LOG CONCAVITY OF THE GROTHENDIECK CLASS OF $\overline{\mathcal{M}}_{0,n}$

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Abstract. Using a known recursive formula for the Grothendieck classes of the moduli spaces $\overline{\mathcal{M}}_{0,n}$, we prove that they satisfy an asymptotic form of ultra-log-concavity as polynomials in the Lefschetz class. We also observe that these polynomials are γ-positive. Both properties, along with numerical evidence, support the conjecture that these polynomials only have real zeros. This conjecture may be viewed as a particular case of a possible extension of a conjecture of Ferroni-Schröter and Huh on Hilbert series of Chow rings of matroids.

We prove asymptotic ultra-log-concavity by studying differential equations obtained from the recursion, whose solutions are the generating functions of the individual betti numbers of $\overline{\mathcal{M}}_{0,n}$. We obtain a rather complete description of these generating functions, determining their asymptotic behavior; their dominant term is controlled by the coefficients of the Lambert W function. The γ-positivity property follows directly from the recursion, extending the argument of Ferroni et al. proving γ-positivity for the Hilbert series of the Chow ring of matroids.

1. Introduction

As a straightforward consequence of the Hard Lefschetz theorem, the sequence of (even) betti numbers of a smooth complex projective variety is *unimodal.* This fact is discussed in detail in [Sta89, Theorem 18]. The sequence is not necessarily *log-concave,* but there are situations where it is expected to be; for example, this is discussed in [MMPR23] for the case of configuration spaces, providing log-concavity results for e.g., the space of ordered n-uples of points in $\mathbb{C}$.

The object of study of this note is the moduli space $\overline{\mathcal{M}}_{0,n}$ of stable n-pointed rational curves for $n \geq 3$. We prove an asymptotic log-concavity property for the Poincaré polynomials of these varieties. We also remark that these polynomials are 'γ-positive'. These results may be viewed as evidence for a conjecture stating that the Poincaré polynomials only have real zeros, see below.

We focus on the class of $\overline{\mathcal{M}}_{0,n}$ in the Grothendieck group of varieties $K(\mathrm{Var}_{\mathbb{C}})$. This is a universal Euler characteristic, therefore *a priori* a more fundamental object. It is known (cf. [MM16], and §2 below) that the class of $\overline{\mathcal{M}}_{0,n}$ is a polynomial with integer coefficients in the Lefschetz-Tate class $\mathbb{L} = [\mathbb{A}^1]$; we denote this class by

$$[\overline{\mathcal{M}}_{0,n}] = a_{n,0} + a_{n,1}\mathbb{L} + \cdots + a_{n,n-3}\mathbb{L}^{n-3}\,.$$

The Poincaré polynomial is given by specializing $\mathbb{L}$ to t^2. Thus, $\overline{\mathcal{M}}_{0,n}$ only has even cohomology (also cf. [Kee92, p. 549]), and the integers $a_{n,k}$ may be interpreted as the ranks of the cohomology groups of $\overline{\mathcal{M}}_{0,n}$. Log-concavity of these polynomials amounts to the statement that $a_{n,i}^2 \geq a_{n,i-1}a_{n,i+1}$ for all $i \geq 1$ and all $n \geq 3$. The stronger condition of

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ultra-log-concavity is the inequality

$$\left(\frac{a_{n,i}}{\binom{n-3}{i}}\right)^2 \geq \frac{a_{n,i-1}}{\binom{n-3}{i-1}} \cdot \frac{a_{n,i+1}}{\binom{n-3}{i+1}}\,.$$

for all $i \geq 1$, all $n \geq 3$.

Theorem 1.1. *With notation as above, $\forall i \geq 1$ $\exists N$ s.t. $\forall n \geq N$*

$$\left(\frac{a_{n,i}}{\binom{n-3}{i}}\right)^2 \geq \frac{a_{n,i-1}}{\binom{n-3}{i-1}} \cdot \frac{a_{n,i+1}}{\binom{n-3}{i+1}}\,. \tag{1.1}$$

Thus, an asymptotic log-concavity property holds for the coefficients of the Grothendieck class of $\overline{\mathcal{M}}_{0,n}$, hence for the betti numbers $a_{n,k} = \mathrm{rk}\, H^{2k}(\overline{\mathcal{M}}_{0,n}, \mathbb{Q})$.

The class $[\overline{\mathcal{M}}_{0,n}]$ is explicitly known recursively, see [Kee92], [MM16, Proposition 3.2], and (2.1) below. The first several expressions for this class are

$$\begin{gathered}
1\\
\mathbb{L}+1\\
\mathbb{L}^2+5\mathbb{L}+1\\
\mathbb{L}^3+16\mathbb{L}^2+16\mathbb{L}+1\\
\mathbb{L}^4+42\mathbb{L}^3+127\mathbb{L}^2+42\mathbb{L}+1\\
\mathbb{L}^5+99\mathbb{L}^4+715\mathbb{L}^3+715\mathbb{L}^2+99\mathbb{L}+1\\
\mathbb{L}^6+219\mathbb{L}^5+3292\mathbb{L}^4+7723\mathbb{L}^3+3292\mathbb{L}^2+219\mathbb{L}+1\\
\mathbb{L}^7+466\mathbb{L}^6+13333\mathbb{L}^5+63173\mathbb{L}^4+63173\mathbb{L}^3+13333\mathbb{L}^2+466\mathbb{L}+1\\
\mathbb{L}^8+968\mathbb{L}^7+49556\mathbb{L}^6+429594\mathbb{L}^5+861235\mathbb{L}^4+429594\mathbb{L}^3+49556\mathbb{L}^2+968\mathbb{L}+1
\end{gathered}$$

Numerical evidence supports the following conjecture. We have learned that the same conjecture was independently formulated by Luis Ferroni.

Conjecture 1. *The polynomial $P_n(t) \in \mathbb{Z}[t]$, such that $[\overline{\mathcal{M}}_{0,n}] = P_n(\mathbb{L})$, has only real zeros.*

Due to a standard result attributed to Newton ([Sta89, Theorem 2]), this conjecture would imply ultra-log-concavity of the polynomials. Thus, Theorem 1.1 gives some support to Conjecture 1.

Related real-rootedness conjectures are listed in [FMSV22, Conjecture 1.6]. Specifically, the first of these conjectures, due independently to Ferroni-Schröter and Huh, posits that the Hilbert series of the Chow ring of an arbitrary matroid should only have real roots; see [FS22, Conjecture 8.18]. The conventional definition for the Chow ring of a matroid is given with respect to its *maximal* building set; using the *minimal* rather than the maximal building set, the Chow ring of the braid matroid agrees with the cohomology of $\overline{\mathcal{M}}_{0,n}$, see §5.2 in *loc. cit.* Thus, Conjecture 1 addresses a particular case of a possible extension of [FMSV22, Conjecture 1.6]; but we note that the Hilbert series of the Chow ring of a matroid with respect to the minimal building set is in general not real-rooted.

Poincaré duality implies that the polynomials expressing $[\overline{\mathcal{M}}_{0,n}]$ are palindromic. For palindromic polynomials with nonnegative coefficients, real-rootedness also implies 'γ-positivity', which amounts to the positivity of the coefficients of the polynomials in a basis consisting of polynomials of the type $t^i(1+t)^j$ (see §3 for the precise definition). The following result is a straightforward consequence of the recursive formula (2.1) determining $[\overline{\mathcal{M}}_{0,n}]$, and may be viewed as further evidence for Conjecture 1.

Theorem 1.2. *For all $n \geq 3$, the polynomial $P_n(t) \in \mathbb{Z}[t]$ such that $[\overline{\mathcal{M}}_{0,n}] = P_n(\mathbb{L})$ is γ-positive.*

The recursive formula determining $[\overline{\mathcal{M}}_{0,n}]$ is proved in [MM16] by an argument using a suitable tree-level partition function. This method is modeled on the analogous result for the Poincaré polynomial of $\overline{\mathcal{M}}_{0,n}$ obtained by Y. Manin in [Man95]. The recursion is equivalent to a recursion for the betti numbers stated by S. Keel in [Kee92, p. 550], following from his complete determination of the Chow groups of $\overline{\mathcal{M}}_{0,n}$. For the convenience of the reader, in §2 we reprove the recursion formula for the Grothendieck class of $\overline{\mathcal{M}}_{0,n}$ directly from Keel's description of $\overline{\mathcal{M}}_{0,n}$ as a sequence of blow-ups over $\overline{\mathcal{M}}_{0,n-1} \times \overline{\mathcal{M}}_{0,4}$. In §3 we prove Theorem 1.2 as a direct consequence of the recursive formula (2.1).

Keel's recursion involves the whole set of betti numbers, while in order to prove Theorem 1.1 it is necessary to have information about individual betti numbers $a_{n,k}$ for fixed k. In §4 we explain how to obtain first order linear differential equations satisfied by the generating functions $\alpha_k(z) := \sum_{n\geq 3} a_{k,n} \frac{z^{n-1}}{(n-1)!}$, determining these functions along with the initial condition $\alpha_k(0) = 0$. For instance,

$$\frac{d\alpha_2}{dz} = \alpha_2(z) + 3e^{3z} - \frac{2z^2+10z+10}{2}e^{2z} + \frac{z^3+5z^2+8z+4}{2}e^z\,,$$

from which

$$\begin{aligned}\alpha_2(z) &= \frac{3e^{3z}}{2} - (z+1)(z+2)e^{2z} + \left(\frac{z^4}{8} + \frac{5z^3}{6} + 2z^2 + 2z + \frac{1}{2}\right)e^z \\ &= 1\cdot\frac{z^4}{4!} + 16\cdot\frac{z^5}{5!} + 127\cdot\frac{z^6}{6!} + 715\cdot\frac{z^7}{7!} + 3292\cdot\frac{z^8}{8!} + 13333\cdot\frac{z^9}{9!} + 49556\cdot\frac{z^{10}}{10!} + \cdots\end{aligned}$$

recovering the coefficients of $\mathbb{L}^2$ in the table displayed above.

We use an inductive argument to obtain a general description of these generating functions as a combination of exponentials with (signed) polynomial coefficients $p_m^{(k)}(z) \in \mathbb{Q}[z]$ (Theorem 4.1). These polynomials certainly deserve further study; we establish their degree and that their leading coefficient is positive and conjecture that they are ultra-log-concave.

The dominant terms in the expressions we obtain determine the asymptotic behavior of $\alpha_{k,n}$.

Theorem 1.3. *For all $k \geq 0$,*

$$\alpha_{k,n} = \operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) \sim \frac{(k+1)^{k+n-1}}{(k+1)!}$$

as $n \to \infty$.

We remark here that Theorem 1.3 is equivalent to the statement that, as $n \to \infty$,

$$\sum_{k=0}^{n-3} \frac{\operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n})}{(k+1)^{n-1}}\, t^{k+1} \sim -W(-t)$$

(in the sense that for every $k \geq 0$, the coefficient of t^k in the l.h.s. converges to the corresponding coefficient in the r.h.s. as $n \to \infty$) where $W(t)$ is the *principal branch of the Lambert W function,* characterized by the identity $W(t)e^{W(t)} = t$. The function $-W(-t)$ is the *tree function,* denoted $T(t)$ in [CGH$^+$96]. This function figures prominently in several generating functions associated with the polynomials $p_m^{(k)}(z)$ mentioned above. We will report on such generating functions in future work.

Theorem 1.1 follows easily from Theorem 1.3, see §5. In fact, in §5 we will obtain a more precise result than Theorem 1.3. We will show that there exist polynomials $q_m^{(k)}(n) \in \mathbb{Q}[n]$ of degree $2m$, with positive leading coefficient, such that

$$\operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) = \frac{(k+1)^{k-1}}{(k+1)!} \cdot (k+1)^n + \sum_{m=1}^{k} (-1)^m q_m^{(k)}(n) \cdot (k+1-m)^n$$

(Theorem 5.1, Remark 5.2). The polynomials $q_m^{(k)}$ have straightforward expressions in terms of the coefficients of the polynomials $p_m^{(k)}$ and are also objects of evident interest. In fact, explicit conjectural expressions can be given for the betti numbers $\operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n})$. We will also report on these in future work.

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2. Recursion for the Grothendieck class of $\overline{\mathcal{M}}_{0,n}$

The class $[\overline{\mathcal{M}}_{0,n}]$ in the Grothendieck group $K(\mathrm{Var}_k)$, k any algebraically closed field, is determined by the following recursion.

Theorem 2.1. $[\overline{\mathcal{M}}_{0,3}] = 1$. *For $n > 3$,*

$$[\overline{\mathcal{M}}_{0,n}] = [\overline{\mathcal{M}}_{0,n-1}](1+\mathbb{L}) + \mathbb{L} \sum_{i=3}^{n-2} \binom{n-2}{i-1} [\overline{\mathcal{M}}_{0,i}][\overline{\mathcal{M}}_{0,n+1-i}]\,. \tag{2.1}$$

This formula is equivalent to the statement given in [MM16], proved there by the same method used to prove an analogous statement for the Poincaré polynomial in [Man95], that is, by adding contributions of strata of $\overline{\mathcal{M}}_{0,n}$. Ultimately, the recursion follows from

$$[\mathcal{M}_{0,k}] = (\mathbb{L}-2)\cdots(\mathbb{L}-k+2)\,,$$

which is easily proved directly, and a sum over trees performed by using (to quote [Man95]) 'a general formula of perturbation theory in order to reduce the calculation of the relevant generating functions to the problem of finding the critical value of an appropriate formal potential.'

The recursion is equivalent to a recursive formula determining the set of betti numbers of $\overline{\mathcal{M}}_{0,n}$, given[1] in [Kee92, p. 550]. In this reference, the formulas for the betti numbers are presented as a consequence of the determination of the Chow groups of $\overline{\mathcal{M}}_{0,n}$, [Kee92, Theorem 1, §3]. The literature on such formulas is very rich. We mention that the recursion is equivalent to a functional equation obtained by Getzler as a consequence of [Get95, Theorem 5.9] and presented as a reformulation of a computation of Fulton and MacPherson from [FM94]. An alternative version of the same functional equation is given by Manin in [Man95, (0.7)]. Chen-Gibney-Krashen extend these formulas to the case of pointed projective spaces and to the motivic setting, [CGK09]; Li obtains general motivic formulas for configuration spaces in [Li09].

For the convenience of the reader, we offer a direct derivation of the recursion in the Grothendieck group $K(\mathrm{Var}_k)$ from Keel's description of $\overline{\mathcal{M}}_{0,n}$.

[1] We alert the reader to two typos in the cited formula in [Kee92]: the binomial $\binom{n}{k}$ should be $\binom{n}{j}$, and the expression $n-j-1$ should be $n-j+1$.

Proof of Theorem 2.1. We recall Keel's recursive construction of $\overline{\mathcal{M}}_{0,n}$ from [Kee92]. The space $\overline{\mathcal{M}}_{0,3}$ is a point, and $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. For $n > 4$, $\overline{\mathcal{M}}_{0,n}$ is constructed as a sequence of blow-ups over $\overline{\mathcal{M}}_{0,n-1} \times \overline{\mathcal{M}}_{0,4}$. The centers of the blow-ups are all disjoint, smooth, of codimension 2. In fact, they are isomorphic to products

$$\overline{\mathcal{M}}_{0,|T|+1} \times \overline{\mathcal{M}}_{0,|T^c|+1}\,,$$

where T denotes a subset of $\{1, \dots, n-1\}$ such that the complement T^c contains two of $1, 2, 3$. Note that each center is isomorphic to $\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,n+1-i}$, with $i = 3, \dots, n-2$.

Now, if $\widetilde{V}$ is the blow-up of a variety V along a regularly embedded center B of codimension r, then the Grothendieck class of $\widetilde{V}$ is

$$[\widetilde{V}] = [V] + (\mathbb{L} + \cdots + \mathbb{L}^{r-1})[B]\,.$$

Indeed, the exceptional divisor of the blow-up is isomorphic to the projectivization of $N_B V$, a $\mathbb{P}^{r-1}$-bundle over B. In the case we are considering, we are blowing up $\overline{\mathcal{M}}_{0,n-1} \times \overline{\mathcal{M}}_{0,4}$, with class

$$[\overline{\mathcal{M}}_{0,n-1} \times \overline{\mathcal{M}}_{0,4}] = [\overline{\mathcal{M}}_{0,n-1} \times \mathbb{P}^1] = [\overline{\mathcal{M}}_{0,n-1}](1 + \mathbb{L})$$

and each center has codimension $r = 2$; therefore

$$[\overline{\mathcal{M}}_{0,n}] = [\overline{\mathcal{M}}_{0,n-1}](1 + \mathbb{L}) + \mathbb{L} \sum_k [B_k]\,,$$

where the sum runs through the centers B_k of the blow-ups. Thus, in order to prove (2.1), it suffices to show that

$$\sum_k [B_k] = \sum_{i=3}^{n-2} \binom{n-2}{i-1} [\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,n+1-i}]\,.$$

Since $\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,n+1-i} \cong \overline{\mathcal{M}}_{0,n+1-i} \times \overline{\mathcal{M}}_{0,i}$, the right-hand side equals

$$\begin{aligned}
(2.2)\quad & \binom{n-2}{\frac{n-1}{2}} [\overline{\mathcal{M}}_{0,\frac{n+1}{2}} \times \overline{\mathcal{M}}_{0,\frac{n+1}{2}}] + \sum_{3 \le i < \frac{n+1}{2}} \left(\binom{n-2}{i-1} + \binom{n-2}{n-i} \right) [\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,n+1-i}] \\
&= \binom{n-2}{\frac{n-1}{2}} [\overline{\mathcal{M}}_{0,\frac{n+1}{2}} \times \overline{\mathcal{M}}_{0,\frac{n+1}{2}}] + \sum_{3 \le i < \frac{n+1}{2}} \binom{n-1}{i-1} [\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,n+1-i}]
\end{aligned}$$

where the first summand only appears if n is odd.

According to Keel's construction, for $n \ge 4$, a center isomorphic to $\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,n+1-i}$ is blown up for all sets $T \subseteq \{1, \dots, n-1\}$ such that

- T^c contains at least two of $1, 2, 3$;
- $|T^c| = n - i$ or $|T^c| = i - 1$.

For all k between 2 and $n-3$, the number of subsets T such that $|T^c| = k$ and T^c contains exactly two of $1, 2, 3$ is

$$3\binom{n-4}{k-2}$$

while the number of subsets T such that $|T^c| = k$ and T^c contains all of $1, 2, 3$ is

$$\binom{n-4}{k-3}$$

(in particular, 0 if $k = 2$). For $3 \le i < \frac{n+1}{2}$, the number of centers isomorphic to the product $\overline{\mathcal{M}}_{0,i} \times \overline{\mathcal{M}}_{0,n+1-i}$ is therefore

$$3\binom{n-4}{n-i-2} + \binom{n-4}{n-i-3} + 3\binom{n-4}{i-3} + \binom{n-4}{i-4} = \binom{n-1}{i-1}.$$

(Maybe more intrinsically, there is one such center for every subset $S \subseteq \{1, \dots, n-1\}$ of size $i-1$. Indeed, if S is such a subset, then either S or S^c satisfies the condition posed on T^c in Keel's prescription.) If n is odd and $i = \frac{n+1}{2}$, the number of centers isomorphic to $\overline{\mathcal{M}}_{0,\frac{n+1}{2}} \times \overline{\mathcal{M}}_{0,\frac{n+1}{2}}$ is

$$3\binom{n-4}{\frac{n+1}{2}-3} + \binom{n-4}{\frac{n+1}{2}-4} = \binom{n-2}{\frac{n-1}{2}}.$$

This is as prescribed in (2.2), concluding the verification. □

Remark 2.2. The distributions of products in the sequence of centers and in the corresponding sum in (2.1) differ in general. For example, for $n = 6$ the summation in (2.1) expands to

$$6\,[\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4}] + 4\,[\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3}]$$

while Keel's construction prescribes blowing up along 7 copies of $\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4}$ and 3 copies of $\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3}$.

The recursion for the Poincaré polynomial following directly from Keel's recursion in [Kee92, p. 550] gives yet a different decomposition: $5\,[\overline{\mathcal{M}}_{0,3} \times \overline{\mathcal{M}}_{0,4}] + 5\,[\overline{\mathcal{M}}_{0,4} \times \overline{\mathcal{M}}_{0,3}]$. ⌟

3. $\overline{\mathcal{M}}_{0,n}$ is γ-positive

For a survey on γ-positivity, we refer the reader to [Ath18]; we follow the terminology in [FMSV22, §2.2]. A univariate polynomial $f(t) = \sum a_i t^i$ is 'symmetric', with 'center' $\frac{d}{2}$, if $a_{d-i} = a_i$ for all i. Every symmetric polynomial $f(t) \in \mathbb{Z}[t]$ with center $\frac{d}{2}$ can clearly be written

$$f(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i\, t^i\, (t+1)^{d-2i} \tag{3.1}$$

for unique integers γ_i, $i = 0, \dots, \lfloor \frac{d}{2} \rfloor$.

Definition 3.1. We say that a symmetric polynomial f is γ*-positive* if all the integers γ_i are nonnegative. ⌟

Our interest in this notion is due to the following well-known fact.

Lemma 3.2 ([Ath18], §1; [FNV23], Proposition 5.3)**.** *Real-rooted symmetric polynomials with nonnegative coefficients are γ-positive.*

Thus, γ-positivity may be taken as collateral evidence for real-rootedness. In [FMSV22, Theorem 1.8] it is shown that the Hilbert series of the Chow ring of every matroid is γ-positive. We prove the analogous statement for $\overline{\mathcal{M}}_{0,n}$.

Theorem 3.3. *For all $n \ge 3$, the polynomial $P_n(t) \in \mathbb{Z}[t]$ such that $[\overline{\mathcal{M}}_{0,n}] = P_n(\mathbb{L})$ is γ-positive.*

(This is Theorem 1.2, stated in the introduction.)

Proof. Following [FMSV22], for a symmetric polynomial (3.1) we let

$$\gamma(f) := \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i \, t^i \,.$$

Thus, f is γ-positive if and only if $\gamma(f)$ has nonnegative coefficients. This operation satisfies several properties (see [FMSV22, Lemma 2.10]):

(i) $\gamma(fg) = \gamma(f)\gamma(g)$;
(ii) $\gamma(tf) = t\gamma(f)$
(iii) $\gamma(f(1+t)) = \gamma(f)$
(iv) If f and g have the same center of symmetry, then $\gamma(f+g) = \gamma(f) + \gamma(g)$.

With this understood, the proof of Theorem 3.3 is a straightforward consequence of the recursion (2.1) for the Grothendieck class $[\overline{\mathcal{M}}_{0,n}]$. In terms of the polynomial P_n, this recursion reads

$$P_n(t) = P_{n-1}(t)(1+t) + t \sum_{i=3}^{n-2} \binom{n-2}{i-1} P_i(t) P_{n+1-i}(t) \,.$$

The constant $P_3(t) = 1$ is trivially γ-positive. Arguing by strong induction, assume that $P_k(t)$ is γ-positive for all $k < n$. The degree of $P_k(t)$ is $k-3$ and the polynomial is palindromic, so it is symmetric with center $\frac{k-3}{2}$. It follows that each term $P_i P_{n+1-i}$ is symmetric with center $\frac{n-5}{2}$, and $\gamma(P_i P_{n+1-i}) = \gamma(P_i)\gamma(P_{n+1-i})$ by (i). By (ii) and (iv),

$$\gamma\left(t \sum_{i=3}^{n-2} \binom{n-2}{i-1} P_i(t) P_{n+1-i}(t) \right) = t \sum_{i=3}^{n-2} \binom{n-2}{i-1} \gamma(P_i)\gamma(P_{n+1-i}) \,,$$

and this polynomial has center $\frac{n-5}{2} + 1 = \frac{n-3}{2}$. By (iii),

$$\gamma(P_{n-1}(t)(1+t)) = \gamma(P_{n-1}) \,,$$

and $P_{n-1}(t)(1+t)$ also has center $\frac{n-3}{2}$. By (iv) again, we can conclude

$$\gamma(P_n) = \gamma(P_{n-1}) + t \sum_{i=3}^{n-2} \binom{n-2}{i-1} \gamma(P_i)\gamma(P_{n+1-i}) \,. \tag{3.2}$$

By induction the r.h.s. has nonnegative coefficients, and it follows that P_n is γ-positive, as needed. □

Remark 3.4. The argument is analogous to the proof of [FMSV22, Theorem 1.8], which hinges on a recursion for the Hilbert series of the Chow ring of an arbitrary matroid, defined by means of maximal building sets, that is very similar to (2.1). It is tempting to venture that a similar recursion may hold for Chow rings of some matroids w.r.t. more general building sets (but simple examples show that γ-positivity need not hold for arbitrary building sets). This would immediately imply γ-positivity for the corresponding Hilbert series. Theorem 3.3 would be recovered as the particular case given by the braid matroid with respect to the minimal building set, cf. [FMSV22, §5.2]. ⌟

The first several polynomial $G_n(t) := \gamma(P_n)$ for $n \geq 3$ are

$$\begin{aligned}
&1\\
&1\\
&1+3t\\
&1+13t\\
&1+38t+45t^2\\
&1+94t+423t^2\\
&1+213t+2425t^2+1575t^3\\
&1+459t+11017t^2+25497t^3\\
&1+960t+43768t^2+240066t^3+99225t^4\,.
\end{aligned}$$

It would be interesting to study these polynomials further. The polynomial P_n is real-rooted *if and only if* G_n is real-rooted ([FMSV22, Proposition 2.9]), so in order to prove Conjecture 1, it would suffice to prove that G_n is real-rooted for all $n \geq 3$.

Using the recursion (3.2), it is easy to show that the formal power series

$$G(z) := \sum_{n\geq 3} G_n \frac{z^{n-1}}{(n-1)!}$$

is the unique solution of the differential equation

$$\frac{dG}{dz} = \frac{z+G}{1-tG}$$

satisfying $G(0) = 0$.

4. The coefficient of $\mathbb{L}^k$ in $[\overline{\mathcal{M}}_{0,n}]$

Keel's recursion ([Kee92, p. 550]) relates the betti numbers $a_{k,n}$ of $\overline{\mathcal{M}}_{0,n}$ to the numbers $a_{\ell,m}$ for all $0 \leq \ell \leq k$, $3 \leq m < n$. This does not suffice for investigating log-concavity, since we need specific information about $a_{k,n}$ for individual k. In this section we obtain a precise description of the corresponding generating functions, from which we will extract in §5 the asymptotic behavior of $a_{k,n} = \operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n})$ for fixed k, as $n \to \infty$. Our result below appears to be new in this form, notwithstanding the very extensive literature on the cohomology of $\overline{\mathcal{M}}_{0,n}$.

As in the introduction, set

$$\alpha_k(z) = \sum_{n\geq 3} a_{k,n} \frac{z^{n-1}}{(n-1)!} = \sum_{n\geq 3} \operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) \frac{z^{n-1}}{(n-1)!}\,,$$

a generating function for the coefficients of $\mathbb{L}^k$ in $[\overline{\mathcal{M}}_{0,n}]$.

Theorem 4.1. *We have $\alpha_0(z) = e^z - (z+1)$. For all $k > 0$,*

$$\alpha_k(z) = \frac{(k+1)^k}{(k+1)!}\, e^{(k+1)z} + e^z \sum_{m=1}^{k} (-1)^m p_m^{(k)}(z)\, e^{(k-m)z} \tag{4.1}$$

where $p_m^{(k)}(z) \in \mathbb{Q}[z]$, $1 \leq m \leq k$, is a polynomial of degree $2m$ with positive leading coefficient.

Proof. Since $\operatorname{rk} H^0(\overline{\mathcal{M}}_{0,n}) = 1$ for all $n \geq 3$, we have $\alpha_0(z) = \sum_{n\geq 3} \frac{z^{n-1}}{(n-1)!} = e^z - (1+z)$ as stated. Next, consider the formal power series

$$M(z) := \sum_{n\geq 3} [\overline{\mathcal{M}}_{0,n}] \frac{z^{n-1}}{(n-1)!}$$

with coefficients in $K(\mathrm{Var}_k)$. The recursion (2.1) implies easily that M satisfies the following differential equation:

$$\frac{dM}{dz} = \frac{z + (1+\mathbb{L})M}{1 - \mathbb{L}M}\,. \tag{4.2}$$

(*Mutatis mutandis,* this is equivalent to [Man95, (0.8)].) The function $\alpha_k(z)$ are the coefficients of M as a power series in $\mathbb{L}$:

$$M = \alpha_0(z) + \alpha_1(z)\mathbb{L} + \alpha_2(z)\mathbb{L}^2 + \cdots\,.$$

Imposing that this series satisfies (4.2) and reading off the coefficients of $\mathbb{L}^k$ gives us differential equations for these coefficients. The first few such equations are

$$\begin{aligned}
\frac{d\alpha_0}{dz} &= \alpha_0 + z \\
\frac{d\alpha_1}{dz} &= \alpha_0^2 + \alpha_0 z + \alpha_0 + \alpha_1 \\
\frac{d\alpha_2}{dz} &= \alpha_0^3 + \alpha_0^2 z + \alpha_0^2 + 2\alpha_0\alpha_1 + \alpha_1 z + \alpha_1 + \alpha_2 \\
\frac{d\alpha_3}{dz} &= \alpha_0^4 + \alpha_0^3 z + \alpha_0^3 + 3\alpha_0^2\alpha_1 + 2\alpha_0\alpha_1 z + 2\alpha_0\alpha_1 + 2\alpha_0\alpha_2 + \alpha_1^2 + \alpha_2 z + \alpha_2 + \alpha_3 \\
&\cdots
\end{aligned}$$

and solving them recursively, they take the form

$$\begin{aligned}
\frac{d\alpha_0}{dz} &= \alpha_0 + z \\
\frac{d\alpha_1}{dz} &= \alpha_1 + e^{2z} - e^z z - e^z \\
\frac{d\alpha_2}{dz} &= \alpha_2 + 3e^{3z} - (z^2 + 5z + 5)e^{2z} + \frac{z^3 + 5z^2 + 8z + 4}{2} e^z \\
&\cdots\,.
\end{aligned}$$

The theorem will be an easy consequence of the following result.

Lemma 4.2. *For $k \geq 1$, the function $\alpha_k(z)$ satisfies a differential equation of the form*

$$\frac{d\alpha_k}{dz} = \alpha_k + k\,\frac{(k+1)^k}{(k+1)!}\,e^{(k+1)z} + e^z \sum_{m=1}^{k} (-1)^m f_m^{(k)}(z)\,e^{(k-m)z} \tag{4.3}$$

where $f_m^{(k)}(z) \in \mathbb{Q}[z]$ denotes a polynomial of degree $2m$ with positive leading coefficient for $m = 1, \cdots, k-1$, and $f_k^{(k)}(z) \in \mathbb{Q}[z]$ is a polynomial of degree $2k-1$ with positive leading coefficient.

To see that (4.3) implies (4.1), set $\alpha_k = e^z A_k$; by (4.3),

$$\frac{dA_k}{dz} = k\,\frac{(k+1)^k}{(k+1)!}\,e^{kz} + \sum_{m=1}^{k} (-1)^m f_m^{(k)}(z) e^{(k-m)z}\,,$$

from which

$$A_k = \frac{(k+1)^k}{(k+1)!}\, e^{kz} + \sum_{m=1}^{k} (-1)^m p_m^{(k)}(z) e^{(k-m)z}$$

where $p_m^{(k)}(z) \in \mathbb{Q}[z]$ are determined by integration by parts and we absorb the constant of integration in the summation. For $k - m > 0$, $\deg p_m^{(k)} = \deg f_m^{(k)} = 2m$; for $m = k$, $\deg p_m^{(k)} = 1 + \deg f_k^{(k)} = 2k$. The leading coefficient of $p_m^{(k)}(z)$ has the same sign as the leading coefficient of $f_m^{(k)}$. The expression (4.1) for $\alpha_k = e^z A_k$ given in Theorem 4.1 follows.

Therefore, we only need to prove Lemma 4.2.

Proof of Lemma 4.2. For all $i > 0$, consider the two statements

$$(L_i) \qquad \frac{d\alpha_i}{dz} = \alpha_i + e^z \sum_{m=0}^{i} (-1)^m f_m^{(i)}(z)\, e^{(i-m)z}$$

$$(T_i) \qquad \alpha_i(z) = e^z \sum_{m=0}^{i} (-1)^m p_m^{(i)}(z)\, e^{(i-m)z}$$

where $f_0^{(i)}(z) = i\, \frac{(i+1)^i}{(i+1)!}$, $p_0^{(i)}(z) = \frac{(i+1)^i}{(i+1)!}$, and the other polynomials $f_m^{(i)}(z)$, $p_m^{(i)}(z)$ satisfy the conditions listed in Lemma 4.2 and Theorem 4.1.

We have to prove that (L_k) holds for all $k > 0$. As shown above, (L_1) and (L_2) hold. We work by strong induction. By the argument preceding this proof, $(L_i) \implies (T_i)$. Therefore, in proving (L_k) we may assume the truth of both (L_i) and (T_i) for all $1 \le i < k$, as well as the expression $\alpha_0(z) = e^z - (z+1)$, which we have already verified.

Rewrite (4.2) as

$$\begin{aligned} \frac{dM}{dz} &= \frac{z + (1+\mathbb{L})M}{1 - \mathbb{L}M} = (z + M) + \mathbb{L}M \frac{1 + z + M}{1 - \mathbb{L}M} \\ &= z + M + \sum_{\ell \ge 1} \mathbb{L}^\ell M^\ell (1 + z + M)\,. \end{aligned}$$

The equation satisfied by the coefficient of $\mathbb{L}^k$ for $k > 0$ is

$$\frac{d\alpha_k}{dz} = \alpha_k + \sum_{\ell=1}^{k} \text{coefficient of } \mathbb{L}^{k-\ell} \text{ in } M^\ell (1 + z + M)\,. \tag{4.4}$$

The coefficients of $\mathbb{L}^{k-\ell}$ in M^ℓ and $M^{\ell+1}$ are respectively

$$\sum_{i_1+\cdots+i_\ell = k-\ell} \alpha_{i_1} \cdots \alpha_{i_\ell}\,; \qquad \sum_{i_1+\cdots+i_{\ell+1} = k-\ell} \alpha_{i_1} \cdots \alpha_{i_{\ell+1}}\,. \tag{4.5}$$

Since $\ell \ge 1$, the expressions only involve terms α_i with $i < k$, which by induction may be assumed to satisfy (T_i) for $i > 0$ and equal $e^z - (z+1)$ for $i = 0$.

It is clear (by induction) that the summation in (4.4) is a linear combination of exponentials with coefficients in $\mathbb{Q}[z]$. In order to prove (L_k), we have to prove the following.

Claim 4.3. *With notation as above, the coefficient of e^{rz} in*

$$\sum_{\ell=1}^{k} \left((1+z) \sum_{i_1+\cdots+i_\ell = k-\ell} \alpha_{i_1} \cdots \alpha_{i_\ell} + \sum_{i_1+\cdots+i_{\ell+1} = k-\ell} \alpha_{i_1} \cdots \alpha_{i_{\ell+1}} \right) \tag{4.6}$$

equals

$$\begin{cases} 0 & \text{if } r<1 \text{ or } r>k+1; \\ \text{a polynomial of degree } 2k-1 \text{ and sign of l.c. } (-1)^k & \text{if } r=1; \\ \text{a polynomial of degree } 2(k+1-r) \text{ and sign of l.c. } (-1)^{k+1-r} & \text{if } 1<r\le k+1\,. \end{cases}$$

Further, the coefficient of the dominant term $e^{(k+1)z}$ *equals* $f_0^{(k)} = k\,\dfrac{(k+1)^k}{(k+1)!}$.

Since our main application (to asymptotic log-concavity) concerns the dominant term, we focus on the last statement in Claim 4.3 first. For this, note that by the induction hypothesis the two terms in (4.5) respectively equal

$$\left(\sum_{i_1+\cdots+i_\ell=k-\ell}\prod_{m=1}^{\ell}\frac{(i_m+1)^{i_m}}{(i_m+1)!}\right)e^{kz}+\text{lower order terms}$$

and

$$\left(\sum_{i_1+\cdots+i_{\ell+1}=k-\ell}\prod_{m=1}^{\ell+1}\frac{(i_m+1)^{i_m}}{(i_m+1)!}\right)e^{(k+1)z}+\text{lower order terms}$$

where 'lower order terms' stands for a linear combination with polynomial coefficients of exponentials e^{mz} with $m<k$, resp., $m<k+1$. Therefore, the equation satisfied by α_k is

$$\frac{d\alpha_k}{dz}=\alpha_k+\sum_{\ell=1}^{k}\left(\sum_{i_1+\cdots+i_{\ell+1}=k-\ell}\prod_{m=1}^{\ell+1}\frac{(i_m+1)^{i_m}}{(i_m+1)!}\right)e^{(k+1)z}+\text{lower order terms.}\tag{4.7}$$

Lemma 4.4.

$$\sum_{\ell=1}^{k}\left(\sum_{j_1+\cdots+j_{\ell+1}=k-\ell}\prod_{m=1}^{\ell+1}\frac{(j_m+1)^{j_m}}{(j_m+1)!}\right)=k\frac{(k+1)^k}{(k+1)!}\,.$$

Proof. Let

$$W(t):=\sum_{j\ge 0}\frac{(-(j+1))^j}{(j+1)!}t^{j+1}\,.$$

This is the *principal branch of the Lambert W function;* in particular,

$$W(t)e^{W(t)}=t$$

(see e.g., [CGH+96, (3.1)]). By implicit differentiation,

$$\frac{dW}{dt}=\frac{W(t)}{t(1+W(t))}\,,$$

and it follows that

$$t^2\frac{d}{dt}\left(\frac{W(t)}{t}\right)=-\frac{W(t)^2}{1+W(t)}=-W(t)^2+W(t)^3-W(t)^4+\cdots\tag{4.8}$$

The coefficient of t^{k+1} in the l.h.s. of (4.8) is

$$(-1)^k\,k\,\frac{(k+1)^k}{(k+1)!}\,.$$

The coefficient of t^{k+1} in the r.h.s. equals the coefficient of t^{k+1} in

$$\sum_{\ell=1}^{k}(-1)^\ell W(t)^{\ell+1}\,.$$

Now $(j_1+1)+\cdots+(j_{\ell+1}+1)=k+1$ if and only if $j_1+\cdots+j_{\ell+1}=k-\ell$, therefore the coefficient of t^{k+1} in $(-1)^\ell W(t)^{\ell+1}$ equals

$$(-1)^\ell \sum_{j_1+\cdots+j_{\ell+1}=k-\ell}\prod_{m=1}^{\ell+1}\frac{-(j_m+1)^{j_m}}{(j_m+1)!}=(-1)^k \sum_{j_1+\cdots+j_{\ell+1}=k-\ell}\prod_{m=1}^{\ell+1}\frac{(j_m+1)^{j_m}}{(j_m+1)!}$$

and this concludes the proof. □

By Lemma 4.4, we can rewrite (4.7) as

$$\frac{d\alpha_k}{dz}=\alpha_k+k\,\frac{(k+1)^k}{(k+1)!}\,e^{(k+1)z}+\text{lower order terms}$$

and this concludes the verification that $f_0^{(k)}=k\,\dfrac{(k+1)^k}{(k+1)!}$ as stated in Claim 4.3.

The rest of the proof of Claim 4.3 is a straightforward, but somewhat involved, verification. If all i_j are $<k$ and positive, then by the induction hypothesis

$$\alpha_{i_1}\cdots\alpha_{i_s}=e^{sz}\sum_{m=0}^{i_1+\cdots+i_s}(-1)^m g_m(z)e^{(i_1+\cdots+i_s-m)z}$$

with $g_m(z)\in\mathbb{Q}[z]$ a polynomial of degree $2m$ and positive leading coefficient. The coefficient of e^{rz} in this term is

$$\begin{cases}0 & \text{if } r<s \text{ or } r>i_1+\cdots+i_s+s\\ (-1)^{i_1+\cdots+i_s+s-r}g_{i_1+\cdots+i_s-(r-s)}(z) & \text{if } s\le r\le i_1+\cdots+i_s+s.\end{cases} \tag{4.9}$$

On the other hand,

$$\alpha_0^t=(e^z-(z+1))^t=\sum_{m=0}^{t}\binom{t}{m}(-1)^{t-m}(z+1)^{t-m}e^{mz}\,,$$

therefore the coefficient of e^{rz} in α_0^t is

$$\begin{cases}0 & \text{if } r<0 \text{ or } r>t\\ \binom{t}{r}(-1)^{t-r}(z+1)^{t-r} & \text{if } 0\le r\le t.\end{cases} \tag{4.10}$$

We will frequently refer to (4.9) and (4.10) in the rest of the proof.

First, (4.9) and (4.10) imply that the coefficient of e^{rz} in (4.6) is possibly nonzero only if $0\le r\le k+1$. Indeed, the maximum exponent for $\alpha_0^t\alpha_{i_1}\cdots\alpha_{i_s}$, where all i_j are positive, is

$$t+(i_1+\cdots+i_s)+s$$

by (4.9) and (4.10), so for $t+s=\ell+1$ and $i_1+\cdots+i_s=\ell-k$ it equals $k+1$.

Next, consider the case $r=0$, that is, the term in (4.6) not involving exponentials. By (4.9), the only possibly nonzero contributions to $r=0$ in (4.6) come from terms with all i_j equal to 0. However, in this case $\sum i_j=0$, that is, $k-\ell=0$, and the corresponding summands in (4.6) are

$$(1+z)\alpha_0^k+\alpha_0^{k+1}=\alpha_0^k(1+z+\alpha_0)=(e^z-(z+1))^k e^z\,. \tag{4.11}$$

This is a multiple of e^z, therefore the contribution to $r = 0$ vanishes, as stated in Claim 4.3.

For $r = 1$: By (4.9), at most one index may be nonzero. If all indices equal 0, then $\ell = k$ as in the previous case, the corresponding part of (4.6) is (4.11), and the coefficient of e^z equals $(-1)^k(1+z)^k$. For $1 \le \ell < k$, the corresponding contribution to (4.6) is

$$\ell(1+z)\alpha_0^{\ell-1}\alpha_{k-\ell} + (\ell+1)\alpha_0^{\ell}\alpha_{k-\ell} = \alpha_0^{\ell-1}\alpha_{k-\ell}\left((\ell+1)e^z - (1+z)\right).$$

Now $\alpha_{k-\ell}$ is a multiple of e^z, so the coefficient of e^z in this expression is the coefficient in $-\alpha_0^{\ell-1}\alpha_{k-\ell}(1+z)$, that is,

$$(-1)^k(1+z)^{\ell}p_{k-\ell}^{(k-\ell)}(z).$$

These polynomials have degrees $k+1, k+2, \dots, 2k-1$ as $\ell = k-1, k-2, \dots, 1$.

The conclusion is that the coefficient of e^z in (4.6) has degree $2k-1$ and sign of leading coefficient $(-1)^k$, as stated in Claim 4.3.

Finally, we consider the case $1 < r \le k+1$. Each α_i with $i > 0$ is a multiple of e^z, so the coefficient of e^{rz} is nonzero only for terms in (4.6) with at most r indices $i_j > 1$. These terms are of two types. First, we have terms

$$(1+z)\alpha_0^t\alpha_{i_1}\cdots\alpha_{i_s} \tag{4.12}$$

with $s \le r$, all i_j positive, $i_1 + \cdots + i_s = k - \ell$, and $s + t = \ell$. The maximum r for which e^{rz} appears in (4.12) is

$$t + (i_1+1) + \cdots + (i_s+1) = s + t + \sum i_j = \ell + k - \ell = k.$$

Therefore (4.12) does not contribute to the coefficient of e^{rz} if $r = k+1$.

For $1 < r \le k$, (4.9) and (4.10) imply that the coefficient of e^{rz} in (4.12) equals

$$\sum_{r_1+r_2=r}\binom{t}{r_1}(-1)^{k-r}(z+1)^{t-r_1+1}g_{k-r-(t-r_1)}(z).$$

For an individual summand to be nonzero we need $r_1 \le t$, i.e., $r_1 - t \le 0$, as well as $s \le r_2$, i.e., $r_1 - t \le r - \ell$. Each nonzero summand has degree

$$t - r_1 + 1 + 2(k - r - (t - r_1)) = 2k - 2r + 1 + r_1 - t;$$

since $r_1 - t \le \min(0, r-\ell)$ for nonzero summands, the maximum of this expression is

$$2k - 2r + 1 + \min(0, r-\ell) < 2(k - r + 1).$$

Therefore, if $1 < r \le k$, the coefficient of e^{rz} in each term of type (4.12) is a polynomial of degree strictly less than $2(k-r+1)$.

The other possible type is

$$\alpha_0^t\alpha_{i_1}\cdots\alpha_{i_s} \tag{4.13}$$

with $s \le r$, all i_j positive, $i_1 + \cdots + i_s = k - \ell$, and $s + t = \ell + 1$. By (4.9) and (4.10), the coefficient of e^{rz} in this term equals

$$\sum_{r_1+r_2=r}\binom{t}{r_1}(-1)^{k-r+1}(z+1)^{t-r_1}g_{k+1-r-(t-r_1)}.$$

We argue as above: nonzero individual summands have $r_1 - t \le 0$ and $s \le r_2$, i.e., $r_1 - t \le r - (\ell+1)$. Now note that as $1 < r$, for all r the sum (4.6) will include terms (4.13) with $\ell + 1 \le r$. For these terms, the condition $r_1 - t \le 0$ implies the condition $s \le r_2$; the degree of the summand,

$$(t - r_1) + 2(k + 1 - r - (t - r_1)) = 2(k+1-r) + (r_1 - t),$$

achieves its maximum for $r_1 = t$ and equals $2(k+1-r)$. All these summands are of the form

$$(-1)^{k-r+1} g_{k+1-r}\,,$$

so the sign of their leading coefficient is $(-1)^{k+1-r}$.

We conclude that, for $1 < r \le k$, the coefficient of e^{rz} in (4.6) is the sum of polynomials of degree $< 2(k+1-r)$ obtained from terms of type (4.12) and from terms of type (4.13) with $\ell \ge r$, and of polynomials of degree exactly $2(k+1-r)$ and sign of leading coefficient $(-1)^{k+1-r}$, from terms of type (4.13) with $\ell < r$.

Therefore in this case the coefficient of e^{rz} in (4.6) is a polynomial of degree $2(k+1-r)$ and sign of leading coefficient $(-1)^{k+1-r}$, and this completes the verification of Claim 4.3. $\square$

This concludes the proof of Lemma 4.2 and therefore of Theorem 4.1. $\square$

Theorem 4.1 identifies the degrees and signs of leading coefficients of the coefficients $p_m^{(k)}(z)$, $m = 1, \dots, k$, in the expression

$$\sum_{n\ge 3} \operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) \frac{z^{n-1}}{(n-1)!} = \frac{(k+1)^k}{(k+1)!} e^{(k+1)z} + e^z \sum_{m=1}^{k} (-1)^m p_m^{(k)}(z)\, e^{(k-m)z}\,,$$

valid for $k \ge 1$. The first several such coefficients are

$$p_1^{(1)} = \frac{1}{2} z^2 + z + 1$$

$$p_2^{(1)} = z^2 + 3z + 2$$
$$p_2^{(2)} = \frac{1}{8} z^4 + \frac{5}{6} z^3 + 2z^2 + 2z + \frac{1}{2}$$

$$p_3^{(1)} = \frac{9}{4} z^2 + \frac{15}{2} z + 5$$
$$p_3^{(2)} = \frac{1}{2} z^4 + \frac{11}{3} z^3 + 9z^2 + 9z + 3$$
$$p_3^{(3)} = \frac{1}{48} z^6 + \frac{7}{24} z^5 + \frac{35}{24} z^4 + \frac{7}{2} z^3 + \frac{17}{4} z^2 + \frac{5}{2} z + \frac{2}{3}$$

$$p_4^{(1)} = \frac{16}{3} z^2 + \frac{56}{3} z + \frac{38}{3}$$
$$p_4^{(2)} = \frac{27}{16} z^4 + \frac{51}{4} z^3 + \frac{129}{4} z^2 + \frac{65}{2} z + \frac{45}{4}$$
$$p_4^{(3)} = \frac{1}{6} z^6 + \frac{13}{6} z^5 + \frac{21}{2} z^4 + \frac{74}{3} z^3 + 30z^2 + 18z + \frac{13}{3}$$
$$p_4^{(4)} = \frac{1}{384} z^8 + \frac{1}{16} z^7 + \frac{5}{9} z^6 + \frac{49}{20} z^5 + \frac{289}{48} z^4 + \frac{103}{12} z^3 + \frac{85}{12} z^2 + \frac{19}{6} z + \frac{13}{24}$$

The leading coefficients of these polynomials are positive by Theorem 4.1, but note that the polynomials themselves appear to be positive. In fact, numerical evidence suggests the following.

Conjecture 2. *For all $k \ge 1$, the polynomials $p_m^{(k)}$, $m = 1, \dots, k$, have positive coefficients and are log-concave with no internal zeros. All but $p_1^{(1)}$, $p_3^{(3)}$, $p_5^{(5)}$ are ultra-log-concave.*

(Note: $p_1^{(1)}(z)$, $p_3^{(3)}(z)$, $p_5^{(5)}(z)$ are log-concave.)

A refinement of the techniques used in this paper yields candidate generating functions for these coefficients which, if confirmed, would imply the positivity part of Conjecture 2 and provide further evidence for their (ultra-)log-concavity. We will report on this elsewhere.

5. Proof of Theorems 1.1 and 1.3

Theorem 1.3 follows easily from Theorem 4.1. In fact, Theorem 4.1 implies the following more precise statement. Denote by $c_{mj}^{(k)} \in \mathbb{Q}$ the coefficients of $p_m^{(k)}(z)$:

$$p_m^{(k)}(z) = \sum_{j=0}^{2m} c_{mj}^{(k)} z^j \,.$$

Theorem 5.1. *Let $n \geq 3$. For every $k \geq 1$:*

$$a_{k,n} = \operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) = \frac{(k+1)^{k+n-1}}{(k+1)!} + \sum_{m=1}^{k} (-1)^m \sum_{j=0}^{2m} \binom{n-1}{j} c_{mj}^{(k)} \, j!(k-m+1)^{n-1-j} \,.$$

Proof. By definition, $a_{k,n}$ is the coefficient of $\frac{z^{n-1}}{(n-1)!}$ in the expansion of $\alpha_k(z)$. The stated formula follows from Theorem 4.1. $\square$

Remark 5.2. By Theorem 4.1, $c_{m,2m}^{(k)} > 0$. By Theorem 5.1,

$$\operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n}) = \frac{(k+1)^{k-1}}{(k+1)!} \cdot (k+1)^n + \sum_{m=1}^{k} (-1)^m q_m^{(k)}(n) \cdot (k+1-m)^n$$

where

$$q_m^{(k)}(n) = \sum_{j=0}^{2m} \frac{c_{mj}^{(k)}}{(k-m+1)^{j+1}} (n-1)\cdots(n-j)$$

is a polynomial in $\mathbb{Q}[n]$ of degree $2m$ and with positive leading coefficient.

If the positivity claim in Conjecture 2 holds, then all coefficients $c_{mj}^{(k)}$ are positive for $k \geq 1$, $1 \leq m \leq k$, $0 \leq j \leq 2m$. ⌟

Example 5.3. We have:

$$\operatorname{rk} H^2(\overline{\mathcal{M}}_{0,n}) = \frac{1}{2} \cdot 2^n - \frac{n^2 - n + 2}{2} \qquad \text{(cf. [Kee92, p. 550])}$$

$$\operatorname{rk} H^4(\overline{\mathcal{M}}_{0,n}) = \frac{1}{2} \cdot 3^n - \frac{n^2 + 3n + 4}{8} \cdot 2^n + \frac{3n^4 - 10n^3 + 33n^2 - 26n + 12}{24}$$

and

$$\operatorname{rk} H^6(\overline{\mathcal{M}}_{0,n}) = \frac{2}{3} \cdot 4^n - \frac{(n+4)(n+3)}{12} \cdot 3^n + \frac{3n^4 + 14n^3 + 57n^2 + 118n + 96}{192} \cdot 2^n - \frac{n^6 - 7n^5 + 35n^4 - 77n^3 + 120n^2 - 72n + 32}{48} \,.$$

In future work we will discuss an explicit conjectural expression for $\operatorname{rk} H^{2k}(\overline{\mathcal{M}}_{0,n})$ for all $k \geq 0$, $n \geq 3$. ⌟

Proof of Theorem 1.3. The statement is true for $k = 0$. For $k > 0$, it is an immediate consequence of Theorem 5.1, since (with notation as in Remark 5.2)

$$\lim_{n\to\infty} q_m^{(k)}(n)\frac{(k-m+1)^n}{(k+1)^n} = 0$$

for $1 \le m \le k$. □

Finally, we deduce Theorem 1.1 from Theorem 1.3.

Proof of Theorem 1.1. We verify the stronger claim that for any fixed $k > 0$, the limit of the ratio

$$\left(\frac{a_{k,n}}{\binom{n-3}{k}}\right)^2 \bigg/ \left(\frac{a_{k-1,n}}{\binom{n-3}{k-1}}\cdot\frac{a_{k+1,n}}{\binom{n-3}{k+1}}\right) \tag{5.1}$$

as $n \to \infty$ is $+\infty$. The terms involved in the ratio are of type

$$a_{i,n} \bigg/ \binom{n-3}{i} \ ,$$

and by Theorem 1.3 this is asymptotic to

$$\frac{(i+1)^{i+n-1}}{(i+1)!}\bigg/\frac{(n-3)!}{i!(n-i-3)!} = \frac{i!(i+1)^{i+n-1}(n-i-3)!}{(i+1)!(n-3)!} = \frac{(i+1)^{i+n-2}(n-i-3)!}{(n-3)!}\,.$$

Thus, the limit of (5.1) as $n \to \infty$ equals the limit of

$$\left(\frac{(k+1)^{k+n-2}(n-k-3)!}{(n-3)!}\right)^2 \bigg/ \left(\frac{k^{k+n-3}(n-k-2)!}{(n-3)!}\frac{(k+2)^{k+n-1}(n-k-4)!}{(n-3)!}\right)$$

as $n \to \infty$. This expression equals

$$\frac{(k+1)^{2(k+n-2)}(n-k-3)}{k^{k+n-3}(k+2)^{k+n-1}(n-k-2)} = \frac{(k+1)^{2(k-2)}}{k^{k-3}(k+2)^{k-1}}\cdot\left(\frac{(k+1)^2}{k(k+2)}\right)^n\cdot\frac{n-k-3}{n-k-2}$$

and converges to ∞ as claimed as $n \to \infty$, since $\frac{(k+1)^2}{k(k+2)} = \frac{k^2+2k+1}{k^2+2k} > 1$. □

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