Algebraic Geometry
Lectures 11 and 12
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Recall: If Y is a variety we a sheaf of regular functions $\mathcal{O}(U)$. If $p \in Y$ we had a local ring $\mathcal{O}_p \ni (U \ni p, f)$ if Y is affine we have that $\mathcal{O}(Y) \supseteq A(Y) = k[x_1, x_2, \ldots, x_n]/I(Y)$ and $A(Y)_{\mathfrak{m}_p} \cong \mathcal{O}_p$.

1.1 Morphisms

1.1.1 Varieties

Suppose Y is a variety. Consider functions F which are regular on some open subset (depending on f). These are pairs (U, f) such that $U \subseteq Y$ is open and nonempty and f is regular on U. We say that $(U, f) \sim (V, f)$ if $f = g$ on $U \cap V$.

Definition 1.1.1. Denote the set of such equivalence classes by $K(Y)$. Since the intersection of any two nonempty open subsets in Y is an open nonempty subset we can make $K(Y)$ into a ring.

Example 1.1.2. $(U, f) + (V, g) = (U \cap V, f + g)$ and $(U, f) \cdot (V, g) = (U \cap V, f \cdot g)$ But now we can also invert. If $f \neq 0$ is regular on U then $\frac{1}{f}$ is regular on $V = U \cap (Y \setminus \{f \neq 0\}) \neq \emptyset$ and so $(U, f)^{-1} = (V, \frac{1}{f})$.

Definition 1.1.3. $K(Y)$ is a field, called the function field of Y.

Remark 1.1.4. If Y is a variety and $p \in Y$ then there are natural inclusions.

$$\begin{array}{rcl} \mathcal{O}(Y) \to & \mathcal{O}_p & \to K_Y \\ f \mapsto & (Y, f) & \\ & (U, f) & \mapsto (U, f) \end{array}$$

These 3 objects depend only on the isomorphism class of Y, thus they are invariants of Y

Theorem 1.1.5. *Let $Y \subset \mathbb{A}^n$ be an affine variety then:*

1. $\mathcal{O}(Y) \cong A(Y)$

2. *We have a one to one correspondence*

$$\begin{array}{rcl} \{ \textit{ points of } Y\} & \leftrightarrow & \{ \textit{ maximal ideals of } A(Y)\} \\ p & \mapsto & \mathfrak{m}_p = \{f \in \mathbb{A}(Y) \Big| f(p) = 0\} \end{array}$$

3. $\forall p \ \mathcal{O}_p \cong A(Y)_{\mathfrak{m}_p}$

4. *$K(Y)$ is the quotient field of $A(Y)$.*

Proof. (3) is done by previous result.
(2) A point in $\mathbb{A}^n$ is a minimal irreducible algebraic set. So points in $\mathbb{A}^n \leftrightarrow$ maximal ideals of $k[x_1, x_2, \ldots, x_n]$ (Under $X \mapsto I(X)$). Irreducible sets go to $I(X)$. If $Y \subseteq \mathbb{A}^n$ is an algebraic set then points in $Y \leftrightarrow$ maximal ideals of $k[x_1, x_2, \ldots, x_n]$ that contain $T(Y)$. (maximal ideals of $A(Y)$)
(3) Any element f in the quotient field of $A(Y)$ is regular on some nonempty open $U \subseteq Y$ map this to (U, f) in $K(Y)$. Conversely, any element of $K(Y)$ is regular at some point $p \in Y$ so it is in $\mathcal{O}_p$, i.e. in $A(Y)_{\mathfrak{m}_p}$ which is contained in the quotient field of $A(Y)$.

$$\left(\frac{f}{g} \in A(Y) \quad \text{with} \quad f \in A(Y), \quad g \in \mathfrak{m}_p \right)$$

(1) Notice $A(Y) \subseteq \mathcal{O}(Y)$ (showed previously)

$$\mathcal{O}(Y) = \bigcap_{p \in Y} \mathcal{O}_p =_{\text{part c}} \bigcap_{p \in Y} A(Y)_{\mathfrak{m}_p} = A(Y)$$

By the exercise and below and part (2) above the proof will be complete. ■

Exercise If B is a domain then B is the intersection (contained in its quotient field) of the localizations of B at all maximal ideals of B.

1.1.2 Projective Varieties

Proposition 1.1.6. *$\forall\ i = 0, 1, \ldots, n$ let $u_i = \{x_i \neq 0\} \subseteq \mathbb{R}^n$. The map $\varphi_i : u_i \to \mathbb{A}^n$ via $[x_0 : x_1 : \cdots : x_n] \mapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \frac{x_n}{x_i}\right)$ is an isomorphism of varieties.*

Proof. We checked that this is a homeomorphism. The regular functions correspond by homogenization and de-homogenization. ■

Remark 1.1.7. $\mathcal{O}_p$

Theorem 1.1.8. *Let Y be a projective variety in $\mathbb{P}^n$ then:*
(a) $\mathcal{O}(Y) = k$
(c) $K(Y)$ = set of elements in the quotient field of $k[x_1, \ldots, x_n]/I(Y)$ that are ratios of homogeneous polynomials of some degree.

Proof. (c) follows from 1.1.6
(a) see [Hartshore] ■

Remark 1.1.9. (a) in 1.1.8 is the analog of the only holomorphic functions on $\mathbb{C}$ (bounded) on a compact Riemann surface are constants.

Remark 1.1.10. we have a map:

$$\begin{array}{rcl} \text{Affine varieties} & \rightarrow & \text{Rings} \\ Y \subseteq \mathbb{A}^n & \mapsto & A(Y) = k[x_1, \ldots, x_n]/I(Y) \end{array}$$

$k[x_1, \ldots, x_n]$ is finitely generated over k. image $\subseteq$ finitely generated domain over k. Conversely, suppose B is a finitely generated domain over k. Suppose there are n generators then $\exists \varphi : k[x_1, \ldots, x_n] \rightarrow B$ then $\ker \varphi = \mathfrak{a}$ is a prime ideal. Take $Y = Z(\mathfrak{a}) \subseteq \mathbb{A}^n$ then $I(Y) = I(Z(\mathfrak{a})) = \mathfrak{a}$ thus, $A(Y) = k[x_1, \ldots, x_n]/I(Y) = k[x_1, \ldots, x_n]/\mathfrak{a} = B$

Now we have a 1-1 correspondence:

$$\text{Affine varieties} \leftrightarrow \text{finietly generated domains over } k$$

Proposition 1.1.11. *Let X, Y be Affine varieties, then $\exists$ 1-1 correspondence*

$$\begin{array}{rcl} \{\textit{morphisms } X \rightarrow Y\} & \leftrightarrow & \{\textit{homomorphisms of } k \textit{ algebras } A(Y) \rightarrow A(X)\} \\ \varphi & \mapsto & (f \rightarrow f\varphi) \end{array}$$

Proof. See [Hartshore] ■

Definition 1.1.12. If $\varphi : X \rightarrow Y$ is a morphism of affine varieties then define $\varphi^* : A(Y) \rightarrow A(X)$ as $\varphi^*(f) = f\varphi$ called pullbacks of φ

Example 1.1.13. Consider $\varphi : \mathbb{A}^1 \rightarrow Y = \{y = x^2\} \subseteq \mathbb{A}^2$ via $x \mapsto (x, x^2)$ then
$A(X) = k[x]$ and $A(Y) = k[x, y]/(y - x^2) \ni \overline{X}, \overline{Y}$

Where $\overline{X}$ = $x-$coordinate on $Y \mapsto X$ on X and $\overline{Y}$ = $y-$coordinate on $Y \mapsto x^2$ on X

What is φ^* ? $\varphi^*(\overline{x}^2 + 2\overline{y}) = x^2 + 2x^2 = 3x^2$

How do we recover φ ? $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ via $x \mapsto x$ coordinate $= x$, y coordinate $= x^2$ i.e. (x, x^2)

Corollary 1.1.14. *We have an "arrow reversing" equivalence of categories:*

$$\begin{array}{rcl} \{\textit{affine varieties over } k\} & \leftrightarrow & \{\textit{ finitely generated domains over } k\} \\ X \mapsto A(X) & & \varphi \mapsto \varphi^* \end{array}$$

Chapter 2

Schemes

2.1 Schemes

2.1.1 Motivation

Motivation for going from varieties to schemes (for a different motivation see [Danilov]).

In 1.1.14 we restricted to irreducible affine algebraic sets. Non irreducible sets are also important however.

Example 2.1.1. The intersection of two irreducible sets need not be irreducible. For $k = \mathbb{C}$ take $Y_1 := y = x^2$ and $Y_2 := y = 4$ then $Y_1 \cap Y_2 = \{(-2, 4), (2, 4)\} \subseteq \mathbb{A}^2$ which is closed and is NOT reducible.

For a general affine algebraic set $X \subseteq \mathbb{A}^n$ we still have $A(X) = k[x_1, x_2, \ldots, x_n]/I(X)$

Definition 2.1.2. An element r in a ring R is **nilpotent** if $r^n = 0$ for some $n \in \mathbb{N}$.

Then $A(X)$ has no non-zero nilpotent elements.

In fact if $\mathfrak{a}$ is an ideal of a ring R then:

$$\begin{aligned} \mathfrak{a} \text{ is radical} \quad &\Longleftrightarrow \quad R/\mathfrak{a} \text{ has no non-zero nilpotents} \\ &\Longleftrightarrow \quad \forall\, x \in R \ \ (x + \mathfrak{a})^n = 0 + \mathfrak{a} \quad \text{then } x + \mathfrak{a} = 0 + \mathfrak{a} \\ &\Longleftrightarrow \quad \forall\, x \in R \ \ x^n \in \mathfrak{a} \\ &\Rightarrow \quad x \in \mathfrak{a} \end{aligned}$$

Definition 2.1.3. A ring R is **reduced** if it has no non-zero nilpotent elements

Remark 2.1.4. domain implies reduced

So $A(X)$ is a finitely generated reduced $k-$algebra. Conversely, if B is a finitely generated reduced $k-$algebra then $\exists\ k[x_1,\ldots,x_n] \twoheadrightarrow B$ call it's kernel $\mathfrak{a}$ then $\mathfrak{a}$ is radical and $Y = Z(\mathfrak{a})$ satisfies $I(Y) = \mathfrak{a}$ so

$$\{\text{affine algebraic sets over } k\} \leftrightarrow \{\text{finitely generated reduced } k-\text{algebras}\}$$

But restricting to reduced rings is not enough.

Example 2.1.5. Consider the family $\{y^2 = x\ ,\ x = c\} \subseteq \mathbb{A}^2_{\mathbb{C}}$ depending on $c \in \mathbb{C}$ if $c \neq 0\ \exists$ two solutions to $y^2 = c$ this corresponds to the coordinate ring $k[x,y]/(y^2 - x\ ,\ x - c) \cong k[y]/(y^2 - c)$. If $c = 0$ there is only one point. This corresponds to the coordinate ring $k[x,y]/(y^2 - x\ ,\ x) \cong k[y]/(y^2)$ In the latter the image $\overline{y}$ of y satisfies $(\overline{y})^2 = 0$

"As $c \rightarrow 0$ the two points get 'closer' in the limit, we get a double point and the ring $k[y]/(y^2)$ remember this."

This example shoes we want to include non-reduced rings as well.

Also why restrict to k being algebraically closed?

Example 2.1.6. We may be interested in integer solutions to $y^2 = x^3 - x$. $(1,0)$ is an integral point on it which is needed in number theory. Grothendieck: In 1-1 correspondence

$$\begin{array}{rcl} \{\text{affine varieties }\} & \hookrightarrow & \{\text{ finitely generated domains over } k\} \\ Y & \mapsto & A(Y) \end{array}$$

Replace right hand side by any ring R and define a "geometric object" on the left hand side (called Spec$R$) and get an equivalence of categories.

2.1.2 First attempt

Recall that if Y is an affine variety then

$$\begin{array}{rcl} \{\text{points of } Y\} & \leftrightarrow & \{\text{maximal ideals of } A(Y)\} \\ P & \mapsto & \{f \in A(Y) \mid f(p) = 0\} \end{array}$$

Thus if R is a ring define geometric object associated to it as $\text{Spec}_{\mathfrak{m}}R = \{$ maximal ideals of $R\}$

What about a topology on $\mathrm{Spec}_{\mathfrak{m}} R$? If $Y = \mathbb{A}^n$ and $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ then

$$\begin{aligned} Z(\mathfrak{a}) &= \{p \in \mathbb{A}^n \,\Big|\, f(p) = 0 \;\forall\; f \in \mathfrak{a}\} \\ &= \{p \in \mathbb{A}^n \,\Big|\, f \in \mathfrak{m}_p \;\forall\; f \in \mathfrak{a}\} \\ &= \{p \in \mathbb{A}^n \,\Big|\, p \in \mathfrak{a} \subseteq \mathfrak{m}_p\} \end{aligned}$$

So if R is a ring and $\mathfrak{a}$ is an ideal define
$Z(\mathfrak{a}) = \{\mathrm{maximal ideals}\,\mathfrak{m}\,\mathrm{such that}\,\mathfrak{a} \subseteq \mathfrak{m}\}$
It is an easy check to verify that this forms a topology, taking closed sets as $Z(\mathfrak{a})$ for some ideal $\mathfrak{a}$ gives a topology.

If $R = A(Y)$ for some affine variety Y then $\mathrm{Spec}_{\mathfrak{m}} R = Y$ with the Zariski topology.

But still there is a problem. Let X and Y be affine varieties and $\psi : A(X) \leftarrow A(Y)$ be a homomorphism of rings, then ψ^{-1} :$\mathrm{Spec}_{\mathfrak{m}} A(X) \to \mathrm{Spec}_{\mathfrak{m}} A(Y)$ via $\mathfrak{m} \mapsto \psi^{-1}(\mathfrak{m})$
Fact: $\psi^{-1}(\mathfrak{m})$ is a maximal ideal.
This is precisely the map $X \to Y$ that is induced by Y

If $g \in A(Y)$ is in $\mathfrak{m}_p$ then $\psi(g) \in I(p)$ and $I(p) = \mathfrak{m}_p$. Thus $\mathfrak{m}_{\varphi(p)} \subseteq \psi^{-1}(\mathfrak{m}_p)$

Lemma 2.1.7. *If A and B are finitely generated domains over k, $\varphi : A \to B$ is a homomorphism, and $\mathfrak{b}$ is a maximal ideal in B then $\varphi^{-1}(\mathfrak{b})$ is a maximal ideal. Thus $\psi^{-1}(\mathfrak{m})$ is a maximal in definition of ψ^{-1} and $\mathfrak{m}_{\varphi(p)} = \psi^{-1}(\mathfrak{m}_p)$*

Remark 2.1.8. If R and S are rings and $\varphi : R \to S$ is a homomorphism we want to define $\mathrm{Spec}_{\mathfrak{m}} S \to \mathrm{Spec}_{\mathfrak{m}} R$ via $\mathfrak{m} \mapsto \varphi^{-1}(\mathfrak{m})$ but $\varphi^{-1}(\mathfrak{m})$ need not be maximal!

Example 2.1.9. $\varphi : \mathbb{Z} \to \mathbb{Q}$ by $\varphi^{-1}(0) = (0)$ which is not maximal.

But observe pull backs of prime ideal are prime.

Proof. Indeed, if $P \subseteq S$ is prime $a, b \in R$ such that $ab \in \varphi^{-1}(P)$ then $\varphi(ab) = \varphi(a)\varphi(b) \in P$ since P is prime we have either $\varphi(a) \in P$ or $\varphi(b) \in P$ thus $a \in \varphi^{-1}(P)$ or $b \in \varphi^{-1}(P)$ ■

Instead of $\mathrm{Spec}_{\mathfrak{m}} R$ consider $\mathrm{Spec} R = \{$ prime ideals of $R\}$

Definition 2.1.10. For an ideal $\mathfrak{a}$ define $V(\mathfrak{a}) = Z(\mathfrak{a}) = \{$ prime ideals that contain $\mathfrak{a}\}$

Bibliography

[Hartshore] Robin Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics. Springer-Verlag, 1977. The standard text in English in this field.

[Danilov] Danilov. *Unknown Title*. Unknown details.