Algebraic Geometry Notes ♯16, 17

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Recall: If A is a ring, the spectrum of A, (Spec$A$, $\mathcal{O}$), is the pair consisting of the topological space Spec$A = \{$ prime ideals of A $\}$ together with the sheaf of rings $\mathcal{O}$ (i.e. the structure sheaf). What should be the morphisms? If f is a continuous map f : Spec$B \to$ Spec$A$, let V be an open subset of Spec$A$, we have an induced map

$$f^\sharp : \mathcal{O}_{SpecA}(V) \to \mathcal{O}_{SpecB}(f^{-1}(V)) = (f_*\mathcal{O}_{SpecB})(V),$$

i.e. $f^\sharp : \mathcal{O}_{SpecA} \to f_*\mathcal{O}_{SpecB}$.

1 Schemes

1.1 Schemes

It turns out that we need some restriction on $f^\sharp$. Suppose $\phi : R \to S$ is a homomorphism of rings, this induces

$$f : SpecS \to SpecR; \quad \mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$$

and $f^\sharp$ is induced by "pullback of regular functions".

Remark: By convention, a homomorphism of rings takes identity to identity. In particular $\phi^{-1}(\mathfrak{p}) \neq R$, because if it were, then $1_R \in \phi^{-1}(\mathfrak{p})$. So $\phi(1_R) = 1_S \in \mathfrak{p}$, a contradiction!

Observation: if $\mathfrak{p} \in$ Spec$S$, then for every open subset V of Spec$R$ containing $f(\mathfrak{p})$. We have a map

$$f^\sharp : \mathcal{O}_{SpecR}(V) \to \mathcal{O}_{SpecS}(f^{-1}(V))$$

This induces a map on direct limit:

$$\begin{aligned}\mathcal{O}_{SpecR,f(\mathfrak{p})} &= \varinjlim_{f(\mathfrak{p})\in V}\mathcal{O}_{SpecR}(V)\\ &\to \varinjlim_{f(\mathfrak{p})\in V}\mathcal{O}_{SpecS}(f^{-1}(V))\\ &\to \varinjlim_{\mathfrak{p}\in U}\mathcal{O}_{SpecS}(U))\\ &= \mathcal{O}_{SpecS,\mathfrak{p}}\end{aligned}$$

Suppose P corresponds to the prime ideal $\mathfrak{p} \in \mathrm{Spec}S$, then $f(\mathfrak{p}) = \phi^{-1}(P)$. Also $\mathcal{O}_{SpecR,f(\mathfrak{p})} = R_{\phi^{-1}(P)}$ and $\mathcal{O}_{SpecS,\mathfrak{p}} = S_P$. The map induced by $f^\sharp$ is

$$R_{\phi^{-1}(P)} \to S_P; \quad (r,s) \mapsto (\phi(r), \phi(s))$$

Note that $f^\sharp(\phi^{-1}(P)) \subseteq P$.

Remark: For affine varieties, the analogous statement is: if $f : X \to Y$ is a morphism of affine varieties and $p \in X$, then we have a pullback map $\mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$. If a regular function g vanishes at $f(p)$ (i.e. g is in $\mathfrak{m}_{f(p)}$), then its pullback $g \circ f$ vanishes at p (i.e. $g \circ f$ is in $\mathfrak{m}_p$). We need to put this restriction on $f^\sharp$

Definition 1.1. If A, B are local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$, respectively, then a homomorphism $\psi : A \to B$ of rings is called a local homomorphism if $\psi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

Remark: $$\begin{aligned}\psi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B &\Leftrightarrow \mathfrak{m}_A \subseteq \psi^{-1}(\mathfrak{m}_B) \subsetneqq A\\ &\Leftrightarrow \mathfrak{m}_A = \psi^{-1}(\mathfrak{m}_B)\end{aligned}$$

Definition 1.2. If A, B are rings, then a morphism from Spec$B$ to Spec$A$ is a pair $(f, f^\sharp)$ consisting of a continuous map $f : \mathrm{Spec}B \to \mathrm{Spec}A$ and a morphism of sheaves $f^\sharp : \mathcal{O}_{SpecA} \to f_*\mathcal{O}_{SpecB}$ such that for each prime ideal $\mathfrak{p} \in \mathrm{Spec}B$, $f^\sharp(f(\mathfrak{p})) \subseteq \mathfrak{p}$.

Proposition 1.3. *(see (a) in Harsthorne Proposition 2.3)*
(b) If $\phi : A \to B$ is a homomorphism of rings, then pullback by ϕ induces a morphism $(f, f^\sharp)$ from SpecB to SpecA.
(c) If A, B are local rings, then any morphism $(f, f^\sharp)$ from SpecB to SpecA arises as a pullback by a homomorphism ϕ from A to B.

Upshot, we have an equivalence of categories:

$$\{rings + ring\ homs\} \leftrightarrow \{spectra\ of\ rings + morphisms\ of\ spectra\}$$

Recall that we already have an equivalence of categories:

$$\{f.\,g.\ domains + ring\ homs\} \leftrightarrow \{affine\ varieties + morphisms\ of\ varieties\}$$

which allows us to "transport commutative algebra into geometry" with "ease".

Proof. (b) we proved everything but continuity of f. For $\phi : A \to B$,

$$f : SpecB \to SpecA; \quad P \mapsto \phi^{-1}(P)$$

A closed set in Spec$A$ is $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of A.

$$\begin{aligned} f^{-1}(V(\mathfrak{a})) &= f^{-1}(\{prime\ ideals\ \mathfrak{q}\ \ of\ A\ that\ contain\ \mathfrak{a}\}) \\ &= \{prime\ ideals\ \mathfrak{p}\ \ of\ B\ such\ that\ \phi^{-1}(\mathfrak{p})\ \supseteq\ \mathfrak{a}\} \\ &= \{prime\ ideals\ of\ B\ that\ contain\ \phi(\mathfrak{a})\} \\ &= V(\phi(\mathfrak{a})), \text{ which is closed in SpecB.} \end{aligned}$$

(c) Now for each open subset $U \subseteq$ Spec$B$, we have

$$f^{\sharp}(U) : \mathcal{O}_{SpecA}(U) \to (f_*\mathcal{O}_{SpecB})(U) = \mathcal{O}_{SpecB}(f^{-1}(U))$$

Take $U =$ Spec$B$, we get a homomorphism

$$\phi : \mathcal{O}_{SpecA}(SpecA) \to \mathcal{O}_{SpecB}(SpecB).$$

Let $\mathfrak{p} \in$ Spec$B$, then $f(\mathfrak{p}) \in$ Spec$A$.

Consider the commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{f(\mathfrak{p})} & \xrightarrow[f^{\sharp}]{} & B_{\mathfrak{p}} \end{array} \qquad \begin{array}{ccc} f(\mathfrak{p}) = \phi^{-1}(\mathfrak{p}) & \longrightarrow & \mathfrak{p} \\ \downarrow & & \downarrow \\ f(\mathfrak{p})A_{f(\mathfrak{p})} & \longrightarrow & \mathfrak{p}B_{\mathfrak{p}} \end{array}$$

Recall that if A is a ring and S is a multiplicatively closed subset, then the map $A \to S^{-1}A$ given by $a \mapsto a/1$ induces a map on ideals given by $I \mapsto S^{-1}I$. This gives a bijection

$$\{ideals\ I\ of\ A\ such\ that\ I \cap S = \emptyset\} \leftrightarrow \{ideals\ of\ S^{-1}A\}$$

In particular, if $\mathfrak{p}$ is a prime ideal and $S = A \backslash \mathfrak{p}$, then

$$\{ideals\ I\ of\ A\ contained\ in\ \mathfrak{p}\} \leftrightarrow \{ideals\ of\ A_{\mathfrak{p}}\}$$

So f is the pullback by ϕ.

Also $f^{\sharp}$ is induced on stalks by ϕ and hence at the level of sheaves as well. □

If A is a domain and S is a multiplicatively closed subset, then there exists a bijection

$$\{ideals\ I\ of\ A\ such\ that\ I \cap S = \emptyset\} \leftrightarrow \{ideals\ of\ S^{-1}A\}$$

Lemma 1.4. *If A is a ring, $\mathfrak{p}$ is a prime ideal and $\phi : A \to A_{\mathfrak{p}}$ is the map $a \mapsto a/1$, then $\phi^{-1}(\mathfrak{p}A_{\mathfrak{p}}) = \mathfrak{p}$.*

Proof. Obviously, $\phi^{-1}(\mathfrak{p}A_{\mathfrak{p}}) \supseteq \mathfrak{p}$
Conversely, if $a \in A$ such that $\phi(a) = a/1 \in \mathfrak{p}A_{\mathfrak{p}}$, then $a/1 = p/s$ for some $p \in \mathfrak{p}$ and $s \notin \mathfrak{p}$.

$$\mathfrak{p}A_{\mathfrak{p}} = \{\sum_i \frac{p_i}{1}\frac{a_i}{s_i} | p_i \in \mathfrak{p}, a_i \in \mathfrak{p}, s_i \notin \mathfrak{p}\}$$

so for some $t \notin \mathfrak{p}, t(as - p) = 0$ i.e. $tas = p \in \mathfrak{p}$.
Since $\mathfrak{p}$ is a prime ideal, $a \in \mathfrak{p}$ □

Recall: If A is a ring, Spec$A$ = prime ideals of A + Zariski topology + structure sheaf. Morphisms: if f is a continuous map f : Spec$B \to$ Spec$A$, let V be an open subset of Spec$A$, and an induced map

$$f^\sharp : \mathcal{O}_{SpecA}(V) \to \mathcal{O}_{SpecB}(f^{-1}(V)) = (f_*\mathcal{O}_{SpecB})(V),$$

i.e. $f^\sharp : \mathcal{O}_{SpecA} \to f_*\mathcal{O}_{SpecB}$. $f^\sharp$ induces a map from $\mathcal{O}_{SpecA,f(\mathfrak{p})}$ to $\mathcal{O}_{SpecB,\mathfrak{p}}$. If we have the extra condition: f^{-1}(maximal ideal of $\mathcal{O}_{SpecA,f(\mathfrak{p})}$) $\subseteq$ maximal ideal of $\mathcal{O}_{SpecB,\mathfrak{p}}$, the morphism is called local homomorphism.

Claim(Danilov): The above condition holds if and only if for each $\mathfrak{p} \in X$ and open set $V \subseteq Y$ with $f(\mathfrak{p}) \in V$ and $S \in \mathcal{O}_{SpecA}$, if $S(f(\mathfrak{p})) = 0$, then $(f^\sharp(S))(\mathfrak{p}) = 0$

Lemma 1.5. *If $X = SpecA$ for a ring A, $P \in X$ and $U \subseteq X$ is open such that $P \in U$, $S \in \mathcal{O}_X$, then $S \notin \{$ maximal ideal of $\mathcal{O}_P$ $\}$ if and only if $S(P) \neq 0$.*

Proof. There exist $g, h \in A$ with h nowhere zero on some open set V such that $P \in V \subseteq U$ such that $S = g/h$ on V.
($\Rightarrow$) If $S(P) \neq 0$, then $g(P) \neq 0$ and so S is invertible in $V \bigcap D(g) = X - V(g) \ni P$ so S is invertible in $\mathcal{O}_P \cong A_{\mathfrak{p}}$, where $\mathfrak{p} \subseteq A$ is the prime ideal corresponding to P. So $S \notin \{$ maximal ideal of $\mathcal{O}_P\}$
($\Leftarrow$) Suppose $S \notin \{$ maximal ideal $\}$, then it is invertible on some open $V \ni P$, which contradicts $S(P) = 0$. □

$$\{rings{+}ring\ homs\} \leftrightarrow \{spectra\ of\ rings{+}morphisms\ satisfyextracondition\}$$

Definition 1.6. A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$, where $f : X \to Y$ is a continuous map and $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a map of sheaves on Y.

Remark: A manifold is a topological space which "locally looks like discs in $\mathbb{R}^n$, for some n", i.e. for each $P \in X$, there exists a neighborhood U of P that is homeomorphic to a disc in $\mathbb{R}^n$.

Definition 1.7. The ringed space $(X, \mathcal{O}_X)$ is said to be a locally ringed space if for each $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring.

Example 1.8. If A is a ring, then $(SpecA, \mathcal{O}_{SpecA})$ is a locally ringed space. (Since if $\mathfrak{p} \in \mathrm{Spec}A$, then $\mathcal{O}_{SpecA,\mathfrak{p}} = A_\mathfrak{p}$)

Definition 1.9. A morphism of locally ringed spaces is a morphism $(f, f^\sharp)$ of ringed spaces such that for each $P \in X$, the map of local rings $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$ induced by $f^\sharp$ is a local homomorphism of local rings. An isomorphism is a morphism with a two-sided inverse.

Example 1.10. Morphisms of spectra of rings are morphisms of locally ringed spaces.

Definition 1.11. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$,(where $\mathcal{O}_X|_U$ is the sheaf on U given by $\mathcal{O}_X|_U(V) = \mathcal{O}_X(V)$, for open $V \subseteq U$) is isomorphic as a locally ringed space to the spectrum of some ring. Sometimes we refer to $(X, \mathcal{O}_X)$ by just $X = (SpX, \mathcal{O}_X)$. Here, X is called the underlying topological space of $(X, \mathcal{O}_X)$ and sometimes denoted SpX, read "space of X" and $\mathcal{O}_X$ is called the structure sheaf of $(X, \mathcal{O}_X)$. A morphism of schemes is a morphism as a locally ringed space.

Remark: Analog of an abstract algebraic set is a locally ringed space $(X, \mathcal{O}_X)$ such that every point $P \in X$ has an open neighborhood U such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(Y, \mathcal{O}_Y)$ for an affine variety Y.

Example 1.12. $\mathbb{P}^n$ is an abstract algebraic set, we will have an anolog of $\mathbb{P}^n$ for schemes.