Measuring the Curl of Paper

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The following problem came to us from an employee of a paper manufacturer. It can be posed to an introductory trigonometry class and provides a good illustration of how mathematics can be used in many fields.

In the paper manufacturing industry, one indicator of the quality of paper produced is its *curl*. Paper with too much curl for its intended use is considered defective and is discarded. To measure the curl, a paper sample is held up to a chart that shows circular arcs of different curvature, each arc associated with a number d which is the *depth* of the arc (Fig. 1). The depth of the matching arc is recorded as the curl of the sample. For example, an $8\frac{1}{2}$ by 11 inch sheet of paper would be held up at the midpoint of a long edge and the opposite edge would be compared with the arcs on the chart and matched to one of them to determine the curl of the paper. The allowable curl for the 11 inch sample would differ from the allowable curl of a

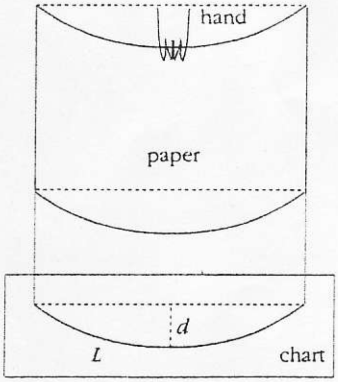

Figure 1. Illustration of the way in which curl is measured. The paper is held at the midpoint of one end and the dangling free end is compared to a chart lying flat on the table. The *depth* of the arc is d.

sample of a different size and would depend on the intended use. Note that the curl d is not the depth of the sample but the depth of the arc (of fixed length L) on the chart.

One could use curvature (by definition, the curvature of a circular arc is the reciprocal of the radius) as an unambiguous measure of the curl of a paper sample. That is, paper samples of different curl will correspond to arcs of different curvature and *vice versa*. However, the depth of an arc on the chart, rather than its curvature, is currently used in the paper manufacturing industry as a measure for the curl of a paper sample. Is there a one-to-one correspondence between the curvature of arcs on the chart and the arc depth? If not, then there will be two different arcs with the same depth and, hence, two samples with different curvatures but the same curl.

For all arcs up to the half circle d is an increasing function of curvature, so there is a one-to-one correspondence between depth and curvature. However, something unusual happens beyond this point. When the arc forms a half circle the depth d is the radius of a circle with circumference $2L$. Thus, $2\pi d = 2L$ or $d = L/\pi$. When the arc forms a complete circle, then d is the diameter of a circle with circumference L, or $d = L/\pi$ as before. Apparently, beyond the half circle the depth d is no longer a strictly increasing function of the curvature. What is the relationship between d and the curvature? At what point does d reach its maximum? Beyond the point of maximization there is no longer a one-to-one correspondence between d and the curvature, and this system of classifying the curl of paper cannot be used.

Consider the angle ϕ subtended by the arc of length L. To find a formula for d there are two cases to consider: $\phi \leq \pi$ and $\phi > \pi$. See Figures 2 and 3 for

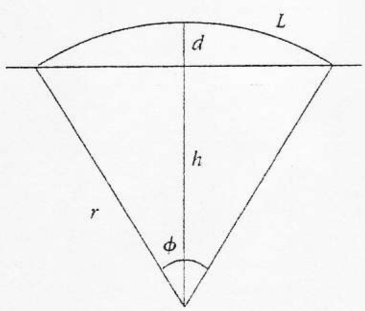

Figure 2. Illustration of problem for $\phi \leq \pi$.

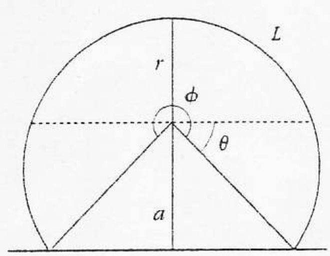

Figure 3. Illustration of problem for $\phi > \pi$. The dashed line runs through the center of the circular arc. The depth of the arc is $d = r + a$.

illustrations of the two cases. If $\phi \leq \pi$, then $\cos(\phi/2) = h/r$ and thus

$$d = r - h = r\left(1 - \cos\left(\frac{\phi}{2}\right)\right).$$

If $\phi > \pi$, then with θ as indicated in Figure 3 we have $\sin\theta = a/r$ and $\theta = \phi/2 - \pi/2$. Now

$$\begin{aligned} d &= r + a = r + r\sin\theta = r(1 + \sin(\phi/2 - \pi/2)) \\ &= r(1 - \sin(\pi/2 - \phi/2)) = r\left(1 - \cos\left(\frac{\phi}{2}\right)\right), \end{aligned}$$

just as in the earlier case. Thus the formula $d = r(1 - \cos(\phi/2))$ expresses the depth in terms of r and ϕ. Since L is the same for all arcs and $L = r\phi$, we have

$$d = \frac{L}{\phi}\left(1 - \cos\left(\frac{\phi}{2}\right)\right),$$

giving d as a function of the single variable ϕ, valid for $0 < \phi \leq 2\pi$. Finally, note that the curvature is $1/r = \phi/L$ so that ϕ is just a scaled form of the curvature.

The ratio d/L is plotted as a function of ϕ in Figure 4. On the interval $0 < \phi \leq \pi$ the depth d increases monotonically, while on the interval $\pi < \phi < 2\pi$ it first

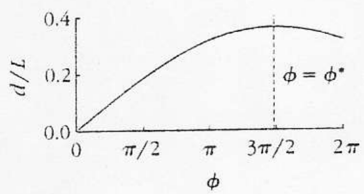

Figure 4. Dependence of the scaled arc depth on the angle ϕ. The dashed line indicates location of the maximum.

increases and then decreases. The angle ϕ at which d is maximized, ϕ^*, is determined by setting the first derivative to 0:

$$d' = \frac{L\left[\frac{\phi^*}{2}\sin\left(\frac{\phi^*}{2}\right) + \cos\left(\frac{\phi^*}{2}\right) - 1\right]}{(\phi^*)^2} = 0,$$

which simplifies to

$$\frac{\phi^*}{2}\sin\left(\frac{\phi^*}{2}\right) + \cos\left(\frac{\phi^*}{2}\right) - 1 = 0.$$

Using a numerical method to solve for ϕ^* we obtain

$$\phi^* \approx 4.66 \text{ rad} = 267.12°.$$

Our analysis shows that using the depth d as a measure of the curl of paper is appropriate only for angles no greater than $\phi^* \approx 267.12°$. On larger intervals the relation between d and the scaled curvature ϕ is not one-to-one.