8.3
Complex Numbers
eGrade question #147-153

$z = x + yi$ is complex number in standard form
The graph of z is the point (x, y).
x is the "real part" of z and y is the "imaginary part" of z

Plot the points in the complex plane:

A) 3 B) $-2i$ C) $3 - 2i$

The distance from the origin to z is called the modulus or magnitude of z and is found by:

$$|z| = \sqrt{x^2 + y^2}$$

The polar form of z is found using $x = r\cos\theta$ and $y = r\sin\theta$:

$$z = r\cos\theta + i\, r\sin\theta = r(\cos\theta + i\, r\sin\theta)$$

θ is called the argument of z and r is the magnitude of z ($r = \sqrt{x^2 + y^2}$)

NOTE: you will see many values for θ but you will use $0 \le \theta \le 2\pi$ when you are asked to find θ. Also, when referring to complex numbers you will use $r \ge 0$.

EXAMPLES: Plot z and write z in polar form.

1) $z = 3 - 3i$

2) $z = 3i$

3) $z = \sqrt{3} - i$

4) Plot and write in standard form $z = 2(\cos 30° + i\sin 30°)$

PRODUCTS AND QUOTIENTS

$z_1 = r_1(\cos \theta_1 + i\sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i\sin \theta_2)$

PRODUCT OF 2 COMPLEX NUMBERS: $z_1 z_2 = r_1 r_2[\cos (\theta_1+\theta_2) + i\sin (\theta_1+\theta_2)]$

proof: $r_1(\cos \theta_1 + i\sin \theta_1) \times r_2(\cos \theta_2 + i\sin \theta_2)$

$= r_1 r_2(\cos \theta_1 + i\sin \theta_1)(\cos \theta_2 + i\sin \theta_2)$

$= r_1 r_2(\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + i^2 \sin \theta_1 \sin \theta_2)$

$= r_1 r_2(\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 + (-1)\sin \theta_1 \sin \theta_2)$

$= r_1 r_2(\cos \theta_1 \cos \theta_2 + i \cos \theta_1 \sin \theta_2 + i \sin \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)$

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QUOTIENT OF 2 COMPLEX NUMBERS: $z_2 \neq 0$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

EXAMPLES

Given $z = 3 [\cos (-\pi/2) + i\sin (-\pi/2)]$ and $w = 2 [\cos (2\pi/3) + i\sin (2\pi/3)]$.

1) Find zw.

2) Find $\frac{z}{w}$.

3) Find $\frac{w}{z}$.

POWERS

$z^n = r^n(\cos n\theta + i\sin n\theta), n \geq 1$ (you can use an inductive proof to prove this one)
n is a positive integer
Recall meaning of $x^{1/n}$
EXAMPLES

1) If $z = \sqrt{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ find z^8.

2) If $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ find z^5.

FINDING COMPLEX ROOTS

Let $z = r(\cos\theta + i\sin\theta)$. if $z \neq 0$, there are n distinct complex roots of z, given by the formula:

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$ where k = 0, 1, 2, . . ., n−1.

EXAMPLE 1: Find all the values of $z^{1/2}$ for z = 16i.

Solution:

1st you need to find r and θ for z. Graph these and it is easy to see what these values are. (Remember: the standard form of z = 0 + 16i which is plotted as (0,16)).

So r = 16 and θ = π/2. Also note that for this problem n = 2 (since we are finding square roots).

There will be TWO square roots of any complex number.

To use the formula above we will need to find θ/n. And we will use k=0 to find the 1st root and k=1 to find the 2nd.

$$\frac{\theta}{n} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{2} \bullet \frac{1}{2} = \frac{\pi}{4}$$

Using k = 0, the values we have already found, and plugging into the formula we get:

$$z_0 = \sqrt{16}\left(\cos\left(\frac{\pi}{4} + \frac{2(0)\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(0)\pi}{2}\right)\right) = 4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Using k=1, the values we have already found, and plugging into the formula we get:

$$z_1 = \sqrt{16}\left(\cos\left(\frac{\pi}{4} + \frac{2(1)\pi}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{2(1)\pi}{2}\right)\right) = 4\left(\cos\left(\frac{\pi}{4} + \frac{4\pi}{4}\right) + i\sin\left(\frac{\pi}{4} + \frac{4\pi}{4}\right)\right) = 4\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$$

EXAMPLE 2: Find all the values of $z^{1/3}$ for z = 16i.

Note the r & θ are the same as in the first example. BUT, now we are finding cube roots so n = 3.

Also, there will be THREE cube roots of any complex number. So we will use the formula again making the change in "n" and using k=0, k=1, and k=2.

r = 16, and θ = π/2, and n = 3

$$\frac{\theta}{n} = \frac{\frac{\pi}{2}}{3} = \frac{\pi}{2} \bullet \frac{1}{3} = \frac{\pi}{6}$$

Using k = 0, the values we have already found, and plugging into the formula we get

$$z_0 = \sqrt[3]{16}\left(\cos\left(\frac{\pi}{6} + \frac{2(0)\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2(0)\pi}{3}\right)\right) = 2\sqrt[3]{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

Using k=1, the values we have already found, and plugging into the formula we get:

$$z_1 = \sqrt[3]{16}\left(\cos\left(\frac{\pi}{6} + \frac{2(1)\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2(1)\pi}{3}\right)\right) = 2\sqrt[3]{2}\left(\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right)\right) =$$

$$2\sqrt[3]{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$$

Using k=2, the values we have already found, and plugging into the formula we get:

$$z_2 = 2\sqrt[3]{2}\left(\cos\left(\frac{\pi}{6} + \frac{2(2)\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2(2)\pi}{3}\right)\right) = 2\sqrt[3]{2}\left(\cos\left(\frac{9\pi}{6}\right) + i\sin\left(\frac{9\pi}{6}\right)\right)$$

$$= 2\sqrt[3]{2}\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right)$$

I have left a few algebra/pre-algebra steps for you to fill in. Come see me if you need help!!

(Such as, why $\sqrt[3]{16} = 2\sqrt[3]{2}$?)

EXAMPLE 3: Given $z = 4\sqrt{3} - 4i$. Find all the values of $z^{1/2}$.

NOTE: eGrade will ask for only one root.