Algebraic Geometry I Lectures 14 and 15

October 22, 2008

Recall from the last lecture the following correspondences

$$\{\text{points on an affine variety } Y\} \longleftrightarrow \{\text{maximal ideals of } A(Y)\}$$

$$\begin{array}{ccc} Spec A & \longleftrightarrow & A \\ \mathcal{P} \in Z(\mathfrak{a}) & \longleftrightarrow & \text{maximal ideal } \supseteq \mathfrak{a} \end{array}$$

Also from the last time, recall that the closed subsets defined as

$$V(\mathfrak{a}) = \{\mathcal{P} \subseteq A | \mathcal{P} \text{ is a prime ideal of } A \text{ and } \mathfrak{a} \subseteq \mathcal{P}\}$$

gives a Zariski topology on $Spec A$.

$$\{\text{closed irreducible subsets of } Y\} \longleftrightarrow \{\text{prime ideals of }\} A(Y)$$

So if Y is an affine variety, then $Spec\,(A(Y))$ contains more elements than Y.

What is the *"geometry"* of $A(Y)$?

Example 0.1. $Y = \mathbb{A}^n$, then elements of $Spec\,(A(Y)) = k[X_1, \ldots, X_n]$ are prime ideals in $k[X_1, \ldots, X_n]$ that contains irreducible polynomials in $k[X_1, \ldots, X_n]$. These latter elements of $Spec\,(A(Y))$ can be visualized as $Z(f)$ where f is irreducible.

Definition 0.2. Let A be a ring. A point $\mathcal{P} \in Spec A$ is closed if and only if $\mathcal{P} = V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ in A.

Remark 0.3.

1. In the above definition, since $\mathcal{P}$ is the only prime ideal that contains $\mathfrak{a}$, $\mathfrak{a}$ is maximal. Therefore closed point in $Spec A$ correspond to maximal ideals of A.

2. In particular, closed points in $Spec\,(A(Y))$ correspond to usual points in Y.

Fact 0.4. Closure of $\mathcal{P} \in SpecA$ is $Z(\mathcal{P})$

Proof. $\mathcal{P} \in Z(\mathfrak{a}) \Leftrightarrow \mathfrak{a} \subseteq \mathcal{P} \Leftrightarrow Z(\mathfrak{a}) \supseteq Z(\mathcal{P})$ □

Definition 0.5. If X is a topological space and Z is an irreducible closed subset then a geometric point for Z is a point $P \in Z$ such that $\overline{P} = Z$

Example 0.6. If R is a domain, so that (0) is prime then $\overline{(0)} = Z((0)) =$ prime ideal that contains $0 = SpecR$

Moral is that instead of looking for geometric intuition, it is often better to work abstractly using algebra.

What should be the morphisms?

We need an analog of regular functions for $SpecA$ for any ring A.

Recall that for an affine variety $Y \subseteq \mathbb{A}^n$ and $U \subseteq Y$ open, $f : Y \rightarrow k$ is regular on U if for any $P \in U$ there exists an open $P \in V \subseteq U$ and $g, h \in k[X_1, \ldots, X_n]$ with $h \neq 0$ on V such that $f = \frac{g}{h}$ on V. Also note that, if f is regular on U then for all $P \in U$, $f \in \mathcal{O}_P = A(Y)_{\mathfrak{m}_P}$. In other words, at each point $P \in U$, f gives an element of $A(Y)_{\mathfrak{m}_P}$ therefore f gives a map

$$\begin{array}{ccc} U & \xrightarrow{f} & \prod A(Y)_{\mathfrak{m}_P} \\ P & \longmapsto & f(P) \end{array}$$

where $f(P) \in A(Y)_{\mathfrak{m}_P}$.

We want to generalize this to $SpecA$. There is no obvious single field k where functions on $SpecA$ should take values. We reverse the process above as follows;

$$\begin{array}{ccc} \text{points} & \longleftrightarrow & \text{prime ideals } \mathcal{P} \\ \text{values in } A(Y)_{\mathfrak{m}_P} & \longleftrightarrow & \text{values in } A_{\mathcal{P}} \end{array}$$

Definition 0.7. If A is a ring and $S \subseteq A$ is a multiplicatively closed (i.e $1 \in S$ and $a, b \in S$ then $ab \in S$) then $S^{-1}A$ is the set of equivalence classes of $(a, b) \in A \times S$ such that $(a_1, s_1) \sim (a_2, s_2)$ if $\exists b \neq 0 \in A$ such that $b(a_1 s_2 - a_2 b_1) = 0$

This motivates the following

Definition 0.8. If $U \subseteq SpecA$ is open, then *regular function* on U is a function

$$\begin{array}{ccc} U & \xrightarrow{f} & \prod_{\mathcal{P}\in U} A_{\mathcal{P}} \\ P & \longmapsto & f(P) \end{array}$$

where $f(P) \in A_{\mathcal{P}}$ such that f is locally a quotient of elements of A.

In other words, for all $\mathcal{P}$ in U, there exists an open V with $\mathcal{P} \in V \subseteq U$ and $g, h \in A$ with for all $\mathcal{P} \in V$ $h \notin \mathcal{P}$ such that $f(\mathcal{P}) = \frac{g}{h}$ in $A_{\mathcal{P}}$ for all $\mathcal{P}$.

We can add and multiply regular functions on an open set. It is easy to see that the assignment of regular functions on an open set U gives a presheaf of rings on A denoted $\mathcal{O}_{SpecA}$ or $\mathcal{O}$. This presheaf satisfies the following;

(*) If U is an open subset and I is a set such that for all i there exists open $V_i \subseteq U$ such that $U = \bigcup_{i\in I} V_i$ (i.e $\{V_i\}_{i\in I}$ is an open cover of U) and $\forall i \in I$, $s_i \in \mathcal{O}(V_i)$ such that

$$\forall i, j \quad s_{i|_{V_i\cap V_j}} = s_{j|_{V_i\cap V_j}}$$

then there exists unique $s \in \mathcal{O}(U)$ such that $s_{|V_i} = s_i$ for all i.

Definition 0.9. A presheaf, like $\mathcal{O}$ above, that satisfies $(*)$ is called a sheaf.

Definition 0.10. $\mathcal{O}_{SpecA}$ is called the structure sheaf of $SpecA$.

So far we have associated to ring A the space $SpecA$ where the topology is defined by the closed sets

$$V(\mathfrak{a}) = \{\mathcal{P} \subseteq A | \mathcal{P} \text{ is a prime ideal of } A \text{ and } \mathfrak{a} \subseteq \mathcal{P}\}$$

for any $\mathfrak{a}$ ideal of A.

Example 0.11. For any field k, $Spec(k) = \{(0)\}$

Remark 0.12. It can happen that for two distinct rings A_1 and A_2, $SpecA_1 = SpecA_2$ as sets even though *"geometrically"* they are different.

Example 0.13. If k is an algebraically closed field then $Spec(k[X]/(X^n))$ consists of a single point.

Proof. Consider $\phi : k[X] \to k[X]/(X^n)$. Let $\mathcal{P}$ is a prime ideal in $k[X]/(X^n)$ containing (X^n). Then $X^n \in \phi^{-1}(\mathcal{P})$, $X \in \phi^{-1}(\mathcal{P})$. So $(X) \subseteq \phi^{-1}(\mathcal{P})$. But (X) is maximal then $\phi^{-1}(\mathcal{P}) = (X)$. The structure sheaf sees the geometry $\mathcal{O}_{Spec(k[X]/(X^n))} = A_{(X)}" = " \frac{a_0+a_1x+\ldots+a_{n-1}x^{n-1}}{b_0+b_1x+\ldots+b_{n-1}x^{n-1}}$ where $b_0 \neq 0$. □

Example 0.14. $X^2 = 0$ in $\mathcal{O}_{Spec(k[X]/(X^n))}((X))$ for $n = 2$ but not for $n = 3$.

Definition 0.15. The spectrum of ring A is the pair $(SpecA, \mathcal{O}_{SpecA})$

Definition 0.16. If $f \in A$ define the open subset $D(f) = SpecA \setminus V((f))$ where (f) is the ideal generated by f.

Example 0.17. Let $Y = \{(X,Y) \in \mathbb{A}^2 | \quad Y = X^2\}$ be a variety and $f(X,Y) = X - Y$. Then $D(f) = Y \setminus Z(f)$.

Let X be a set with topology. Recall that a collection $\mathcal{C}$ of open sets in X is said to be basis for the topology on X if for all $P \in X$ and $P \in U \subseteq X$ open there exists $U' \in \mathcal{C}$ such that $P \in U' \subseteq U$.

Example 0.18. In $\mathbb{R}^n$, $B_\epsilon = \{P \in \mathbb{R}^n | \quad |P| < \epsilon\}$ forms a basis, for various ϵ.

Lemma 0.19. *The $D(f)$'s form a basis for the Zariski topology on $SpecA$*

Proof. Homework □

Proposition 0.20.

(a) *For all $\mathcal{P} \in SpecA$, $\mathcal{O}_\mathcal{P} \simeq A_\mathcal{P}$*

(b) *For all $f \in A$, $\mathcal{O}(D(f)) \simeq A_f$ where $A_f = S^{-1}A$ with $S = \{1, f, f^2, ...\}$*

(c) *In particular $\mathcal{O}(Spec(A)) \simeq A$*

Proof. See Hartshorne □

Remark 0.21.

1. We could have defined the Zariski topology on $SpecA$ as the one generated by the $D(f)$'s(i.e $U \subseteq SpecA$ is called open if for all $P \in U$ there exists f such that $P \in D(f) \subseteq U$)

2. We could also have built the structure sheaf $\mathcal{O}$ by using $\mathcal{O}(D(f)) = A_f$ and forcing the sheaf axioms.

Now by the proposition, we have a morphism

$$\begin{array}{ccc} \{\text{spectra of rings}\} & \longleftrightarrow & \{\text{rings}\} \\ (SpecA, \mathcal{O}_{SpecA}) & \longleftrightarrow & \mathcal{O}_{SpecA}(SpecA) \end{array}$$

What should be the morphisms between spectra of rings?

For varieties, a morphism was a continous map such that pullbacks of regular functions are regular.

If $f : X \to Y$ is a morphism of varieties and $V \subseteq Y$ is open, then we have a map on Y defined as;

$$\text{for any } V \subseteq Y \text{ open, } f_*\mathcal{O}_X(V) := \mathcal{O}_X(f^{-1}(V))$$

Definition 0.22. If $f : X \to Y$ is a continous map of topological spaces and $\mathcal{F}$ is a sheaf on X, define the direct image of $\mathcal{F}$, denoted $f_*\mathcal{F}$, on Y by

$$\text{for any } V \subseteq Y \text{ open, } (f_*\mathcal{F})(V) := \mathcal{F}(f^{-1}(V))$$

Homework 0.23. Verify that $f_*\mathcal{F}$ is a sheaf

So if $f : X \to Y$ is a morphism of varieties then for all $V \subseteq Y$ open, there is a homomorphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V)) = f_*\mathcal{O}_X(V)$

Definition 0.24. If $\mathcal{F}$ and $\mathcal{G}$ are presheaves(resp. sheaves) on topological space Y then a morphism $\phi : \mathcal{F} \to \mathcal{G}$ of presheaves (resp. sheaves) consists of homomorphism $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open subset U of X such that if $V \subseteq U$ an inclusion of open subsets of Y then the diagram

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ {\scriptstyle \rho_{UV}} \downarrow & & \downarrow {\scriptstyle \rho'_{UV}} \\ \mathcal{F}(V) & \xrightarrow[\phi(V)]{} & \mathcal{G}(V) \end{array}$$

commutes.

Definition 0.25. A morphism of sheaves is an isomorphism if it has a *"two-sided"* inverse.

Example 0.26. If $f : X \to Y$ is a morphism of varieties then f induces a morphism of sheaves $\mathcal{O}_Y \to f_*\mathcal{O}_X$ which is denoted $f^\sharp$. For varieties f determines $f^\sharp$ but not for spectra of rings.

Example 0.27. Consider

$$\begin{array}{ccc} k[X]/(X^2) & \xrightarrow{\phi} & k[X]/(X^2) \\ X & \longmapsto & 2X \\ (\frac{X}{2}) & \longleftarrow & (X) \end{array}$$

with $char(k) \neq 2$. On points in $Spec\left(k[X]/(X^2)\right)$ the map ϕ is identity but on structure sheaf it is different. So for spectra of rings, f and $f^\sharp$ has to be specified.