http://topology.auburn.edu/tp/

TOPOLOGY PROCEEDINGS

Volume 44 (2014)
Pages 315-324

E-Published on December 9, 2013

PENNER SEQUENCES AND ASYMPTOTICS OF MINIMUM DILATATIONS FOR SUBFAMILIES OF THE MAPPING CLASS GROUP

ERIKO HIRONAKA

Abstract. Let $\mathcal{F}_m \subset \text{Mod}(S_m)$ be a collection of subsets of the mapping class group of a compact oriented surface S_m of genus g_m, where g_m is unbounded. We say $\mathcal{F} = \bigcup_m \mathcal{F}_m$ admits asymptotically small dilatations if there exists a sequence $\phi_m \in \mathcal{F}_m$ of pseudo-Anosov elements so that $\lambda(\phi_m)^{g_m}$ is bounded. In this paper, we describe Penner's construction for producing sequences of pseudo-Anosov mapping classes whose normalized dilatations converge and apply the construction to the setting of handlebody mapping class groups and mapping classes with trivial homological dilatation.

1. Introduction

Let S_g be a closed oriented surface of genus g, and let $\text{Mod}(S_g)$ be its mapping class group. Robert C. Penner [16] shows that for each genus g, the minimum dilatation δ_g of pseudo-Anosov mapping classes in $\text{Mod}(S_g)$ satisfies

$$\log \delta_g \asymp \frac{1}{g}. \tag{1.1}$$

There are several naturally defined subgroups and subcollections of the mapping class group for which this asymptotic behavior on minimum dilatations does not hold (see, for example, [4], [3], [18]). Let g_m be a strictly monotone increasing sequence of integers $g_m \geq 2$. A collection

2010 *Mathematics Subject Classification.* 57M50, 57M27.

Key words and phrases. handlebody mapping classes, Penner sequences, pseudo-Anosov mapping classes, small dilatations.

This work was partially supported by a grant from the Simons Foundation (#209171 to Eriko Hironaka).

$\mathcal{F} = \bigcup_m \mathcal{F}_{g_m}$ of subsets $\mathcal{F}_{g_m} \subset \mathrm{Mod}(S_g)$ *admits asymptotically small dilatation pseudo-Anosov maps* if there is a sequence $\phi_{g_m} \in \mathcal{F}_{g_m}$ of pseudo-Anosov mapping classes that satisfy

$$\log(\lambda(\phi_{g_m})) \asymp \frac{1}{g_m}.$$

In this paper, we describe a generalization of Penner's construction of small dilatation mapping classes and apply it to find two "naturally defined" collections of mapping classes that admit asymptotically small dilatation pseudo-Anosov maps.

Our first example is the handlebody subgroups. A mapping class ϕ on a surface S is a *handlebody mapping class* if there is an identification of S with the boundary of a handlebody H so that ϕ extends to H. Howard Masur [12] shows that the limit set of the handlebody subgroup has measure zero in Thurston's sphere of measured foliations. Thus, these subgroups are *small* in this sense. On the other hand, the following theorem shows that the handlebody subgroups are *large* in the sense of the range of dilatations of pseudo-Anosov elements.

Theorem 1.1. *Let $\mathcal{H}_g \subset Mod(S_g)$ be the set of handlebody mapping classes on a genus g surface. Then $\mathcal{H}_g$ admits asymptotically small dilatation pseudo-Anosov maps.*

Our second example is the collection of mapping classes with trivial homological dilatation. In [4], Benson Farb, Christopher J. Leininger, and Dan Margalit prove that the set of dilatations of pseudo-Anosov mapping classes in the Torelli subgroup of $\mathrm{Mod}(S_g)$ is bounded from below by a constant greater than one. Thus, mapping classes that act trivially on first homology do not admit small dilatations. If we look, however, at mapping classes whose action on first homology has spectral radius equal to one, the behavior of minimum dilatations is different.

Theorem 1.2. *Let $\mathcal{F}_g \subset Mod(S_g)$ be the subcollections of mapping classes whose homological dilatation equals one. Let $g_m = 2m$ range over the even numbers ≥ 2. Then $\mathcal{F}_{g_m}$ admits asymptotically small dilatation pseudo-Anosov maps.*

Dilatations of pseudo-Anosov mapping classes $\phi_g \in \mathrm{Mod}(S_g)$, where g is the genus of S_g, are bounded from below by the following inequality [16] (see also [14, p. 44])

$$\frac{\log(2)}{12g - 12} \leq \log(\lambda(\phi_g)).$$

Thus, to prove statements like Theorem 1.1 and Theorem 1.2, it suffices to find a sequence of pseudo-Anosov mapping classes $\phi_{g_m} : S_{g_m} \to S_{g_m}$, so that the *genus-normalized dilatation*

$$L_{\text{genus}}(S_m, \phi_m) = \lambda(\phi_{g_m})^{g_m}$$

is bounded.

In [16], Penner developed techniques for constructing sequences of this kind which are sometimes known as *Penner sequences* (see also [2], [18], [19]). Penner sequences have the following useful properties:

(i) ϕ_m is pseudo-Anosov,
(ii) the genus of S_m is linear in m, and
(iii) $L_{\text{genus}}(S_m, \phi_m)$ is bounded.

We define generalized Penner sequences in section 2 and use this in section 3 to prove Theorem 1.1 and Theorem 1.2 using explicit constructions. Section 4 contains further questions about small dilatation mapping classes.

2. Generalized Penner Sequences

In this section, we define a generalization of Penner's example in [16], which we will use in section 3. This generalization is a special case of the ones studied in [19] and [8].

Let S be a compact surface of finite type with negative topological Euler characteristic. A *simple closed curve* on S is the image of an embedded circle on S. A *relative closed curve* is the image of an interval on S whose endpoints lie on the boundary of S. A simple closed or relative closed curve on S is *essential* if it does not bound a disk on S and it is not homotopic to a curve on the boundary of S. A *multi-curve* on S is a finite union of pairwise disjoint essential closed curves on S. A *relative multi-curve* on S is a finite union of pairwise disjoint closed and relative closed curves on S.

A pair of multi-curves a and b *fills* S if

(i) a and b intersect minimally, and
(ii) the complementary components of $a \cup b$ are either open disks or boundary parallel annuli.

Given a multi-curve a, let δ_a be the composition of right Dehn twists centered at the components of a. (See, for example, [5] for definition and properties of Dehn twists.)

Let a and b be multi-curves, let c be a simple closed curve that is disjoint from b, and assume that the pair of multi-curves a and $b \cup c$ fills S. Let $d \subset S$ be a relative closed multi-curve. Assume the following:

(i) c is connected,
(ii) d is disjoint from a and b,
(iii) $S \setminus d$ is connected, and

(iv) the algebraic intersection of c and d is zero.
We call d the *cutting curve.*

Lemma 2.1. *The mapping class $\phi : S \to S$ defined by $\phi = \delta_c \delta_a^{-1} \delta_b$ is pseudo-Anosov.*

Proof. This follows from Penner's semi-group criterion [15]. □

Let $\alpha : \pi_1(S) \to \mathbb{Z}$ be the map sending loops on S to their algebraic intersection with d. By composing with the quotient map $Z \to \mathbb{Z}/m\mathbb{Z}$, we have regular m-cyclic coverings $\rho_m : S_m \to S$. Let Σ be the surface with boundary obtained by cutting S along d and let Σ_0 be the closure of a lift of the interior of Σ to S_m.

Let d^+ and d^- be the loci on the boundary of Σ corresponding to the two sides of d in S and let $d_0^\pm$ be their lifts to Σ_0. Then there is a generator r_m of the group of deck transformations of S_m over S so that $r_m(d_0^-) = d_0^+$. Let $\Sigma_i = r_m^i(\Sigma_0)$ and $d_i^\pm = r_m^i(d_0^\pm)$. Then S_m is the union

$$S_m = \bigcup_{i=0}^{m-1} \Sigma_i,$$

where Σ_i is attached to Σ_{i+1} by gluing d_i^+ to d_{i+1}^-, the sum "$i+1$" being taken modulo m.

By construction, each of a, b, and c has m disjoint lifts in S_m. Let $a^{(0)}$ and $b^{(0)}$ be the lifts of a and b that are strictly contained in Σ_0, and let $c^{(0)}$ be the lift of c that intersects Σ_0, but does not intersect Σ_{m-1}. Define the *Penner sequence* $\phi_m : S_m \to S_m$ *associated to* (S, a, b, c, d) to be the mapping class

$$\phi_m = r_m \delta_{c^{(0)}} \delta_{a^{(0)}}^{-1} \delta_{b^{(0)}}.$$

Theorem 2.2. *Let (S_m, ϕ_m) be a Penner sequence associated to (S, a, b, c, d). Then*

(1) *the topological Euler characteristic of S_m satisfies*

$$\chi(S_m) = m\chi(S),$$

(2) *the mapping class ϕ_m is pseudo-Anosov,*
(3) *the χ-normalized dilatations $L_\chi(S_m, \phi_m) = \lambda(\phi_m)^{|\chi(S_m)|}$ form a convergent sequence*

$$\lim_{m \to \infty} L_\chi(S_m, \phi_m) = L_\chi(S, \phi),$$

and
(4) *the genus normalized dilatations $L_{genus}(S_m, \phi_m)$ are bounded.*

Proof. The topological Euler characteristic of Σ is given by

$$\chi(\Sigma) = \chi(S) - \chi(d).$$

The covering S_m contains m copies of d and m copies of the interior of Σ, and hence

$$\chi(S_m) = m\chi(d) + m(\chi(S) - \chi(d)) = m\chi(S).$$

The mapping classes ϕ_m lie in the Penner semigroup generated by negative Dehn twists on a and positive Dehn twists on $b \cup c$. By Penner's semigroup criterion, it follows that ϕ_m is pseudo-Anosov [15], proving (2). William P. Thurston's fibered face theory [17] gives a correspondence between pseudo-Anosov mapping classes on surfaces and rational points on fibered faces of hyperbolic 3-manifolds. In [8], it is shown that each Penner sequence (S_m, ϕ_m) corresponds to a convergent sequence on a fibered face whose limit is the point associated to (S, ϕ). By a result of David Fried [6] (see also [13], [14]), the normalized dilatation extends to a continuous (and convex) function on fibered faces, implying (3).

Let r_m be the number of boundary components of S_m. Then, since $r_m \geq 0$, we have $|\chi(S_m)| = 2g_m + r_m - 2 \geq 2g_m - 2$. Thus,

$$L_{\text{genus}}(S_m, \phi_m) = \lambda(\phi_m)^{g_m} \leq \lambda(\phi_m)^{|\chi(S_m)|/2+1} = L(S_m, \phi_m)^{\frac{1}{2}} \lambda(\phi_m).$$

Since, by Theorem 2.2(3), $\lambda(\phi_m)$ and $L(S_m, \phi_m)$ are bounded, $L_{\text{genus}}(S_m, \phi_m)$ is bounded. □

Remark 2.3. That the normalized dilatations $L_\chi(S_m, \phi_m)$ are bounded was proved for a special case in [16] and generalized in [19]. Theorem 2.2 is stronger because it also gives information about the limiting value of $L(S_m, \phi_m)$.

Let (S_m, ϕ_m) be a Penner sequence. Let $\overline{S}_m$ be the closed surface obtained by filling each boundary component of S_m with a disk. Let $\overline{\phi}_m$ be the mapping class on $\overline{S}_m$ induced by ϕ_m. We call $(\overline{S}, \overline{\phi}_m)$ the closures of (S_m, ϕ_m).

By the construction, the multi-curves a and $b \cup c$ divide S into disk or boundary parallel annular regions bounded by polygons whose sides alternate between lying on a and lying on $b \cup c$, and each polygon bounding a disk is even-sided with at least four sides.

Proposition 2.4. *Let (S, a, b, c, d) have the additional property that each boundary parallel disk in the complement of $a \cup b \cup c$ is bounded by a polygon with at least four sides. Then the closures $(\overline{S}_m, \overline{\phi}_m)$ are pseudo-Anosov mapping classes with*

$$\lambda(\overline{S}_m, \overline{\phi}_m) = \lambda(S_m, \phi_m),$$

and the genus-normalized dilatations $L_{genus}(\overline{S}_m, \overline{\phi}_m)$ are bounded.

Proof. For the covering S_m two things can happen locally. If a complementary component is homeomorphic to a disk, then the number of sides of the bounding polygon stays the same. If the complementary component is a boundary parallel annulus, the number of sides of the polygon either stays the same or increases. It follows (see [15]) that ϕ_m cannot have any one-pronged boundary components, and hence $(\overline{S}_m, \overline{\phi}_m)$ is pseudo-Anosov with the same dilatation as (S_m, ϕ_m) (see, for example, [9, Lemma 2.6]).

Since the genus of S_m and of $\overline{S}_m$ are the same, we have

$$L_{\text{genus}}(S_m, \phi_m) = L_{\text{genus}}(\overline{S}_m, \overline{\phi}_m).$$

The rest follows from Theorem 2.2(4). □

3. Applications

Here, we prove the theorems stated in the introduction using families of examples that satisfy the conditions of Theorem 2.2 and Proposition 2.4.

3.1. Handlebody mapping classes.

We now construct Penner sequences consisting of handlebody mapping classes and prove Theorem 1.1.

Let (S, a, b, c, d) be the surface and curves shown in Figure 1. Then S is a closed genus-2 surface and b is the empty curve. Let $p : S \to H$ be the inclusion of S as the boundary of the genus-2 handlebody. Then $(S, a, \emptyset, c, d)$ defines a Penner sequence $\phi_m : S_m \to S_m$ with no boundary components.

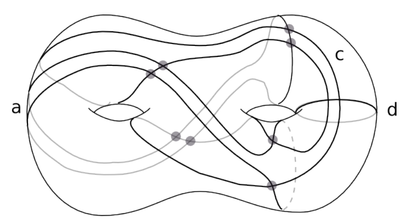

Figure 1. Pair of multi-curves a and c and cutting curve d on the surface S.

By Theorem 2.2, we have

$$\log \lambda(\phi_m) \asymp \frac{1}{m}.$$

Figure 2 gives a picture of the mapping classes $\phi_m : S_m \to S_m$. One observes that S_m has genus $g = m + 1$, and ϕ_m is a union of Dehn twists that bound disks in the interior of the handlebody. This proves Theorem 1.1.

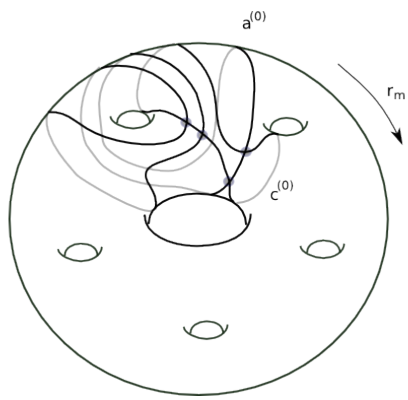

FIGURE 2. The mapping class (S_m, ϕ_m).

Using the theory of fibered faces, it is possible to compute the dilatations of the mapping classes in these examples explicitly.

By studying the action of ϕ on the curves a and c (and the associated train track defined in the proof of Penner's semi-group criterion), one can find the dilatation $\lambda(\phi)$ as the largest eigenvalue of

$$\begin{bmatrix} 1 & 8 \\ 8 & 65 \end{bmatrix}.$$

Thus, $\lambda(\phi)$ is the largest root of the characteristic polynomial

$$x^2 - 66x + 1 = 0.$$

By Theorem 2.2, we have

$$\lim_{m\to\infty} L_\chi(\overline{S}_m, \overline{\phi}_m) = \lambda(\phi)^2 \approx (65.98)^2 \approx 4353.99.$$

Remark 3.1. Using the McMullen polynomial [14], one can also find the dilatations of each of the mapping classes $(\overline{S}_m, \phi_m)$: the dilatations of ϕ_m is the largest root of

$$x^{2m} - 16x^{m+1} - 34x^m - 16x^{m-1} + 1 = 0.$$

(See [8] for more detailed descriptions of the computational techniques.)

3.2. Mapping classes with homological dilatation equal to one.

Consider the surface and curves shown in Figure 3 where $a = a_1 \cup a_2$. (The base example shown in Figure 3 also appears in [5, Figure 14.1] and [11, Lemma 5.1].) Then (S, a, b, c, d) satisfies the conditions of Theorem 2.2 and gives rise to Penner sequence (S_m, ϕ_m) where S_m has genus $2m$.

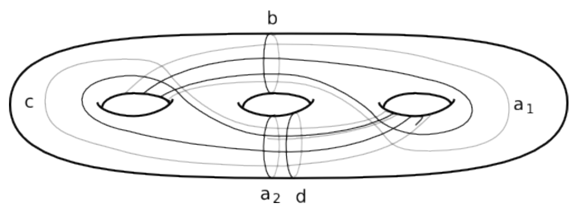

Figure 3. Example generating a Penner sequence with trivial homological dilatation.

The mapping classes $\delta_{a_2}^{-1}\delta_b$, δ_{a_1}, and δ_c are all elements of the Torelli subgroup of $\mathrm{Mod}(S_3)$, as are their lifts to S_m (see Figure 4).

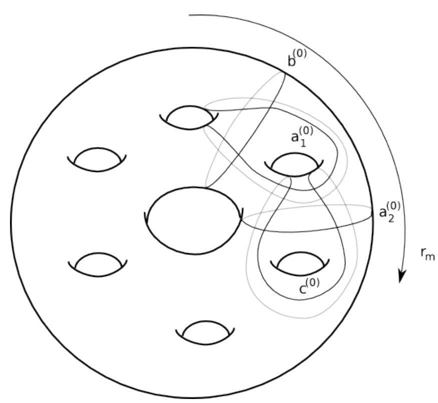

Figure 4. Penner sequence of pseudo-Anosov mapping classes with trivial homological dilatation on even genus surfaces.

The rotation r_m is a rotation by "two clicks" and has order $m = g_m/2$. Since ϕ_m is a composition of an element in the Torelli group and a rotation

$$\phi_m = r_m \delta_{c^{(0)}} \delta^{-1}_{a_1^{(0)}} \delta^{-1}_{a_2^{(2)}} \delta_{b^{(0)}}$$

it has trivial homological dilatation. This completes the proof of Theorem 1.2.

4. Further Questions

Let $\mathcal{P} = \bigcup_g \mathcal{P}(S_g)$, where $\mathcal{P}(S_g)$ is the set of pseudo-Anosov mapping classes on the closed surface S_g of genus g. The smallest known accumulation point for the genus-normalized dilatation

$$L_{\text{genus}}(\phi_g) = \lambda(\phi_g)^g$$

equals

$$\mu = \gamma_0^2,$$

where γ_0 is the golden mean (see [7], [1], [10]).

Question 4.1. Is μ the smallest accumulation point for L_{genus} on $\mathcal{P}$?

The results of this paper lead to the following more specific questions.

Question 4.2. What is the smallest accumulation point for genus-normalized dilatation restricted to handlebody mapping classes or to mapping classes with trivial homological dilatation?

For example, one would expect the smallest accumulation point for handlebody subgroups to relate to the geometry of fibered 4-dimensional manifolds, and perhaps this further restriction gives a lower bound for normalized dilatation that is higher than μ.

Question 4.3. Can the smallest accumulation point for genus-normalized dilatations of pseudo-Anosov mapping classes be achieved as the limit of dilatations for closures of a generalized Penner sequence?

So far there is no known example of a Penner sequence whose genus-normalized dilatations converge to μ.

Acknowledgments. The author is grateful to U. Hamenstädt, S. Hensel, C. McMullen, and A. Valdivia for stimulating conversations, and to the referee for helpful comments.

References

[1] John W. Aaber and Nathan Dunfield, *Closed surface bundles of least volume*,Algebr. Geom. Topol. **10** (2010), no. 4, 2315–2342.

[2] Max Bauer, *Examples of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **330** (1992), no. 1, 333Ű359.

[3] Corentin Boissy and Erwan Lanneau, *Dynamics and geometry of the Rauzy-Veech induction for quadratic differentials*, Ergodic Theory Dynam. Systems **29** (2009), no. 3, 767–816.

[4] Benson Farb, Christopher J. Leininger, and Dan Margalit, *The lower central series and pseudo-Anosov dilatations*, Amer. J. Math. **130** (2008), no. 3, 799–827.

[5] Benson Farb and Dan Margalit, *A Primer on Mapping Class Groups.* Princeton Mathematical Series, 49. Princeton, NJ: Princeton University Press, 2012.

[6] David Fried, *Flow equivalence, hyperbolic systems and a new zeta function for flows*, Comment. Math. Helv. **57** (1982), no. 2, 237–259.

[7] Eriko Hironaka, *Small dilatation mapping classes coming from the simplest hyperbolic braid*, Algebr. Geom. Topol. **10** (2010), no. 4, 2041–2060.

[8] ———, *Quotient families of mapping classes.* arXiv:1212.3197 [math.GT].

[9] Eriko Hironaka and Eiko Kin, *A family of pseudo-Anosov braids with small dilatation*, Algebr. Geom. Topol. **6** (2006), 699–738.

[10] Eiko Kin and Mitsuhiko Takasawa, *Pseudo-Anosovs on closed surfaces having small entropy and the Whitehead sister link exterior*, J. Math. Soc. Japan **65** (2013), no. 2, 411–446.

[11] Christopher J. Leininger and Dan Margalit, *On the number and location of short geodesics in moduli space*, J. Topol. **6** (2013), no. 1, 30–48.

[12] Howard Masur, *Measured foliations and handlebodies*, Ergodic Theory Dynam. Systems **6** (1986), no. 1, 99–116.

[13] Shigenori Matsumoto, *Topological entropy and Thurston's norm of atoroidal surface bundles over the circle*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34** (1987), no. 3, 763–778.

[14] Curtis T. McMullen, *Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 4, 519–560.

[15] Robert C. Penner, *A construction of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **310** (1988), no. 1, 179–197.

[16] ———, *Bounds on least dilatations*, Proc. Amer. Math. Soc. **113** (1991), no. 2, 443–450.

[17] William P. Thurston, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130.

[18] Chia-Yen Tsai, *The asymptotic behavior of least pseudo-Anosov dilatations*, Geom. Topol. **13** (2009), no. 4, 2253–2278.

[19] Aaron D. Valdivia, *Sequences of pseudo-Anosov mapping classes and their asymptotic behavior*, New York J. Math. **18** (2012), 609–620.

Department of Mathematics; Florida State University; Tallahassee, Florida 32306

E-mail address: hironaka@math.fsu.edu