivates the following definition:

Definition 2.8. Let X be a space.Let $t(X)$ denote the set of non-empty irreducible closed subsets of X. Define a topology on $t(X)$ by letting the closed subsets of $t(X)$ be $t(Y)$ for each closed subset $Y \subset X$. A continuous map $f : X_1 \to X_2$ induces a continuous map between $t(X_i)$ by taking a closed subset of X_1 to the closure of its image under f. We also have a continuous map $\alpha_X : X \to t(X)$ taking points to their closure. See proposition 2.6 in

[H]. Thus a variety is not necessarily a scheme, but may be viewed as on. As a *set* V is the set of all closed points if $t(V)$.

3 Functor of Points

Let $k \subset L$ be fields, L algebraic over k. Let $\wp \subset k[x_1, \dots, x_n]$, $V = \mathrm{Z}(\wp) \subset \mathbb{A}^n$. Suppose $a_1, \dots, a_n \in L$ such that for all $f \in \wp$, $f(a_1, \dots, a_n) = 0 \in L$. Then $p = (a_1, \dots, a_n)$ is a point on V "defined over L". Consider $\phi : k[x_1, \dots, x_n] \to L$ by $x_i \mapsto a_i$. If $f \in \wp$, then $\phi(f(x_1, \dots, x_n)) = 0$. Therefore $f \in \mathrm{Ker}\phi$, that is $\wp \subset \mathrm{Ker}\phi$. So the induced map $\bar{\phi} : k[x_1, \dots, x_n]/\wp \to L$ is a k-algebra homomorphism. Conversely, given such a k-algebra map ϕ, the point $(\phi(x_1), \dots, \phi(x_n))$ is a point on V defined over L; for all $f \in \wp$, $f(\phi(x_1), \dots, \phi(x_n)) = \phi(f(x_1), \dots, f(x_n)) = 0$. So as a set, the L-valued points on V are k-algebra homomorphisms of $k[x_1, \dots, x_n]/\wp$ to L, i.e. $\mathrm{Hom}_{\mathrm{Sch}(k)}(\mathrm{Spec}L, \mathrm{Spec}(k[x_1, \dots, x_n]/\wp))$.