FROM MOTIVIC CHERN CLASSES OF SCHUBERT CELLS TO THEIR HIRZEBRUCH AND CSM CLASSES

PAOLO ALUFFI, LEONARDO C. MIHALCEA, JÖRG SCHÜRMANN, AND CHANGJIAN SU

ABSTRACT. The equivariant motivic Chern class of a Schubert cell in a complete flag manifold $X = G/B$ is an element in the equivariant K-theory ring of X to which one adjoins a formal parameter y. In this paper we prove several folklore results about motivic Chern classes, including finding specializations at $y = -1$ and $y = 0$; the coefficient of the top power of y; how to obtain Chern-Schwartz-MacPherson (CSM) classes as leading terms of motivic classes; divisibility properties of the Schubert expansion of motivic Chern classes. We collect several conjectures on the positivity, unimodality, and log concavity of CSM and motivic Chern classes of Schubert cells, including a conjectural positivity of structure constants of the multiplication of Poincaré duals of CSM classes. In addition, we prove a 'star duality' for the motivic Chern classes, showing how they behave under the involution taking a vector bundle to its dual. We use the motivic Chern transformation to define two equivariant variants of the Hirzebruch transformation, which appear naturally in the Grothendieck-Hirzebruch-Riemann-Roch formalism. We utilize the Demazure-Lusztig recursions from the motivic Chern class theory to find similar recursions giving the Hirzebruch classes of Schubert cells, their Poincaré duals, and their Segre versions. We explain the functoriality properties needed to extend the results to partial flag manifolds G/P.

CONTENTS

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1. Introduction

Let X be a quasi-projective complex algebraic variety and denote by $\mathrm{K}_0(\mathrm{var}/X)$ the Grothendieck motivic group consisting of equivalence classes of morphisms $[f : Z \to X]$ modulo the usual additivity relations. Also, denote by $\mathrm{K}(X)$ the Grothendieck ring of vector bundles on X. The *motivic Chern transformation* defined by Brasselet, Schürmann and Yokura [BSY10] is the assignment for every such X of a group homomorphism

$$\mathrm{MC}_y : \mathrm{K}_0(\mathrm{var}/X) \to \mathrm{K}(X)[y],$$

uniquely determined by the fact that it commutes with proper push-forwards and that it satisfies the normalization condition

$$\mathrm{MC}_y[\mathrm{id}_X : X \to X] = \lambda_y(T^*_X) = \sum y^i[\wedge^i T^*X] \in \mathrm{K}(X)[y]$$

if X is nonsingular. If $Z \hookrightarrow X$ is a locally closed subset, the *motivic Chern class* of Z (regarded in X) is defined by

$$\mathrm{MC}_y(Z) := \mathrm{MC}_y[Z \hookrightarrow X] \in \mathrm{K}(X)[y];$$

here y is a formal indeterminate.

If X admits a torus action, there is an equivariant version $\mathrm{MC}^T_y : \mathrm{K}^T_0(\mathrm{var}/X) \to \mathrm{K}_T(X)[y]$ defined in [FRW21, AMSS19]. We will work in this context, and omit the superscript T from the notation to increase legibility.

Our main object of study in this paper will be the classes $\mathrm{MC}_y(X(w)^\circ)$, the (torus equivariant) motivic Chern classes of Schubert cells $X(w)^\circ$ in the flag manifolds G/B, for G a complex, semisimple, Lie group, and $B \subseteq G$ a Borel subgroup. By functoriality, these determine the motivic Chern classes in the 'partial' flag manifolds G/P, with $P \supseteq B$ a standard parabolic subgroup.

The motivic classes $\mathrm{MC}_y(X(w)^\circ)$ are closely related to the study of representation theory of the Hecke algebra of G, and through this connection they play a prominent role in several related topics: (K-theoretic) stable envelopes and integrable systems [RTV15, AMSS19, FRW21], Whittaker functions from p-adic representation theory [MS22], characteristic classes of singular varieties [FRW17]. In interesting situations they recover point counting over finite fields [MS22] (see also §4.3 below), and are closely related to the study of the intersection (co)homology and the Riemann-Hilbert correspondence for arbitrary complex projective manifolds [Sch09]. In Schubert Calculus, the motivic Chern classes, and their (co)homological counterparts, the *Chern-Schwartz-MacPherson (CSM)* classes, provide deformations of the usual Schubert classes, which, provably or conjecturally, satisfy remarkable positivity, unimodality, and log-concavity properties; see §8 below.

Among the main goals of this paper is to gather in a single place several folklore results concerning properties of (torus equivariant) motivic classes $\mathrm{MC}_y(X(w)^\circ)$. These include results on the specializations at $y = -1$ and $y = 0$ of the parameter y; the coefficient of $y^{\dim X(w)}$ in $\mathrm{MC}_y(X(w)^\circ)$; how to recover the CSM classes as the initial terms of the motivic Chern classes; divisibility properties of Schubert expansions. We also state several conjectures and prove a new duality for motivic Chern classes.

Our main new contribution is a treatment of the (torus equivariant) *Hirzebruch transformation* $\mathrm{Td}^T_{y,*}$, and in particular a study of the *Hirzebruch classes* $\mathrm{Td}^T_{y,*}(X(w)^\circ)$ of Schubert cells, as an application of properties of motivic Chern classes. Similarly to the motivic Chern transformation, the Hirzebruch transformation $\mathrm{Td}^T_{y,*} : \mathrm{K}^T_0(\mathrm{var}/X) \to \widehat{H}^T_*(X; \mathbb{Q}[y])$ is a functorial transformation defined uniquely by a normalization property, with values in a completed (Borel-Moore or Chow) homology group; see §6. In the non-equivariant context this transformation was defined in [BSY10], and it arises naturally in the context of the Grothendieck-Hirzebruch-Riemann-Roch (GHRR) formalism. The 'unnormalized' variant of this transformation was studied by Weber [Web16, Web17]. As in the case of MC, we will omit the superscript T from the notation, since all the classes considered in this paper are equivariant by default.

In this paper we extend the definition of the Hirzebruch transformation to the equivariant context, for arbitrary quasi-projective complex algebraic varieties X with a torus action. As hinted above, there are two variants of the Hirzebruch transformation. The 'unnormalized' variant is defined in Theorem 6.1 as the composition

$$\widetilde{\mathrm{Td}}_{y,*} := \mathrm{td}_* \circ \mathrm{MC}_y : \mathrm{K}^T_0(\mathrm{var}/X) \to \widehat{H}^T_*(X; \mathbb{Q}[y])$$

of the (equivariant) Todd transformation td_* constructed by Edidin and Graham [EG00] with the motivic Chern transformation. The 'normalized' version of Definition 6.4 is the composition

$$\mathrm{Td}_{y,*} := \psi^{1+y}_* \circ \widetilde{\mathrm{Td}}_{y,*} : \mathrm{K}^T_0(\mathrm{var}/X) \to \widehat{H}^T_*(X; \mathbb{Q}[y]) \subseteq \widehat{H}^T_*(X; \mathbb{Q}[y, (1+y)^{-1}])$$

of a certain Adams operator with the unnormalized transformation. The Adams operator acts by multiplying by powers of $1 + y$ (see §6). A technical subtlety is that *a priori* $\mathrm{Td}_{y,*}$ requires coefficients in $\mathbb{Q}[y, (1 + y)^{-1}]$, but it can be shown that $\mathbb{Q}[y]$ suffices. In fact, an important property of $\mathrm{Td}_{y,*}$ is that the specialization at $y = -1$ is well-defined. This specialization recovers the (equivariant) MacPherson's transformation [Mac74, Ohm06] $c^T_* : \mathcal{F}^T(X) \to H^T_*(X)$ from the group of (equivariant) constructible functions to homology; see Corollary 6.6 and also [AMSS17, §3.2] for a summary of Ohmoto's definition and a discussion of alternative (equivalent) definitions.

We cover general preliminaries in §2 and then focus on the case of flag manifolds. In §3 we recall the definition of the Demazure-Lusztig (DL) operators determining recursively the motivic Chern classes $\mathrm{MC}_y(X(w)^\circ)$ of Schubert cells in flag manifolds. We show (Proposition 3.5) how one can recover the cohomological DL operators as initial terms of the Chern character applied to the K-theoretic DL operators; in other words, how one recovers the action of the degenerate Hecke algebra as a limit of the action of the Hecke algebra. In §4 we investigate several properties of the motivic Chern classes, including a divisibility property by powers of $1 + y$ of the coefficients arising in their Schubert expansions (illustrated in

Example 4.10), and the fact that the integral of a motivic Chern class of a Schubert cell is equal to the number of points over the finite field $\mathbb{F}_q$ with $q = -y$ elements, see Proposition 4.13 and Remark 4.14. In §5 we study the effect of specializing the parameter y on motivic Chern classes. In a nutshell, $y = -1$ recovers the fixed point classes, and $y = 0$ the ideal sheaf classes, see Theorem 5.1.

In §6 we obtain 'Hirzebruch operators' calculating recursively the (equivariant) Hirzebruch classes $\widetilde{\mathrm{Td}}_{y,*}(X(w)^\circ)$ and $\mathrm{Td}_{y,*}(X(w)^\circ)$ (see Theorem 6.11), their Poincaré duals (see Theorem 6.12), and the Segre-Hirzebruch classes (see Theorem 6.13). Our treatment in §6 is particularly extensive, so we summarize some of our results here for the convenience of the reader.

Let P_i be the minimal parabolic group associated to the i-th simple root, and denote by $p_i : G/B \to G/P_i$ the natural projection. The BGG operator ∂_i^H is defined as $(p_i)^*(p_i)_*$. Define the unnormalized and the normalized variants of the Hirzebruch operators

$$\widetilde{\mathcal{T}}_i^{\mathrm{Hir}}, \mathcal{T}_i^{\mathrm{Hir}} : \widehat{H}_T^*(X, \mathbb{Q})[y] \to \widehat{H}_T^*(X, \mathbb{Q})[y]$$

by

$$\widetilde{\mathcal{T}}_i^{\mathrm{Hir}} := \widetilde{\mathrm{Td}}_y(T_{p_i})\partial_i^H - \mathrm{id}; \quad \mathcal{T}_i^{\mathrm{Hir}} := \mathrm{Td}_y(T_{p_i})\partial_i^H - \mathrm{id}\,.$$

We prove in Theorem 6.11 that these operators determine recursively the (un)normalized Hirzebruch classes of Schubert cells. More precisely, $\widetilde{\mathrm{Td}}_{y,*}(X(\mathrm{id})^\circ) = \mathrm{Td}_{y,*}(X(\mathrm{id})^\circ) = [X(\mathrm{id})]_T$ and

$$\widetilde{\mathcal{T}}_i^{\mathrm{Hir}}(\widetilde{\mathrm{Td}}_{y,*}(X(w)^\circ)) = \widetilde{\mathrm{Td}}_{y,*}(X(ws_i)^\circ) \quad , \quad \mathcal{T}_i^{\mathrm{Hir}}(\mathrm{Td}_{y,*}(X(w)^\circ)) = \mathrm{Td}_{y,*}(X(ws_i)^\circ)$$

for all $w \in W$ and all simple reflections s_i such that $ws_i > w$ in the Bruhat ordering. In Theorem 6.13 we prove analogous statements for the 'Segre-Hirzebruch classes' $\frac{\widetilde{\mathrm{Td}}_{y*}(X(w)^\circ)}{\widetilde{\mathrm{Td}}_y(TX)}$ and $\frac{\mathrm{Td}_{y*}(X(w)^\circ)}{\mathrm{Td}_y(TX)}$. We find the Poincaré duals of these classes, along with the appropriate version of the DL operators which determine them; see Theorem 6.12. Perhaps not surprisingly, the theory we find is essentially equivalent to that of motivic Chern classes. For instance, the Hirzebruch operators are images of the DL operators in K-theory via a Todd transformation. In particular, the Hirzebruch operators satisfy the same relations as the DL operators in K-theory (see Lemma 6.9 and Remark 6.10), implying that they give an action of the Hecke algebra on the equivariant (co)homology of G/B.

In §6.3 we study the specializations at $y = 0$ and $y = -1$ of the Hirzebruch classes and operators, and we recover from this point of view the Todd transformation $\mathrm{td}_*(\mathcal{I}_w^T)$ of the boundary ideal sheaves $\mathcal{I}_w^T$ of Schubert varieties, respectively the CSM classes $c_{\mathrm{SM}}(X(w)^\circ)$ of Schubert cells. Furthermore, we recover the DL operators giving recursions for these classes. To illustrate, the Grothendieck-Hirzebruch- Riemann-Roch implies that

$$\langle \mathrm{td}_*(\mathcal{I}_u^T), \mathrm{ch}(\mathcal{O}^{v,T}) \rangle = \delta_{u,v},$$

where ch denotes the (equivariant) Chern character. The specialization $y = 0$ in Theorem 6.12 gives the operators, listed in Proposition 6.14, which determine the classes $\mathrm{td}_*(\mathcal{I}_u^T)$ and $\mathrm{ch}(\mathcal{O}^{v,T})$ recursively.

In §7 we revisit the procedure giving the degenerate Hecke algebra action as a limit, in order to explain how CSM classes may be computed directly as the leading terms of the motivic Chern classes, in the case of Schubert cells; see Theorem 7.1. The CSM and motivic classes give bases of the (equivariant) cohomology and K theory rings of flag manifolds.

Based on computational evidence, in §8 we discuss several conjectural properties of CSM and motivic classes, concerning positivity and log-concavity of the coefficients of their Schubert expansions and the positivity of their structure constants.

Finally, in §9 we prove a new 'star duality' for motivic Chern classes, for the duality $\star : \mathrm{K}_T(X) \to \mathrm{K}_T(X)$ which sends the class $[E]$ of a vector bundle to $[E^\vee] = [\mathrm{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)]$. The duality for the motivic classes, stated in Theorem 9.1, generalizes a known relation between ideal sheaves and duals of structure sheaves of Schubert varieties proved by Brion [Bri05, Prop. 4.3.4].

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Throughout this project we used the Maple package `Equivariant Schubert Calculator`, written by Anders Buch.[1]

2. Preliminaries

2.1. **Equivariant (co)homology.** Let X be a quasi-projective complex algebraic variety. In this paper we will deal with the Borel-Moore homology group $H_*(X)$ of X and the cohomology ring $H^*(X)$, with rational coefficients. As an alternative, one could use Chow (co)homology; there is a homology degree-doubling cycle map from Chow to Borel-Moore, and our constructions are compatible with this map. This map is an isomorphism in some important situations, such as the complex flag manifolds studied later in this note. We refer to [Ful84, §19.1] and [Gin98, §2.6] for more details about Borel-Moore homology and its relation to the Chow group. In case we speak of (co)dimension we always assume that our spaces are pure dimensional; in addition, by (co)dimension we will mean the *complex* (co)dimension. Any subvariety $Y \subseteq X$ of (complex) dimension k has a fundamental class $[Y] \in H_{2k}(X)$. Whenever X is smooth, we can and will identify the Borel-Moore homology and cohomology via Poincaré duality.

Let T be a torus and let X be a variety with a T-action. Then the equivariant cohomology $H_T^*(X)$ is the ordinary cohomology of the Borel mixing space $X_T := (ET \times X)/T$, where ET is the universal T-bundle and T acts by $t \cdot (e,x) = (et^{-1}, tx)$. The ring $H_T^*(X)$ is an algebra over $H_T^*(\mathrm{pt})$, the polynomial ring $\mathrm{Sym}_{\mathbb{Q}}\, \mathfrak{X}(T) \simeq \mathbb{Q}[t_1, \dots, t_s]$ in the character group $\mathfrak{X}(T)$ (written additively) and with $t_i \in H_T^2(\mathrm{pt})$; see e.g., [Kum02, §11.3.5]. One may also define T-equivariant Borel-Moore homology and Chow groups, related by an equivariant cycle map; see e.g., [EG98]. Every k-dimensional subvariety $Y \subseteq X$ that is stable under the T action determines an equivariant fundamental class $[Y]_T$ in $H_{2k}^T(X)$.

As in the non-equivariant case, whenever X is smooth, we will identify $H_*^T(X)$ and $H_T^*(X)$. In particular, when $X = \mathrm{pt}$ is a point, the identification sends $a \in H_T^*(\mathrm{pt})$ to $a \cap [\mathrm{pt}]_T$. If

[1] The package is available at `https://sites.math.rutgers.edu/~asbuch/equivcalc/`

X is smooth and proper, then there is an $H_T^*(\mathrm{pt})$-bilinear Poincaré (or intersection) pairing $H_T^*(X) \otimes H_T^*(X) \to H_T^*(\mathrm{pt})$ defined by

$$\langle a, b \rangle = \int_X a \cdot b \quad ,$$

where the integral stands for the push-forward to a point. Equivariant vector bundles have equivariant Chern classes $c_i^T(-)$, such that $c_j^T(E) \cap -$ is an operator $H_i^T(X) \to H_{i-2j}^T(X)$; see [And12, §1.3], [EG98, §2.4].

We address the reader to [And12, Knu, AF23] for background on equivariant cohomology and homology.

2.2. **Equivariant K theory.** For any T-variety X, the equivariant K theory ring $\mathrm{K}_T(X)$ is the Grothendieck ring generated by symbols $[E]$, where $E \to X$ is a T-equivariant vector bundle, modulo the relations $[E] = [E_1] + [E_2]$ for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles. For any proper morphism $f : X \to Y$ there is a functorial push-forward $f_* : \mathrm{K}_T(X) \to \mathrm{K}_T(Y)$ defined by $f_*([E]) = \sum (-1)^i [R^i f_*(E)]$. The ring $\mathrm{K}_T(X)$ is an algebra over $\mathrm{K}_T(\mathrm{pt}) = R(T)$, the representation ring of T. This may be identified with the Laurent polynomial ring $\mathbb{Z}[e^{\pm t_1}, \dots, e^{\pm t_r}]$ where e^{t_i} are characters corresponding to a basis of the character lattice in the Lie algebra of T. We will often denote the tuple $(e^{\pm t_1}, \dots, e^{\pm t_r})$ simply by e^t. An introduction to equivariant K theory may be found in [CG09, Chapter 5]. The equivariant K ring of X admits a 'vector bundle duality' involution $\star : \mathrm{K}_T(X) \to \mathrm{K}_T(X)$ mapping the class $[E]$ of a vector bundle to the class $[E^\vee]$ of its dual. This is not an involution of $\mathrm{K}_T(\mathrm{pt})$-algebras; it satisfies $(e^\lambda)^\vee = e^{-\lambda}$. Under mild hypotheses (e.g., X projective) there is also a 'Serre duality' involution $\mathcal{D} : \mathrm{K}_T(X) \to \mathrm{K}_T(X)$ inherited from (equivariant) Grothendieck-Serre duality and defined by

$$\mathcal{D}([F]) := [R\,\mathrm{Hom}(F, \omega_X^\bullet)] = [\omega_X^\bullet] \otimes [F]^\vee \in \mathrm{K}_T(X)$$

for $[F] \in \mathrm{K}_T(X)$, where $\omega_X^\bullet \simeq \omega_X[\dim X]$ is the (equivariant) dualizing complex of X. Thus if X is nonsingular, $[\omega_X^\bullet] = (-1)^{\dim X}[\omega_X]$ with ω_X the equivariant canonical bundle of X. Observe the multiplicativity

$$\mathcal{D}([E] \otimes [F]) = \mathcal{D}([E]) \otimes [F]^\vee.$$

In later sections of this paper we will primarily be concerned with flag manifolds $X = G/B$, with T acting on X by left multiplication. In this case X is a smooth projective variety and the ring $\mathrm{K}_T(X)$ is naturally isomorphic to the Grothendieck group $K_0(\mathfrak{coh}^T(\mathcal{O}_X))$ of T-linearized coherent sheaves on X. This follows from the fact that every such coherent sheaf has a finite resolution by T-equivariant vector bundles. There is a $\mathrm{K}_T(\mathrm{pt})$-bilinear pairing

$$\langle -, - \rangle : \mathrm{K}_T(X) \otimes \mathrm{K}_T(X) \to \mathrm{K}_T(\mathrm{pt}) = R(T); \quad \langle [E], [F] \rangle := \int_X E \otimes F = \chi(X; E \otimes F),$$

where $\chi(X; E)$ is the (equivariant) Euler characteristic, i.e., the virtual representation

$$\chi(X; E) = \int_X [E] = \sum_i (-1)^i H^i(X; E).$$

Note that

$$\langle \mathcal{D}[E], [F]^\vee \rangle = \int_X \mathcal{D}([E \otimes F]) = \chi(X; E \otimes F)^\vee = (\langle [E], [F] \rangle)^\vee \ ,$$

by equivariant Grothendieck-Serre duality (the corresponding result [Har77, Chapter III, Theorem 7.6] also holds equivariantly, e.g. as a very special case of [LH09, Part II, Theorem 25.2 and Theorem 28.11]).

Let y be an indeterminate. The *Hirzebruch λ_y-class* of an equivariant vector bundle E is the class

$$\lambda_y(E) := \sum_k [\wedge^k E] y^k \in \mathrm{K}_T(X)[y].$$

The λ_y-class is multiplicative, i.e., for any short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of equivariant vector bundles there is an equality $\lambda_y(E) = \lambda_y(E_1)\lambda_y(E_2)$ in $\mathrm{K}_T(X)[y]$. (As pointed out in [Nie74, §1.2], this is part of the λ-ring structure of $\mathrm{K}_T(X)$, cf. [SGA71, 6.V.2.4].)

2.3. **The Chern character.** For a pure-dimensional T-variety X, Edidin and Graham [EG00] defined an equivariant Chern character

$$\mathrm{ch}_T : \mathrm{K}_T(X) \to \widehat{H}^T_*(X) := \prod_{i \le \dim X} H^T_{2i}(X)$$

such that:

- If $V \subseteq X$ is a T-invariant subvariety, then $\mathrm{ch}_T[\mathcal{O}_V] = [V]_T$+l.o.t. (lower order terms). (Non-equivariantly, see [Ful84, Theorem 18.3(5)].)
- If $\mathcal{L}$ is an equivariant line bundle with first Chern class $c_1^T(\mathcal{L})$, then $\mathrm{ch}_T[\mathcal{L}] = e^{c_1^T(\mathcal{L})} \cap [X]_T$. In particular, $\mathrm{ch}_T(e^\lambda) = e^\lambda \in \hat{H}^{\mathbb{C}*}_*(\mathrm{pt})$ for all characters λ.
- ch_T commutes with pull-backs.
- If X is smooth, then after identifying $H^T_*(X) \simeq H^*_T(X)$ via Poincaré duality, ch_T is a ring homomorphism.

We will generally omit the subscript T in the notation, since the equivariant context is assumed throughout the paper. A fundamental result is the Grothendieck-Hirzebruch-Riemann-Roch (GHRR) theorem. In the equivariant case, this was proved in [EG00]. For now we state the following particular form; in §6 below we will need more general versions. Let $f : X \to Y$ be a smooth proper T-equivariant morphism of smooth T-varieties, and let $a \in \mathrm{K}_T(X)$. Then

$$\mathrm{ch}\, f_*(a) = f_*(\mathrm{ch}(a) \cdot \mathrm{Td}(T_f)).$$

where $\mathrm{Td}(T_f)$ is the equivariant Todd class of the relative tangent bundle of f. Recall that if $E \to X$ is an equivariant vector bundle with Chern roots $x_1, \dots, x_e$, then

$$\mathrm{Td}(E) = \prod_{i=1}^e \frac{x_i}{1 - e^{-x_i}} = \prod_{i=1}^e (1 + \frac{1}{2} x_i + \dots);$$

see e.g., [Ful84, Example 3.2.4].

3. Operators in equivariant cohomology and K-theory of flag manifolds

The goal of this section is to introduce the Schubert basis in the equivariant K ring of flag manifolds and recall the definition and basic properties of the Demazure and (cohomological and K-theoretic) Demazure-Lusztig (DL) operators acting on the K-theory ring. An important fact which we will use later, and for which we could not find a reference, is that the cohomological DL operators may be recovered from certain initial terms of the K-theoretic ones; cf. Proposition 3.5.

3.1. **Schubert data.** Let G be a complex semisimple, simply connected, Lie group, and fix a Borel subgroup B with a maximal torus $T \subseteq B$. Let B^- denote the opposite Borel subgroup. Let $W := N_G(T)/T$ be the Weyl group, and $\ell : W \to \mathbb{N}$ the associated length function. Denote by w_0 the longest element in W; then $B^- = w_0 B w_0$. Let also $\Delta := \{\alpha_1, \dots, \alpha_r\} \subseteq R^+$ denote the set of simple roots included in the set of positive roots for (G, B). Let ρ denote the half sum of the positive roots. The simple reflection for the root $\alpha_i \in \Delta$ is denoted by s_i and the corresponding *minimal* parabolic subgroup is denoted by P_i, containing the Borel subgroup B.

Let $X := G/B$ be the (complete) flag variety. It has a stratification by Schubert cells $X(w)^\circ := BwB/B$ and opposite Schubert cells $Y(w)^\circ := B^- wB/B$. The closures $X(w) := \overline{X(w)^\circ}$ and $Y(w) := \overline{Y(w)^\circ}$ are the Schubert varieties. With these definitions, $\dim_{\mathbb{C}} X(w) = \operatorname{codim}_{\mathbb{C}} Y(w) = \ell(w)$. The Weyl group W admits a partial ordering, called the Bruhat ordering, defined by $u \le v$ if and only if $X(u) \subseteq X(v)$.

Let $\mathcal{O}_w^T := [\mathcal{O}_{X(w)}] \in \mathrm{K}_T(G/B)$ be the Grothendieck class determined by the structure sheaf of $X(w)$ (a coherent sheaf), and similarly $\mathcal{O}^{w,T} := [\mathcal{O}_{Y(w)}]$. The equivariant K-theory ring has $\mathrm{K}_T(\mathrm{pt})$-bases $\{\mathcal{O}_w^T\}_{w \in W}$ and $\{\mathcal{O}^{w,T}\}_{w \in W}$ for $w \in W$. Let $\partial X(w) := X(w) \smallsetminus X(w)^\circ$ be the boundary of the Schubert variety $X(w)$, and similarly $\partial Y(w)$ the boundary of $Y(w)$. It is known that the dual bases of $\{\mathcal{O}_w^T\}$ and $\{\mathcal{O}^{w,T}\}$ are given by the classes of the ideal sheaves $\mathcal{I}^{w,T} := [\mathcal{O}_{Y(w)}(-\partial Y(w))]$, respectively $\mathcal{I}_w^T := [\mathcal{O}_{X(w)}(-\partial X(w))]$. I.e.,

$$\langle \mathcal{O}_u^T, \mathcal{I}^{v,T} \rangle = \langle \mathcal{O}^{u,T}, \mathcal{I}_v^T \rangle = \delta_{u,v}. \tag{1}$$

See e.g., [Bri05, Proposition 4.3.2] for the non-equivariant case; the same proof works equivariantly. See also [GK08, Proposition 2.1]. It is also shown in [Bri05] that

$$\mathcal{O}_w^T = \sum_{v \le w} \mathcal{I}_v^T \quad \text{and} \quad \mathcal{I}_w^T = \sum_{v \le w} (-1)^{\ell(w)-\ell(v)} \mathcal{O}_v^T. \tag{2}$$

(Again, Brion's argument also works in the equivariant context.) For any weight (character) λ of T, we denote by $\mathcal{L}_\lambda$ the G-homogeneous line bundle

$$\mathcal{L}_\lambda = G \times^B \mathbb{C}_\lambda.$$

Let P be a standard parabolic subgroup of G, i.e., $B \subseteq P \subseteq G$. Such a subgroup is determined by a subset $\Delta_P \subseteq \Delta$; for instance, $\Delta_B = \emptyset$. Denote by W_P the subgroup of W generated by the simple reflections s_i such that $\alpha_i \in \Delta \smallsetminus \Delta_P$. Let W^P denote the subset of minimal length representatives of W/W_P. By definition, $\ell(wW_P) = \ell(w)$ for $w \in W^P$. Similarly to G/B, the partial flag manifold G/P has finitely B and B^- many orbits – the Schubert cells – indexed by the elements $w \in W^P$:

$$X(wW_P)^\circ = BwP/P; \quad Y(wW_P)^\circ = B^- wP/P;$$

as before $\dim X(wW_P)^\circ = \operatorname{codim} \ Y(wW_P)^\circ = \ell(wW_P)$. The Schubert varieties $X(wW_P)$, $Y(wW_P)$ in G/P are the closures of the corresponding Schubert cells.

3.2. **BGG, Demazure, and Demazure-Lusztig operators.** Fix a simple root $\alpha_i \in \Delta$ and denote by $P_i \subseteq G$ the corresponding minimal parabolic subgroup. Consider the fiber diagram:

$$\begin{array}{ccc} FP := G/B \times_{G/P_i} G/B & \xrightarrow{pr_1} & G/B \\ \downarrow{\scriptstyle pr_2} & & \downarrow{\scriptstyle p_i} \\ G/B & \xrightarrow{p_i} & G/P_i \end{array}$$

The **Bernstein-Gelfand-Gelfand** (BGG) operator [BGG73] is the $H_T^*(\mathrm{pt})$-linear morphism $\partial_i^H : H_T^i(X) \to H_T^{i+2}(X)$ defined by $\partial_i^H := (p_i)^*(p_i)_*$. The same geometric definition gives the **Demazure** operator $\partial_i : \mathrm{K}_T(X) \to \mathrm{K}_T(X)$ in the (equivariant) K-theory, linear over $\mathrm{K}_T(\mathrm{pt})$; see [Dem74]. These operators satisfy

$$\partial_i^H[X(w)]_T = \begin{cases} [X(ws_i)]_T & \text{if } ws_i > w; \\ 0 & \text{otherwise }. \end{cases} \quad \partial_i(\mathcal{O}_w^T) = \begin{cases} \mathcal{O}_{ws_i}^T & \text{if } ws_i > w; \\ \mathcal{O}_w^T & \text{otherwise }. \end{cases} \tag{3}$$

From this, one deduces that both operators satisfy the same commutation and braid relations as those for the elements of W. In cohomology, $(\partial_i^H)^2 = 0$, while in K-theory $\partial_i^2 = \partial_i$.

The relative cotangent bundle of the projection p_i is $T_{p_i}^* = \mathcal{L}_{\alpha_i}$. Define the $H_T^*(\mathrm{pt})$-algebra automorphism $s_i : H_T^*(G/B) \to H_T^*(\mathrm{pt})$ by

$$s_i = \mathrm{id} + c_1^T(T_{p_i}^*)\partial_i^H = \mathrm{id} + c_1^T(\mathcal{L}_{\alpha_i})\partial_i^H. \tag{4}$$

It was proved by Knutson [Knu] that this is an automorphism induced by the right Weyl group action on G/B; see [AM16] and also [MNS22], where both left and right actions are studied. Using this automorphism, the **cohomological Demazure-Lusztig (DL) operators** are $H_T^*(\mathrm{pt})$-linear endomorphism of $H_T^*(G/B)$ defined by

$$\mathcal{T}_i^H = \partial_i^H - s_i; \quad \mathcal{T}_i^{H,\vee} = \partial_i^H + s_i. \tag{5}$$

These operators satisfy the same braid and commutation relations as the BGG operators, and, in addition $(\mathcal{T}_i^H)^2 = (\mathcal{T}_i^{H,\vee})^2 = \mathrm{id}$; see [AM16, Proposition 4.1]. In other words, these give a twisted representation of the Weyl group W on $H_T^*(G/B)$. This representation was studied earlier by Lascoux, Leclerc and Thibon [LLT96], and by Ginzburg [Gin98] in relation to the degenerate Hecke algebra. The operators are adjoint to each other, in the sense that for any $a, b \in H_T^*(G/B)$,

$$\langle \mathcal{T}_i^H(a), b\rangle = \langle a, \mathcal{T}_i^{H,\vee}(b)\rangle.$$

It is convenient to consider a homogenized version of this operator. Add a formal variable $\hbar$ of cohomological complex degree 1. Then the homogenized operators are

$$\mathcal{T}_i^{H,\hbar} = \hbar\partial_i^H - s_i; \quad \mathcal{T}_i^{H,\vee,\hbar} = \hbar\partial_i^H + s_i. \tag{6}$$

The variable $\hbar$ will arise geometrically from the $\mathbb{C}^*$-action by dilation on $T^*(G/B)$. The restriction of this action to the zero-section $G/B \hookrightarrow T^*(G/B)$ is trivial, and $H_{T\times\mathbb{C}^*}^*(G/B) = H_T^*(G/B)[\hbar]$, where $\hbar$ is interpreted as a generator of $H_{\mathbb{C}^*}^2(\mathrm{pt})$.

We define next the K-theoretic version of the DL operator. Fix an indeterminate y; later, we will set $y = -e^{-\hbar}$. Define the **K-theoretic Demazure-Lusztig (DL) operators**

$$\mathcal{T}_i := \lambda_y(T_{p_i}^*)\partial_i - \mathrm{id}; \quad \mathcal{T}_i^\vee := \partial_i\lambda_y(T_{p_i}^*) - \mathrm{id}\,. \tag{7}$$

The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are $\mathrm{K}_T(\mathrm{pt})[y]$-module endomorphisms of $\mathrm{K}_T(X)[y]$.

Remark 3.1. The operator $\mathcal{T}_i^\vee$ was defined by Lusztig [Lus85, Eq. (4.2)] in relation to affine Hecke algebras and equivariant K theory of flag varieties. The 'dual' operator $\mathcal{T}_i$ arises naturally in the study of motivic Chern classes of Schubert cells [AMSS19] (where the operators are denoted $\mathfrak{R}_i, \mathfrak{R}_i^\vee$). In an algebraic form, $\mathcal{T}_i$ appeared recently in [BBL15, LLL17, MS22], in relation to Whittaker functions. The left versions of these operators are studied in [MNS22]. ⌟

As in the cohomological case, the Demazure and Demazure-Lusztig (DL) operators are adjoint to each other and ∂_i is self-adjoint: for any $a, b \in K_T(X)$,

$$\langle \mathcal{T}_i(a), b\rangle = \langle a, \mathcal{T}_i^\vee(b)\rangle \quad \text{and} \quad \langle \partial_i(a), b\rangle = \langle a, \partial_i(b)\rangle.$$

See [AMSS19, Lemma 3.3] for a proof.

Proposition 3.2 ([Lus85]). *The operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ satisfy the usual commutation and braid relations for the group W. For each simple root $\alpha_i \in \Delta$ the following quadratic formula holds:*

$$(\mathcal{T}_i + \mathrm{id})(\mathcal{T}_i + y) = (\mathcal{T}_i^\vee + \mathrm{id})(\mathcal{T}_i^\vee + y) = 0.$$

An immediate corollary of the quadratic formula is that for $y \neq 0$, the operators $\mathcal{T}_i$ and $\mathcal{T}_i^\vee$ are invertible. In fact,

$$\mathcal{T}_i^{-1} = -\frac{1}{y}\mathcal{T}_i - \frac{1+y}{y}\,\mathrm{id}$$

as operators on $\mathrm{K}_T(X)[y, y^{-1}]$. The same formula holds when $\mathcal{T}_i$ is exchanged with $\mathcal{T}_i^\vee$.

Consider next the localized equivariant K theory ring

$$\mathrm{K}_T(G/B)_{\mathrm{loc}} := \mathrm{K}_T(G/B) \otimes_{\mathrm{K}_T(\mathrm{pt})} \mathrm{Frac}(\mathrm{K}_T(\mathrm{pt}))$$

where Frac denotes the fraction field. The Weyl group elements $w \in W$ are in bijection with the torus fixed points $e_w \in G/B$. Let $\iota_w := [\mathcal{O}_{e_w}] \in \mathrm{K}_T(G/B)_{\mathrm{loc}}$ be the class of the structure sheaf of e_w. By the localization theorem, the classes ι_w form a basis for the localized equivariant K theory ring; we call this basis the *fixed point basis*.

We need the following lemma, whose proof can be found e.g., in [AMSS19, Lemma 3.7].

Lemma 3.3. *The following formulas hold in $\mathrm{K}_T(G/B)_{\mathrm{loc}}$:*

(a) For any weight λ, $\mathcal{L}_\lambda \cdot \iota_w = e^{w\lambda}\iota_w$;

(b) For any simple root α_i,

$$\partial_i(\iota_w) = \frac{1}{1-e^{w\alpha_i}}\iota_w + \frac{1}{1-e^{-w\alpha_i}}\iota_{ws_i};$$

(c) The action of the operator $\mathcal{T}_i$ on the fixed point basis is given by the following formula

$$\mathcal{T}_i(\iota_w) = -\frac{1+y}{1-e^{-w\alpha_i}}\iota_w + \frac{1+ye^{-w\alpha_i}}{1-e^{-w\alpha_i}}\iota_{ws_i}.$$

(d) The action of the adjoint operator $\mathcal{T}_i^\vee$ is given by

$$\mathcal{T}_i^\vee(\iota_w) = -\frac{1+y}{1-e^{-w\alpha_i}}\iota_w + \frac{1+ye^{w\alpha_i}}{1-e^{-w\alpha_i}}\iota_{ws_i}.$$

(e) The action of the inverse operator $(\mathcal{T}_i^\vee)^{-1}$is given by

$$(\mathcal{T}_i^\vee)^{-1}(\iota_w) = -\frac{1+y^{-1}}{1-e^{w\alpha_i}}\iota_w - \frac{y^{-1}+e^{w\alpha_i}}{1-e^{-w\alpha_i}}\iota_{ws_{\alpha_i}}.$$

We also record the action of several specializations of the Demazure-Lusztig operators, see [AMSS19, Lemma 3.8].

Lemma 3.4. *(a) The following specializations hold:*

$$(\mathcal{T}_i)_{y=0} = \partial_i - \mathrm{id}; \quad (\mathcal{T}_i^\vee)_{y=0} = \partial_i - \mathrm{id};$$

Further, for any $w \in W$, the following hold:

$$(\partial_i - \mathrm{id})(\mathcal{I}_w^T) = \begin{cases} \mathcal{I}_{ws_i}^T & \text{if } ws_i > w; \\ -\mathcal{I}_w^T & \text{if } ws_i < w. \end{cases} \quad \partial_i(\mathcal{O}_w^T) = \begin{cases} \mathcal{O}_{ws_i}^T & \text{if } ws_i > w; \\ \mathcal{O}_w^T & \text{if } ws_i < w. \end{cases}$$

(b) Let $w \in W$. Then the specializations at $y = -1$ satisfy

$$(\mathcal{T}_i)_{y=-1}(\iota_w) = \iota_{ws_i}.$$

In other words, this specialization is compatible with the right Weyl group multiplication.

3.3. **Leading terms of DL operators.** Next we use the grading induced by the equivariant Chern character to identify the 'initial terms' of the Demazure-Lusztig operators as certain operators on equivariant (co)homology related to the degenerate Hecke algebra. These operators appear as convolution operators in [Gin98] and determine the Chern-Schwartz-MacPherson classes of Schubert cells [AM16].

As usual, $X = G/B$ but we consider the extended torus $A := T \times \mathbb{C}^*$ where $\mathbb{C}^*$ acts trivially. (This is the restriction of the action of A on $T^*(X)$, where $\mathbb{C}^*$ acts by dilation. The CSM and motivic Chern classes considered later in this paper are naturally $\mathbb{C}^*$-equivariant; this justifies the use of the extended torus.) We now set $y = -e^{-\hbar}$, and therefore (cf. §2.3)

$$\mathrm{ch}_{\mathbb{C}^*}(y) = -e^{-\hbar} = -1 + \hbar + O(\hbar^2) \in \widehat{H}^*_{\mathbb{C}^*}(\mathrm{pt}).$$

We analyze the relation between the cohomological and K-theoretic DL operators.

Proposition 3.5. *Let $w \in W$ and consider the Grothendieck class $\mathcal{O}_w^A \in \mathrm{K}_A(X)$ for the Schubert variety $X(w)$. Then*

$$\mathrm{ch}_A(\mathcal{T}_i(\mathcal{O}_w^A)) = \mathcal{T}_i^{H,\hbar}[X(w)]_A + \mathrm{l.o.t.}$$

and

$$\mathrm{ch}_A(\mathcal{T}_i^\vee(\mathcal{O}_w^A)) = \mathcal{T}_i^{H,\vee,\hbar}[X(w)]_A + \mathrm{l.o.t.},$$

where l.o.t. are terms in $\prod_{i<\ell(w)} H^A_{2i}(X)$.

Proof. Since $X = G/B$ is non-singular, the Chern character is a ring homomorphism, thus for any invariant subvariety $Z \subseteq X$ and any equivariant line bundle $\mathcal{L}$,

$$\mathrm{ch}_A([\mathcal{O}_Z]_A \cdot \mathcal{L}) = [Z]_A + c_1^A(\mathcal{L}) \cdot [Z]_A + \mathrm{l.o.t.}$$

We take $Z = X(w)$, and we have two cases: either $w < ws_i$ or $w > ws_i$. If $w < ws_i$ then $\partial_i(\mathcal{O}_w^A) = \mathcal{O}_{ws_i}^A$. Using this, we obtain

$$\begin{aligned}
\mathrm{ch}_A(\mathcal{T}_i(\mathcal{O}_w^A)) =& \mathrm{ch}_A(\mathcal{O}_{ws_i}^A + y\mathcal{L}_{\alpha_i} \cdot \mathcal{O}_{ws_i}^A - \mathcal{O}_w^A) \\
=& \mathrm{ch}_A(\mathcal{O}_{ws_i}^A) - e^{-\hbar} e^{c_1^A(\mathcal{L}_{\alpha_i})} \mathrm{ch}_A(\mathcal{O}_{ws_i}^A) - [X(w)]_A + \mathrm{l.o.t.} \\
=& \mathrm{ch}_A(\mathcal{O}_{ws_i}^A) - (1-\hbar)(1 + c_1^A(\mathcal{L}_{\alpha_i})) \mathrm{ch}_A(\mathcal{O}_{ws_i}^A) - [X(w)]_A + \mathrm{l.o.t.} \\
=& \hbar[X(ws_i)]_A - c_1^A(\mathcal{L}_{\alpha_i})[X(ws_i)]_A - [X(w)]_A + \mathrm{l.o.t.} \\
=& \hbar\partial_i^H[X(w)]_A - (\mathrm{id} + c_1^A(\mathcal{L}_{\alpha_i})\partial_i^H)[X(w)]_A + \mathrm{l.o.t.}
\end{aligned}$$

where l.o.t. $\in \prod_{i<\ell(w)} H_i^A(X)$. By [AM16, (3)] (which uses a different sign convention) the last expression equals

$$(\hbar\partial_i^H - s_i)[X(w)]_A + \mathrm{l.o.t.} = \mathcal{T}_i^{H,\hbar}[X(w)]_A + \mathrm{l.o.t.}$$

and we are done in this case. If $w > ws_i$ then $\partial_i(\mathcal{O}_w^A) = \mathcal{O}_w^A$, $\partial_i[X(w)]_A = 0$ and $s_i[X(w)]_A = [X(w)]_A$ by [AM16, (4)]. Then a similar, but simpler calculation proves the first part of the proposition.

For the second statement we start by observing that for $a \in \mathrm{K}_T(X)$, by the Grothendieck-Hirzebruch-Riemann-Roch (GHRR)

$$\mathrm{ch}_A \partial_i(a) = \mathrm{ch}_A p_i^*(p_i)_*(a) = p_i^* \mathrm{ch}_A((p_i)_*(a)) = \partial_i^H(\mathrm{ch}_A(a) \mathrm{Td}^A(T_{p_i})). \tag{8}$$

Then the second statement can be proved as follows. By the GHRR theorem,

$$\begin{aligned}\mathrm{ch}_A(\mathcal{T}_i^\vee(\mathcal{O}_w^A)) =& \mathrm{ch}_A\,\partial_i(\mathcal{O}_w^A + y\mathcal{L}_{\alpha_i}\cdot\mathcal{O}_w^A) - \mathrm{ch}_A(\mathcal{O}_w^A)\\ =&\partial_i^H\Big(\mathrm{ch}_A(\mathcal{O}_w^A + y\mathcal{L}_{\alpha_i}\cdot\mathcal{O}_w^A)\,\mathrm{Td}^A(T_{p_i})\Big) - \mathrm{ch}_A(\mathcal{O}_w^A)\\ =&\partial_i^H\Big(\mathrm{ch}_A(\mathcal{O}_w^A)\,\mathrm{Td}^A(T_{p_i})(1-e^{-\hbar+c_1^A(\mathcal{L}_{\alpha_i})})\Big) - \mathrm{ch}_A(\mathcal{O}_w^A)\end{aligned}$$

Observe that $\mathrm{Td}^A(T_{p_i})(1-e^{-\hbar+c_1^A(\mathcal{L}_{\alpha_i})}) = \hbar + c_1^A(\mathcal{L}_{-\alpha_i}) +$ (terms of degree ≥ 2) in *cohomology*. Then the last expression equals

$$\begin{aligned}&\partial_i^H\Big((\hbar + c_1^A(\mathcal{L}_{-\alpha_i})[X(w)]_A\Big) - [X(w)]_A + l.o.t\\ =&\hbar\partial_i^H[X(w)]_A + (\mathrm{id} + c_1^A(\mathcal{L}_{\alpha_i})\partial_i^H)[X(w)]_A + \text{l.o.t.}\\ =&(\hbar\partial_i^H + s_i)[X(w)]_A + \text{l.o.t.}\\ =&\mathcal{T}_i^{H,\vee,\hbar}[X(w)]_A + \text{l.o.t.}\end{aligned}$$

Here the third equation follows from the definition of s_i from (4), and the second from the general fact that for every weight λ, $c_1^A(\mathcal{L}_\lambda)\partial_i^H = \partial_i^H c_1^A(\mathcal{L}_{s_i\lambda}) - \langle\lambda,\alpha_i^\vee\rangle$. This can be proved by e.g., adapting parts (a) and (b) of Lemma 3.3 to the cohomological context. □

4. Equivariant motivic Chern classes

The aim of this section is to introduce and recall the basic properties of the motivic Chern classes - one of the main objects in this note. In the second part of the section we focus on the motivic Chern classes of Schubert cells in flag manifolds. Aside from recalling results proved in e.g., [AMSS19], we prove a new divisibility property of the coefficients in the transition matrix from motivic classes to the Schubert basis (Proposition 4.15). In §4.3 we discuss the relation to point counting over finite fields; this was mentioned briefly in [AMSS19] and [MS22].

4.1. **Preliminaries about motivic Chern classes.** We recall the definition of the motivic Chern classes, following [BSY10]. For now let X be a quasi-projective, complex algebraic variety, with an action of T. First we recall the definition of the (relative) motivic Grothendieck group $\mathrm{K}_0^T(\mathrm{var}/X)$ of varieties over X, mostly following Looijenga's notes [Loo02]; see also Bittner [Bit04]. For simplicity, we only consider the T-equivariant quasi-projective context (replacing the 'goodness' assumption in [Bit04]), which is enough for all applications in this paper. The group $\mathrm{K}_0^T(\mathrm{var}/X)$ is the quotient of the free abelian group generated by symbols $[f: Z\to X]$ where Z is a quasi-projective T-variety and $f: Z\to X$ is a T-equivariant morphism modulo the additivity relations

$$[f: Z\to X] = [f: U\to X] + [f: Z\smallsetminus U\to X]$$

for $U\subseteq Z$ an open invariant subvariety. For every equivariant morphism $g: X\to Y$ of quasi-projective T-varieties there are well-defined push-forwards $g_!: \mathrm{K}_0^T(\mathrm{var}/X)\to\mathrm{K}_0^T(\mathrm{var}/Y)$ (given by composition) and pull-backs $g^*: \mathrm{K}_0^T(\mathrm{var}/Y)\to\mathrm{K}_0^T(\mathrm{var}/X)$ (given by fiber product); see [Bit04, §6]. There are also external products

$$\times: \mathrm{K}_0^T(\mathrm{var}/X)\times\mathrm{K}_0^T(\mathrm{var}/X')\to\mathrm{K}_0^T(\mathrm{var}/X\times X');\quad [f]\times[f']\mapsto[f\times f'],$$

which are $\mathrm{K}_0^T(\mathrm{var}/\mathrm{pt})$-bilinear and commute with push-forward and pull-back. If $X = \mathrm{pt}$, then $\mathrm{K}_0^T(\mathrm{var}/\mathrm{pt})$ is a ring with this external product, and the groups $\mathrm{K}_0^T(\mathrm{var}/X)$ also acquire by the external product a module structure over $\mathrm{K}_0^T(\mathrm{var}/\mathrm{pt})$ such that push-forward $g_!$ and pull-back g^* are $\mathrm{K}_0^T(\mathrm{var}/\mathrm{pt})$-linear.

Remark 4.1. For any variety X, similar functors can be defined on the ring of constructible functions $\mathcal{F}(X)$, and the Grothendieck group $\mathrm{K}_0(\mathrm{var}/X)$ may be regarded as a motivic version of $\mathcal{F}(X)$. In fact, there is a map $e : \mathrm{K}_0(\mathrm{var}/X) \to \mathcal{F}(X)$ sending $[f : Y \to X] \mapsto f_!(1\!\!1_Y)$, where $f_!(1\!\!1_Y)$ is defined using compactly supported Euler characteristic of the fibers. The map e is a group homomorphism, and if $X = \mathrm{pt}$ then e is a ring homomorphism. The constructions extend equivariantly, with $\mathcal{F}^T(X) \subseteq \mathcal{F}(X)$ the subgroup of T-invariant constructible functions. ⌟

The following theorem was proved in the non-equivariant case by Brasselet, Schürmann and Yokura [BSY10, Theorem 2.1]. Minor changes in the argument are needed in the equivariant case – see also [FRW21, AMSS19]. In future work we will reprove the theorem below and relate equivariant motivic Chern classes to certain classes in the equivariant K-theory of the cotangent bundle as defined by Tanisaki [Tan87] with the help of equivariant mixed Hodge modules.

Theorem 4.2. [AMSS19, Theorem 4.2] *Let X be a quasi-projective, non-singular, complex algebraic variety with an action of the torus T. There exists a unique natural transformation $\mathrm{MC}_y : \mathrm{K}_0^T(\mathrm{var}/X) \to \mathrm{K}_T(X)[y]$ satisfying the following properties:*

(1) *It is functorial with respect to push-forwards via T-equivariant proper morphisms of non-singular, quasi-projective varieties.*
(2) *It satisfies the normalization condition*

$$\mathrm{MC}_y[\mathrm{id}_X : X \to X] = \lambda_y(T^*X) = \sum y^i[\wedge^i T^*X] \in \mathrm{K}_T(X)[y].$$

The transformation MC_y satisfies the following properties:

(3) *It is determined by its image on classes $[f : Z \to X] = f_![\mathrm{id}_Z]$ where Z is a non-singular, irreducible, quasi-projective algebraic variety and f is a T-equivariant proper morphism.*
(4) *It satisfies a Verdier-Riemann-Roch (VRR) formula: for every smooth, T-equivariant morphism $\pi : X \to Y$ of quasi-projective and non-singular algebraic varieties, and every $[f : Z \to Y] \in \mathrm{K}_0^T(\mathrm{var}/Y)$, the following holds:*

$$\mathrm{MC}_y[\pi^* f : Z \times_Y X \to X] = \lambda_y(T^*_\pi) \cap \pi^* \mathrm{MC}_y[f : Z \to Y].$$

If one forgets the T-action, then the equivariant motivic Chern class above recovers the non-equivariant motivic Chern class from [BSY10] (either by its construction, or by the properties (1)-(3) from Theorem 4.2 and the corresponding results from [BSY10]).

Remark 4.3. Theorem 4.2 and its proof work more generally for a possibly singular, quasi-projective T-equivariant base variety X, using the Grothendieck group of T-equivariant coherent $\mathcal{O}_X$-sheaves in the target (cf. [FRW21, Remark 2.2]), i.e.,

$$\mathrm{MC}_y : \mathrm{K}_0^T(\mathrm{var}/X) \to \mathrm{K}_0(\mathfrak{coh}^T(\mathcal{O}_X))[y] \,.$$

Moreover, MC_y commutes with exterior products:

$$\mathrm{MC}_y[f \times f' : Z \times Z' \to X \times X'] = \mathrm{MC}_y[f : Z \to X] \boxtimes \mathrm{MC}_y[f' : Z' \to X'] \,.$$

This follows as in the non-equivariant context [BSY10, Corollary 2.1] from part (3) of Theorem 4.2 and the multiplicativity of the equivariant λ_y-class for smooth and quasi-projective T-varieties X, X':

$$\lambda_y(T^*(X \times X')) = \lambda_y(T^*X) \boxtimes \lambda_y(T^*X') \in \mathrm{K}_T(X \times X')[y] \,.$$ ⌟

Remark 4.4. The equivariant χ_y-genus of a T-variety Z is by definition

$$\chi_y(Z) := \mathrm{MC}_y([Z \to \mathrm{pt}]) \in \mathrm{K}_T(\mathrm{pt})[y]\,.$$

By *rigidity* of the χ_y-genus (see [FRW21, §2.5] and [Web16, Theorem 7.2]), it contains no information about the action of T; it is equal to the non-equivariant χ_y-genus under the embedding $\mathbb{Z}[y] \to \mathrm{K}_T(\mathrm{pt})[y]$. ⌟

In what follows, the variety X will usually be understood from the context. If $Y \subseteq X$ is a T-invariant subvariety, not necessarily closed, we set

$$\mathrm{MC}_y(Y) := \mathrm{MC}_y[Y \hookrightarrow X].$$

If $i : Y \subseteq X$ is closed nonsingular subvariety and $Y' \subseteq Y$, then by functoriality

$$\mathrm{MC}_y[Y' \hookrightarrow X] = i_*\mathrm{MC}_y[Y' \hookrightarrow Y]$$

(K-theoretic push-forward). For instance if $Y' = Y$ then

$$\mathrm{MC}_y[i : Y \to X] = i_*(\lambda_y(T^*Y) \otimes [\mathcal{O}_Y])$$

as an element in $\mathrm{K}_T(X)$. We will often abuse notation and suppress the push-forward notation. Similarly for the other transformations discussed in later sections.

4.2. **Motivic Chern classes of Schubert cells.** Assume now that $X = G/B$. The following result from [AMSS19, Corollary 5.2, cf. Remark 5.4], allows us to calculate recursively the motivic Chern classes of Schubert cells.

Theorem 4.5. *Let $w \in W$. Then the motivic Chern class $\mathrm{MC}_y(X(w)^\circ)$ is given by*

$$\mathrm{MC}_y(X(w)^\circ) = \mathcal{T}_{w^{-1}}(\mathcal{O}^T_{\mathrm{id}}).$$

Here, $\mathcal{T}_{w^{-1}} = \mathcal{T}_{i_k} \cdots \mathcal{T}_{i_1}$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition. This operator is well-defined by Proposition 3.2.

Following [AMSS19, Remark 6.4], we introduce an operator which will yield the (Poincaré) duals of motivic Chern classes. For each simple root $\alpha_i \in \Delta$, define

$$\mathfrak{L}_i = \partial_i + y(\partial_i \mathcal{L}_{\alpha_i} + \mathrm{id}) = -y(\mathcal{T}_i^\vee)^{-1} = \mathcal{T}_i^\vee + (1+y)\,\mathrm{id}\,. \tag{9}$$

Since $\mathcal{T}_i^\vee$ satisfy the usual braid and commutativity relations, so do these operators. Hence, $\mathfrak{L}_w$ is well-defined for all $w \in W$, where $\mathfrak{L}_w = \mathfrak{L}_{i_1} \cdots \mathfrak{L}_{i_k}$ if $w = s_{i_1} \cdots s_{i_k}$ is a reduced decomposition. Define the following elements in the equivariant K theory ring:

$$\widetilde{\mathrm{MC}}_y(Y(w)^\circ)) := \mathfrak{L}_{w^{-1}w_0}(\mathcal{O}^{w_0,T}); \quad \widetilde{\mathrm{MC}}_y(X(w)^\circ) = \mathfrak{L}_{w^{-1}}(\mathcal{O}^T_{\mathrm{id}}).$$

By definition $\widetilde{\mathrm{MC}}_y(Y(w)^\circ))$ and $\widetilde{\mathrm{MC}}_y(X(w)^\circ)$ are elements in $\mathrm{K}_T(X)[y]$.[2]

Theorem 4.6. [AMSS19, Theorem 6.2] *The classes $\widetilde{\mathrm{MC}}_y(Y(w)^\circ))$ are orthogonal to the motivic Chern classes: for any $u, v \in W$,*

$$\langle \mathrm{MC}_y(X(u)^\circ), \widetilde{\mathrm{MC}}_y(Y(v)^\circ)\rangle = \delta_{u,v} \prod_{\alpha>0}(1+ye^{-\alpha}),$$

where the pairing on the left hand side is the intersection pairing defined in §2.2.

Note that $\prod_{\alpha>0}(1+ye^{-\alpha}) = \lambda_y(T^*_{w_0}X)$.

[2] If w_0^L denotes the left Weyl group action by w_0, as in [MNS22], then $w_0^L.\widetilde{\mathrm{MC}}_y(X(w)^\circ) = \widetilde{\mathrm{MC}}_y(Y(w_0w)^\circ)$. This generalizes the more familiar formula from Schubert calculus: $w_0^L.[X(w)]_T = [Y(w_0w)]_T$.

Remark 4.7. Another family of classes dual to motivic Chern classes is given by a certain Serre dual variant of Segre motivic classes. Combining [AMSS19, Theorem 8.11] and [MNS22, Theorem 7.1]) (see also [FRW21]), one obtains the remarkable equality:

$$\frac{\widetilde{\mathrm{MC}}_y(Y(v)^\circ)}{\prod_{\alpha>0}(1+ye^{-\alpha})} = (-y)^{\dim G/B-\ell(v)}\frac{\mathcal{D}(\mathrm{MC}_y(Y(v)^\circ))}{\lambda_y(T^*X)} \in K_T(X)[y]_S[y^{-1}].$$

Here $\mathcal{D}$ denotes the (Grothendieck-Serre) duality, extended to the parameter y via $y^n \mapsto y^{-n}$, and $\mathrm{K}_T(X)[y]_S$ is appropriately localized so that $\lambda_y(T^*X)$ is invertible (see [AMSS19, Remark 8.9]). We note that one may define these 'Serre-Segre' motivic classes for any partial flag manifold G/P, and they are always dual to motivic Chern classes of Schubert cells; see [MNS22, Theorem 7.2]. Geometrically, the duality above is expected to arise from a transversality formula, generalizing to K-theory the results from [Sch17]. In cohomology, this is explained in [AMSS17, §7]. ⌟

Consider the expansions of the equivariant motivic Chern classes,

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u\leq w} c_{u,w}(y;e^t)\mathcal{O}_u^T. \tag{10}$$

The equivariant K-Chevalley formula, used to multiply a Schubert class by the line bundle $\mathcal{L}_{\alpha_i}$ [LP07, PR99], and Theorem 4.5, gives a recursive procedure to calculate the motivic Chern classes of Schubert cells. The recursion also implies that the coefficients are polynomials $c_{u,w}(y;e^t) \in \mathbb{Z}[e^{\pm\alpha_1},\dots,e^{\pm\alpha_r}][y] \subseteq \mathrm{K}_T(\mathrm{pt})[y]$ in the characters associated to the simple roots. Note that the inclusion may be strict, reflecting the fact that the root lattice is in general strictly included in the weight lattice. Also, recall that e^t stands for the tuple $(e^{\pm t_1},\dots,e^{\pm t_r})$.

We provide next a few calculations for the motivic Chern classes of the flag manifolds $\mathbb{P}^1$ and Fl(3).

Example 4.8 ($\mathrm{Fl}(2)=\mathbb{P}^1$). The equivariant motivic Chern classes for $\mathbb{P}^1$ are :

$$\mathrm{MC}_y(X(\mathrm{id})) = \mathcal{O}_{\mathrm{id}}^T; \quad \mathrm{MC}_y(X(s)^\circ) = (1+e^{-\alpha_1}y)\mathcal{O}_s^T - (1+(1+e^{-\alpha_1})y)\mathcal{O}_{\mathrm{id}}^T. \quad ⌟$$

Example 4.9 (The Schubert cell $X(s_1s_2)^\circ$). The equivariant motivic Chern classes for larger flag manifolds are much more complicated. For instance, the equivariant motivic Chern class of the Schubert cell $X(s_1s_2)^\circ \subseteq \mathrm{Fl}(3)$ is

$$\begin{aligned}\mathrm{MC}_y(X(s_1s_2)^\circ) =&(1+e^{-\alpha_1}y)(1+e^{-(\alpha_1+\alpha_2)}y)\mathcal{O}_{s_1s_2}^T-\\ &(1+e^{-\alpha_1}y)(1+(1+e^{-(\alpha_1+\alpha_2)})y)\mathcal{O}_{s_1}^T-\\ &(1+(1+e^{-\alpha_1})(1+e^{-\alpha_2})y+e^{-\alpha_2}(1+e^{-\alpha_1}+e^{-2\alpha_1})y^2)\mathcal{O}_{s_2}^T+\\ &(1+(2+e^{-\alpha_1}+e^{-\alpha_2}+e^{-(\alpha_1+\alpha_2)})y)\mathcal{O}_{\mathrm{id}}^T+\\ &(1+e^{-\alpha_1}+e^{-\alpha_2}+e^{-(\alpha_1+\alpha_2)}+e^{-(2\alpha_1+\alpha_2)})y^2\mathcal{O}_{\mathrm{id}}^T.\end{aligned} \quad ⌟$$

The expressions above encode a remarkable amount of information. For instance, a simple verification in Example 4.9 shows that the expression for the *sum* of the coefficients simplifies dramatically and equals y^2, reflecting the geometric fact that we deal with a cell of dimension 2. We will prove this and more in Proposition 4.13 and Theorem 5.1 below. For now, we also provide some examples of non-equivariant motivic Chern classes. These are obtained from the equivariant ones by making the substitution $e^\lambda \mapsto 1$ for each weight λ.

Example 4.10. The following are the non-equivariant motivic Chern classes of Schubert cells in Fl(3). (Recall that we use the notation MC_y for both the notion in ordinary and in equivariant K-theory.)

$$\begin{aligned}
\mathrm{MC}_y(X(\mathrm{id})) =&\mathcal{O}_{\mathrm{id}};\\
\mathrm{MC}_y(X(s_1)^\circ) =&(1+y)\mathcal{O}_{s_1}-(1+2y)\mathcal{O}_{\mathrm{id}};\\
\mathrm{MC}_y(X(s_2)^\circ) =&(1+y)\mathcal{O}_{s_2}-(1+2y)\mathcal{O}_{\mathrm{id}};\\
\mathrm{MC}_y(X(s_1s_2)^\circ) =&(1+y)^2\mathcal{O}_{s_1s_2}-(1+y)(1+2y)\mathcal{O}_{s_1}-(1+y)(1+3y)\mathcal{O}_{s_2}+(5y^2+5y+1)\mathcal{O}_{\mathrm{id}};\\
\mathrm{MC}_y(X(s_2s_1)^\circ) =&(1+y)^2\mathcal{O}_{s_2s_1}-(1+y)(1+2y)\mathcal{O}_{s_2}-(1+y)(1+3y)\mathcal{O}_{s_1}+(5y^2+5y+1)\mathcal{O}_{\mathrm{id}};\\
\mathrm{MC}_y(X(w_0)^\circ) =&(1+y)^3\mathcal{O}_{w_0}-(1+y)^2(1+2y)(\mathcal{O}_{s_1s_2}+\mathcal{O}_{s_2s_1})+\\
&(1+y)(5y^2+4y+1)(\mathcal{O}_{s_1}+\mathcal{O}_{s_2})-(8y^3+11y^2+5y+1)\mathcal{O}_{\mathrm{id}}.
\end{aligned}$$

The classes $\widetilde{\mathrm{MC}}_y(Y(w)^\circ)$ for the Schubert cells in Fl(3) are:

$$\begin{aligned}
\widetilde{\mathrm{MC}}_y(Y(w_0)) =&\mathcal{O}^{w_0};\\
\widetilde{\mathrm{MC}}_y(Y(s_1s_2)^\circ) =&(1+y)\mathcal{O}^{s_1s_2}+y\mathcal{O}^{w_0};\\
\widetilde{\mathrm{MC}}_y(Y(s_2s_1)^\circ) =&(1+y)\mathcal{O}^{s_2s_1}+y\mathcal{O}^{w_0};\\
\widetilde{\mathrm{MC}}_y(Y(s_1)^\circ) =&(1+y)^2\mathcal{O}^{s_1}+y(1+y)\mathcal{O}^{s_1s_2}+2y(1+y)\mathcal{O}^{s_2s_1}+y^2\mathcal{O}^{w_0};\\
\widetilde{\mathrm{MC}}_y(Y(s_2)^\circ) =&(1+y)^2\mathcal{O}^{s_2}+2y(1+y)\mathcal{O}^{s_1s_2}+y(1+y)\mathcal{O}^{s_2s_1}+y^2\mathcal{O}^{w_0};\\
\widetilde{\mathrm{MC}}_y(Y(\mathrm{id})^\circ) =&(1+y)^3\mathcal{O}^{\mathrm{id}}+y(1+y)^2(\mathcal{O}^{s_1}+\mathcal{O}^{s_2})+\\
&2y^2(1+y)(\mathcal{O}^{s_1s_2}+\mathcal{O}^{s_2s_1})+y^3\mathcal{O}^{w_0}.
\end{aligned}$$

An algebra check together with fact that $\langle \mathcal{O}_u, \mathcal{O}^v\rangle = 1$ if and only if $u \geq v$, shows that

$$\langle \mathrm{MC}_y(X(u)^\circ), \widetilde{\mathrm{MC}}_y(Y(v)^\circ)\rangle = (1+y)^3\delta_{u,v}$$

as stated in Theorem 4.6. ⌟

On the basis of these (and other) examples, we can conjecture that the motivic Chern classes and their duals satisfy a certain positivity property. This is discussed in section 8. At this time we note that the positivity of the dual classes in Example 4.10 does *not* extend beyond small cases. For instance, the coefficient of $\mathcal{O}^{s_3s_1s_2}$ in the expansion of $\widetilde{\mathrm{MC}}_y(Y(\mathrm{id})^\circ) \in K(\mathrm{Fl}(4))$ equals $y^2(4y-1)(1+y)^3$.

The 'top' Schubert coefficient is calculated in the following result.

Lemma 4.11. *The coefficient $c_{w,w}(y; e^t)$ is given by*

$$c_{w,w}(y, e^t) = \prod_{\alpha>0, w\alpha<0} (1 + ye^{w\alpha}) = \lambda_y(T^*_{e_w}X(w)).$$

Proof. The localization $\mathrm{MC}_y(X(w)^\circ)|_w$ equals $c_{w,w}(y, e^t)(\mathcal{O}^T_w)|_w$. By Lemma 3.3(c) and Theorem 4.5, we get

$$\mathrm{MC}_y(X(w)^\circ)|_w = \prod_{\alpha>0, w\alpha<0} \frac{1+ye^{w\alpha}}{1-e^{w\alpha}}\iota_w|_w.$$

However, $(\mathcal{O}^T_w)|_w = \frac{\iota_w|_w}{\lambda_{-1}(T^*_wX(w))} = \frac{\iota_w|_w}{\prod_{\alpha>0,w\alpha<0}(1-e^{w\alpha})}$. The claim follows. □

We end this section with an analogue of Theorem 4.5 for the Segre motivic classes

$$\mathrm{SMC}_y(X(w)^\circ, X) := \frac{\mathrm{MC}_y(X(w)^\circ)}{\lambda_y(T^*X)}.$$

These classes live in a localization $\mathrm{K}_T(X)_S$ where the element $\prod_{\alpha>0}(1+ye^{\alpha})(1+ye^{-\alpha}) \in \mathrm{K}_T(\mathrm{pt})[y]$ is invertible; see [AMSS19, Remark 8.9]. We recall [MS22, Theorem 4.2], which will be used below when discussing the Hirzebruch version of the Segre classes.

Theorem 4.12. *For any $w \in W$ one has in $K_T(X)[y]_S$:*

$$\frac{\mathrm{MC}_y(X(w)^\circ)}{\lambda_y(T^*X)} = \frac{\mathcal{T}^\vee_{w^{-1}}(\mathcal{O}^T_{\mathrm{id}})}{\prod_{\alpha>0}(1+ye^{\alpha})} = \mathcal{T}^\vee_{w^{-1}}\left(\frac{\mathcal{O}^T_{\mathrm{id}}}{\prod_{\alpha>0}(1+ye^{\alpha})}\right)$$

and

$$\frac{\mathrm{MC}_y(Y(w)^\circ)}{\lambda_y(T^*X)} = \frac{\mathcal{T}^\vee_{(w_0w)^{-1}}(\mathcal{O}^{w_0,T})}{\prod_{\alpha>0}(1+ye^{-\alpha})} = \mathcal{T}^\vee_{(w_0w)^{-1}}\left(\frac{\mathcal{O}^{w_0,T}}{\prod_{\alpha>0}(1+ye^{-\alpha})}\right). \tag{11}$$

Note that $\prod_{\alpha>0}(1+ye^{\alpha}) = \lambda_y(T^*_{e_{\mathrm{id}}}X)$ and $\prod_{\alpha>0}(1+ye^{-\alpha}) = \lambda_y(T^*_{e_{w_0}}X)$.

4.3. **Integrals of motivic Chern classes and point counting.** By functoriality of motivic Chern classes, the integral of the motivic Chern class of a Schubert cell equals

$$\int_{G/B}\mathrm{MC}_y(X(w)^\circ) := \mathrm{MC}_y[X(w)^\circ \to \mathrm{pt}] = \mathrm{MC}_y[\mathbb{A}^{\ell(w)} \to \mathrm{pt}] = \mathrm{MC}_y[\mathbb{A}^1 \to \mathrm{pt}]^{\ell(w)}.$$

The last equality uses the fact the the map $\mathrm{MC}_y : \mathrm{K}_0^T(\mathrm{var}/\mathrm{pt}) \to \mathrm{K}_0^T(\mathrm{pt})$ is a *ring* homomorphism, with the product given by exterior product of varieties; see [BSY10, Corollary 2.1] extended equivariantly in [AMSS19, Theorem 4.2]. One can calculate $\mathrm{MC}_y[\mathbb{A}^1 \to \mathrm{pt}]$ directly from Example 4.8:

$$\mathrm{MC}_y[\mathbb{A}^1 \to \mathrm{pt}] = \int_{\mathbb{P}^1}\mathrm{MC}_y(\mathbb{A}^1) = -y.$$

Combining all these, we deduce:

Proposition 4.13. *Recall the Schubert expansion* (10)*. Then the following hold:*

(a) $\int_{G/B}\mathrm{MC}_y(X(w)^\circ) = \sum c_{w,u}(y,e^t) = (-y)^{\ell(w)}$.

(b) *The χ_y-genus of G/B equals*

$$\chi_y(G/B) = \int_{G/B}\lambda_y(T^*(G/B)) = \sum_{w\in W}(-y)^{\ell(w)}.$$

Proof. Part (a) follows from the considerations above and because $\int_{G/B}\mathcal{O}^T_w = 1$, since $H^i(X(w),\mathcal{O}_{G/B}) = 0$ for $i>0$, as Schubert varieties are rational with rational singularities. Part (b) follows from (a), using the fact that $\lambda_y(T^*(G/B))\otimes\mathcal{O}_{G/B} = \mathrm{MC}_y(G/B) = \sum_{w\in W}\mathrm{MC}_y(X(w)^\circ)$. □

If one specializes $y=-q$, this proposition shows that the χ_y-genus of a Schubert variety $X(w)$ is equal to the number of points of $X(w)$ over $\mathbb{F}_q$, the field with q elements. This type of arguments are discussed more generally for any G/P, or for smooth, projective T-varieties with finitely many fixed points in [MS22, Theorem A.1]; see Remark 4.14 below for a further generalization.

Utilizing again the specialization $y=-q$ and taking $G/B = \mathrm{Fl}(n)$, one recovers in a natural way q-analogues of classical formulae. In this case, $W = S_n$ (the symmetric group) and

$$\chi_{-q}(\mathrm{Fl}(n)) = \sum_{w\in S_n} q^{\ell(w)}.$$

It is known that this sum equals the q-analogue of the factorial,

$$\sum_{w\in S_n} q^{\ell(w)} = [n]_q! = (1+q)(1+q+q^2)\cdot\ldots\cdot(1+q+\ldots+q^{n-1}).$$

In fact, it is fun to work out a geometric interpretation of this formula. The natural projection $p_n : \mathrm{Fl}(n) \to \mathrm{Gr}(n-1,n)$ sending a flag $(F_1 \subseteq \ldots \subseteq F_n = \mathbb{C}^n)$ to $F_{n-1} \subseteq \mathbb{C}^n$ is a G-equivariant Zariski locally trivial fibration, with fiber isomorphic to $\mathrm{Fl}(n-1)$. By additivity and multiplicativity of motivic Chern classes over a point, it follows that

$$\chi_{-q}(\mathrm{Fl}(n)) = \chi_{-q}(\mathrm{Fl}(n-1))\cdot\chi_{-q}(\mathrm{Gr}(n-1,n)) = \chi_{-q}(\mathrm{Fl}(n-1))\cdot(1+q+q^2+q^3+\ldots+q^{n-1}).$$

The last equality follows from the fact that the (dual) projective space $\mathrm{Gr}(n-1,n)$ is the disjoint union of Schubert cells, each of which with contribution $q^{\dim(\mathrm{cell})}$ to the χ_{-q} genus. Then the equality follows by induction on n. A more detailed analysis of this geometric argument is carried out in §4.5.

Remark 4.14. This relation between $\mathbb{F}_q$ point count and the χ_y-genus of a complex algebraic variety X holds more generally for X of strongly polynomial-count in the sense of Katz [HRV08, 6 Appendix]. By Theorem 6.1.2 in *loc. cit.* one obtains the relation $P_X(q) = E(x,y)$ for $q = xy$, with $P_X(q)$ the polynomial point count over $\mathbb{F}_q$ and E the corresponding E-polynomial of X defined in terms of the mixed Hodge numbers of the compactly supported cohomology of X. By forgetting the weight filtration one gets in these cases $E(y,1) = \chi_{-y}(X)$ by [BSY10, (5.3) and (5.5), p. 43-44]. ⌟

4.4. **Divisibility properties.**

Consider now the expansions of the *non-equivariant* motivic Chern classes,

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u\le w} c_{u,w}(y)\mathcal{O}_u \quad \in \mathrm{K}(X)[y]. \tag{12}$$

The coefficients $c_{u,w}(y)$ from (12) are polynomials in $\mathbb{Z}[y]$. Example 4.10 suggests a divisibility property of these coefficients.

Proposition 4.15. *The coefficient $c_{u,w}(y)$ is divisible by $(1+y)^{\ell(u)}$.*

Proof. We prove the statement by induction on $\ell(u)$. If $u = w$, Lemma 4.11 gives $c_{w,w}(y) = (1+y)^{\ell(w)}$. Now assume $u < w$ and that the statement is true for any $v \le w$ with $\ell(v) > \ell(u)$. Arguing by contradiction, suppose that $(1+y)^\ell \mid c_{u,w}(y)$ and $(1+y)^{\ell+1} \nmid c_{u,w}(y)$ for some $\ell < \ell(u)$. We use the hypotheses of Proposition 3.5, where we only keep the action of $\mathbb{C}^*$. In particular $y = -e^{-\hbar}$, therefore the initial term of $1+y$ is $\hbar$. Consider the expansion:

$$\begin{aligned}\mathrm{ch}_{\mathbb{C}^*}(\mathrm{MC}_{-e^{-\hbar}}(X(w)^\circ)) = &\sum_{z\le w, \ell(z)<\ell(u)} c_{z,w}(-e^{-\hbar})\,\mathrm{ch}_{\mathbb{C}^*}(\mathcal{O}_z)\\ &+ \sum_{z\le w,\ell(z)>\ell(u)} c_{z,w}(-e^{-\hbar})\,\mathrm{ch}_{\mathbb{C}^*}(\mathcal{O}_z) + \sum_{z\le w,\ell(z)=\ell(u)} c_{z,w}(-e^{-\hbar})\,\mathrm{ch}_{\mathbb{C}^*}(\mathcal{O}_z).\end{aligned} \tag{13}$$

Theorem 4.5 and Proposition 3.5 imply that the part with the highest homological degree in $\mathrm{ch}_{\mathbb{C}^*}(\mathrm{MC}_{-e^{-\hbar}}(X(w)^\circ))$ lies in $H_0^{\mathbb{C}^*}(X)$. (In Theorem 7.1 below we will show this is the homogenized Chern-Schwartz-MacPherson class $c^\hbar_{\mathrm{SM}}(X(w)^\circ)$, in particular it is nonzero.) Since $\ell < \ell(u)$, it follows that the coefficient of $\hbar^\ell[X(u)] \in H^{\mathbb{C}^*}_{2\ell(u)-2\ell}(X)$ in the left-hand side of (13) is equal to 0. We analyze this coefficient on the right-hand side. The first summand has no contribution because $\ell(z) < \ell(u)$. By induction, every term in the second summand is divisible by $\hbar^{\ell(u)+1}$, thus again it does not contribute to the coefficient of $\hbar^\ell[X(u)]$. In the last summand, only the term with $z = u$ can contribute. Its contribution equals the coefficient of $\hbar^\ell$ in $c_{u,w}(-e^{-\hbar})$. This coefficient is non-zero, by the hypothesis on ℓ, and it gives a non-zero coefficient of $\hbar^\ell[X(u)]$, contradicting the previous conclusion. □

We end with the following corollary.

Corollary 4.16. *Consider the non-equivariant motivic Chern class*

$$\mathrm{MC}_y(X(w)^\circ) = \sum c_{u,w}(y)\mathcal{O}_u.$$

Then $c_{\mathrm{id},w}(-1) = 1$.

Proof. By the divisibility property, $c_{u,w}(-1) = 0$ for $\ell(u) > 0$. Then

$$1 = \int_{G/B} \mathrm{MC}_{-1}(X(w)^\circ) = c_{\mathrm{id},w}(-1)$$

by Proposition 4.13. □

We invite the reader to verify this corollary for the motivic Chern classes in Fl(3) from Example 4.10 above.

4.5. **The parabolic case.** Consider the (generalized) partial flag manifold G/P, and let $\pi : G/B \to G/P$ be the natural projection. This is a G-equivariant locally trivial fibration in the Zariski topology, with fiber $F := \pi^{-1}(1.P) = P/B$. This fiber is the flag manifold $L/(B \cap L)$, where L is the Levi subgroup of P. The Schubert varieties in F are indexed by the elements in W_P. Furthermore, the image $\pi(X(w)^\circ)$ equals $X(wW_P)^\circ$, and the restriction of π to $X(w)^\circ$ is a trivial fibration, showing that $X(w)^\circ \simeq X(wW_P)^\circ \times (w.F \cap X(w)^\circ)$. It follows that in the Grothendieck group $\mathrm{K}_0(\mathrm{var}/(G/P))$,

$$[X(w)^\circ \to G/P] = [w.F \cap X(w)^\circ \to wP] \boxtimes [X(wW_P)^\circ \to G/P]. \tag{14}$$

The intersection $w.F \cap X(w)^\circ$ is the Schubert cell in $w.F$ indexed by $w_2 \in W_P$, where $w = w_1 w_2$ is the parabolic factorization of w with respect to P; cf. [BCMP22, Theorem 2.8]. This argument allows us calculate the push-forwards of motivic Chern classes.

Proposition 4.17. *The following hold:*

(a) $\pi_*\mathrm{MC}_y(X(w)^\circ) = (-y)^{\ell(w)-\ell(wW_P)}\mathrm{MC}_y(X(wW_P)^\circ)$ *in* $\mathrm{K}_T(G/P)$.

(b) More generally, let $P \subseteq Q$ *be two standard parabolic subgroups, and* $\pi' : G/P \to G/Q$ *the natural projection. Then*

$$\pi'_*\mathrm{MC}_y(X(wW_P)^\circ) = (-y)^{\ell(wW_P)-\ell(wW_Q)}\mathrm{MC}_y(X(wW_Q)^\circ).$$

Proof. Equation (14) and functoriality of motivic Chern classes imply that

$$\pi_*\mathrm{MC}_y(X(w)^\circ) = \mathrm{MC}_y[w.F \cap X(w)^\circ \to wP] \cdot \mathrm{MC}_y(X(wW_P)^\circ),$$

Then the claim follows from Proposition 4.13(a). Part (b) may be obtained by applying (a) to the composition of projections $G/B \to G/Q \to G/P$. □

We end this section by pointing out that the argument from Proposition 4.13 extends *verbatim* to the parabolic case. One obtains:

Proposition 4.18. *Let* $w \in W^P$ *and consider the Schubert expansion* $\mathrm{MC}_y(X(wW_P)^\circ) = \sum_{u \in W^P, u \leq w} c_{u,w}(y, e^t)\mathrm{MC}_y(X(uW_P)^\circ)$. *Then the following hold:*

(a) $\int_{G/P} \mathrm{MC}(X(wW_P)^\circ) = \sum c_{w,u}(y, e^t) = (-y)^{\ell(w)}$.

(b) The χ_y*-genus of* G/P *equals* $\chi_y(G/P) = \sum_{w \in W^P}(-y)^{\ell(w)}$.

To illustrate, let $G/P = \mathrm{Gr}(k,n)$ be the Grassmann manifold of subspaces of dimension k in $\mathbb{C}^n$. The set W^P corresponds to partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ included in the $k \times (n-k)$ rectangle such that $\dim X(\lambda) = |\lambda| = \lambda_1 + \ldots + \lambda_k$. For $q = -y$, the χ_{-q} genus is

$$\chi_{-q}(\mathrm{Gr}(k,n)) = \sum_{\lambda} q^{|\lambda|} = \binom{n}{k}_q,$$

the q-analogue of the binomial coefficient.

5. The parameter y in motivic Chern classes

In this section we discuss some key combinatorial properties of the Schubert expansion of the motivic Chern classes, including specializations of the parameter y and their geometric interpretation.

Theorem 5.1. *Let P be a parabolic subgroup of G, $X = G/P$, and $w \in W^P$. Then the following hold:*

(a) The specialization $y = -1$ gives $\mathrm{MC}_{-1}(X(wW_P)^\circ) = \iota_{wW_P}$, the equivariant class of the unique torus fixed point in $X(wW_P)^\circ$.

(b) The specialization $y = 0$ gives $\mathrm{MC}_0(X(wW_P)^\circ) = \mathcal{I}^T_{wW_P}$, the class of the ideal sheaf $\mathcal{O}_{X(wW_P)}(-\partial X(wW_P))$.

(c) The degree of $\mathrm{MC}_y(X(wW_P)^\circ)$ with respect to y is equal to $\ell(w)$, and the coefficient of $y^{\ell(w)}$ in $\mathrm{MC}_y(X(wW_P)^\circ)$ is the class of $\omega_{X(wW_P)}$, the dualizing sheaf on $X(wW_P)$.

Proof. We first prove all the statements for $P = B$. Parts (a) and (b) follow from Theorem 4.5 and the specializations at $y = -1$ and $y = 0$ of the operator $\mathcal{T}_i$, from Lemma 3.4. Let $\theta : Z \to X(w)$ denote the Bott-Samelson resolution of the Schubert variety $X(w)$ used, e.g., in [AM16]. By functoriality,

$$\mathrm{MC}_y[X(w)^\circ \to G/B] = \theta_* \mathrm{MC}_y[X(w)^\circ \to Z].$$

The restriction of θ to $\theta^{-1}(X(w)^\circ)$ is an isomorphism, and it is known that the complement of $\theta^{-1}(X(w)^\circ)$ in Z is a simple normal crossing divisor. Since Z is smooth, by inclusion-exclusion it follows that the term with highest power of y in $\mathrm{MC}_y(Z \smallsetminus \theta^{-1}(X(w)^\circ))$ is the same as the one in $\mathrm{MC}_y(Z) = \lambda_y(T^*(Z))$, namely $y^{\ell(w)} \wedge^{\dim Z} T^*(Z) = y^{\ell(w)}\omega_Z$, a multiple of the canonical bundle of Z. This finishes the proof of (c), since $\theta_*(\omega_Z) = \omega_{X(w)}$ as $X(w)$ has rational singularities; see e.g., [BK05].

We now turn to the general G/P situation. Since $w \in W^P$, the projection $\pi : G/B \to G/P$ restricts to a birational map $\pi : X(w) \to X(wW_P)$ which is an isomorphism over the Schubert cell $X(wW_P)^\circ$. By Proposition 4.17, $\pi_* \mathrm{MC}_y(X(w)^\circ) = \mathrm{MC}_y(X(wW_P)^\circ)$. Then each claim follows from the corresponding statement for G/B, taking into consideration that $\pi_*(\iota_w) = \iota_{wW_P}$, $\pi_* \mathcal{I}^T_w = \mathcal{I}^T_{wW_P}$ (cf. [Bri02, Proof of Lemma 4]), and $\pi_* \omega_{X(w)} = \omega_{X(wW_P)}$ (since $X(wW_P)$ has rational singularities; see [BK05]). □

Remark 5.2. In the non-equivariant case, the result in (b) can also be proved using the fact that, for the Schubert *variety,* we have $\mathrm{MC}_0(X(w)) = \mathcal{O}^T_w$ since $X(w)$ has rational, hence DuBois, singularities [BSY10, Example 3.2]. The equivariant generalization of this argument would use either an equivariant version of the DuBois complex, which is not available at this time in the literature, or a corresponding equality in $K_0(\mathfrak{coh}^T(\mathcal{O}_X))$. ⌟

The duality in Theorem 4.6 allows us to calculate the specializations of $\widetilde{\mathrm{MC}}_y(Y(w)^\circ)$.

Corollary 5.3. *Let $w \in W$. Then the following hold:*

(a) The specialization $y = -1$ gives

$$\widetilde{\mathrm{MC}}_{-1}(Y(w)^\circ) = \frac{\lambda_{-1}(T^*_{w_0}(G/B))}{\lambda_{-1}(T^*_w(G/B))}\iota_w = \frac{\prod_{\alpha>0}(1-e^{-\alpha})}{\prod_{\alpha>0}(1-e^{w\alpha})}\iota_w.$$

(b) The specialization $y = 0$ gives $\widetilde{\mathrm{MC}}_0(Y(w)^\circ) = \mathcal{O}^{w,T}$.

Proof. Theorem 4.6 and Theorem 5.1 imply that $\kappa := \widetilde{\mathrm{MC}}_{-1}(Y(w)^\circ)$ is a class in $\mathrm{K}_T(G/B)$ with localizations

$$\kappa|_u = \delta_{u,w}\lambda_{-1}(T^*_{w_0}(G/B)).$$

Then (a) follows from the injectivity of the localization map, see [Nie74, Theorem 3.2], and the fact that $\langle \iota_w, \iota_u\rangle = \delta_{u,w}\lambda_{-1}(T^*_w(G/B))$. Part (b) is a consequence of the duality between the ideal sheaves and structure sheaves recalled in (1), combined with Theorem 5.1(b). ☐

6. Equivariant Hirzebruch classes

6.1. **The equivariant Hirzebruch transformation.** In the non-equivariant case, it is useful to define and study several transformations associated with motivic Chern classes. These transformations provide a unified point of view from which, for instance, the classical Chern-Schwartz-MacPherson (CSM) classes may be obtained as a subproduct of a 'normalized' Todd transformation together with Grothendieck-Riemann-Roch. This process is explained in [BSY10, §0 and §3]. In this paper we use the equivariant Riemann-Roch theorem proved by Edidin and Graham [EG00, Theorem 3.1] to explain the equivariant counterparts of these 'unnormalized' and 'normalized' transformations. We note that Weber [Web16, Web17] first studied the equivariant unnormalized Hirzebruch class we will consider below.

Since many of the results explained here are not available in this generality in the literature, we find it useful to recall the precise hypotheses we utilize. We hope this section may be used as a reference in the future. We will consider more in detail the case of (equivariant) CSM classes in §7. There we will also provide an alternative way to obtain CSM classes from motivic Chern classes.

By X we denote a complex algebraic variety. For a commutative ring R, the completions $\widehat{A}^T_*(X,R), \widehat{H}^T_*(X,R)$ denote the product of the equivariant Chow groups [EG98], respectively the equivariant Borel-Moore homology groups (where the degree is doubled), with coefficients in R:

$$\widehat{A}^T_*(X,R) := \prod_{i\le \dim X} A^T_i(X)\otimes R \quad \text{or} \quad \widehat{H}^T_*(X,R) := \prod_{i\le \dim X} H^T_{2i}(X)\otimes R \quad .$$

The Chow and Borel-Moore homology are related by a cycle map $cl : A^T_i(X) \to H^T_{2i}(X)$, and one may work directly in the Chow context, or in the image under this map; cf. [EG98, §2.8]. In what follows we have chosen to work in the Borel-Moore context. The coefficients ring R will be mostly $\mathbb{Z}, \mathbb{Q}, \mathbb{Q}[y]$ or $\mathbb{Q}[y,(1+y)^{-1}]$. In case no coefficients are mentioned, we are using $R = \mathbb{Q}$ (as before).

For a T-variety X let

$$\mathrm{td}_* : K_0(\mathfrak{coh}^T(\mathcal{O}_X)) \to \widehat{H}^T_*(X)$$

be the equivariant Todd class transformation to the completion $\widehat{H}^T_*(X)$, constructed in [EG00, §3.2]. Then td_* is covariant for proper T-equivariant morphisms. Also note that

$$\mathrm{td}_*(\mathcal{O}_X) = \mathrm{Td}(TX)\cap [X]_T \tag{15}$$

for X smooth by [EG00, Theorem 3.1(d)(i)] and [EG05, Remark 6.10 and Lemma A.1], since T is abelian so that the adjoint action of T on its Lie algebra is trivial (see also [MS15, p. 2218-2219] and compare with [AMSS17, Proof of Theorem 3.3] for the counterpart of Ohmoto's equivariant Chern class transformation). We also have the equivariant Chern character (§2.3)

$$\mathrm{ch} : \mathrm{K}_T(X) \to \widehat{H}^*_T(X).$$

This is a contravariant ring homomorphism for T-equivariant morphisms. Then the Todd transformation satisfies the module property

$$\mathrm{td}_*([E]_T \otimes [\mathcal{F}]_T) = \mathrm{ch}([E]_T) \cap \mathrm{td}_*([\mathcal{F}]_T) \tag{16}$$

for $[E]_T \in \mathrm{K}_T(X)$ and $[\mathcal{F}]_T \in \mathrm{K}_0(\mathfrak{coh}^T(\mathcal{O}_X))$ ([EG00, Theorem 3.1]). Recall that for a T-equivariant vector bundle E, the cohomological Todd class $\mathrm{Td}(E) := \mathrm{Td}([E]_T)$ is multiplicative in short exact sequences, and for an equivariant line bundle $\mathcal{L}$ it is defined by

$$\mathrm{Td}(\mathcal{L}) := \frac{c_1^T(\mathcal{L})}{1 - e^{-c_1^T(\mathcal{L})}} = 1 + \frac{1}{2} c_1^T(\mathcal{L}) + \dots$$

Define the **unnormalized** (respectively **normalized**) **cohomological Hirzebruch class** $\widetilde{\mathrm{Td}}_y$ (resp., Td_y) by

$$\widetilde{\mathrm{Td}}_y(\mathcal{L}) := \mathrm{ch}(\lambda_y(\mathcal{L}^\vee))\,\mathrm{Td}(\mathcal{L}) = \frac{c_1^T(\mathcal{L})(1 + y e^{-c_1^T(\mathcal{L})})}{1 - e^{-c_1^T(\mathcal{L})}}$$

and

$$\mathrm{Td}_y(\mathcal{L}) := \frac{\widetilde{\mathrm{Td}}_y((1+y)[\mathcal{L}]_T)}{1+y} = \frac{c_1^T(\mathcal{L})(1 + y e^{-c_1^T(\mathcal{L})(1+y)})}{1 - e^{-c_1^T(\mathcal{L})(1+y)}}$$

Then extend these definitions to any equivariant vector bundle using the splitting principle, by requiring that they be multiplicative on short exact sequences. Note the specializations:

$$\widetilde{\mathrm{Td}}_{y=0}(\mathcal{L}) = \mathrm{Td}_{y=0}(\mathcal{L}) = \mathrm{Td}(\mathcal{L}), \tag{17}$$

and

$$\widetilde{\mathrm{Td}}_{y=-1}(\mathcal{L}) = c_1^T(\mathcal{L}) \quad , \quad \mathrm{Td}_{y=-1}(\mathcal{L}) = c^T(\mathcal{L}) = 1 + c_1^T(\mathcal{L}). \tag{18}$$

The power series $\mathrm{Td}_y(\mathcal{L}) = 1 + \dots$ has constant coefficient 1, whereas $\widetilde{\mathrm{Td}}_y(\mathcal{L}) = 1 + y + \dots$ has constant coefficient $1 + y$, explaining the name 'unnormalized'.

Combining Theorem 4.2 with the equivariant Riemann-Roch theorem proved by Edidin and Graham [EG00, Theorem 3.1] one obtains the following results about the unnormalized equivariant Hirzebruch class transformation.

Theorem 6.1. *Let X be a quasi-projective, non-singular, complex algebraic variety with an action of the torus T. The unnormalized (equivariant) Hirzebruch transformation*

$$\widetilde{\mathrm{Td}}_{y,*} := \mathrm{td}_* \circ \mathrm{MC}_y : \mathrm{K}_0^T(\mathrm{var}/X) \to \widehat{H}_*^T(X)[y] \subseteq \widehat{H}_*^T(X; \mathbb{Q}[y, (1+y)^{-1}])$$

is the unique natural transformation satisfying the following properties:

(a) *It is functorial with respect to T-equivariant proper morphisms of non-singular, quasi-projective varieties.*
(b) *It satisfies the normalization condition*
$$\widetilde{\mathrm{Td}}_{y,*}([\mathrm{id}_X]) = \widetilde{\mathrm{Td}}_y(TX) \cap [X]_T.$$
(c) *It is determined by its image on classes $[f : Z \to X] = f_![\mathrm{id}_Z]$ where Z is a non-singular, irreducible, quasi-projective algebraic variety and f is a T-equivariant proper morphism.*

(d) *It satisfies a Verdier-Riemann-Roch (VRR) formula: for any smooth, T-equivariant morphism $\theta : X \to Y$ of quasi-projective and non-singular algebraic varieties, and any $[f : Z \to Y] \in \mathrm{K}_0^T(\mathrm{var}/Y)$,*

$$\widetilde{\mathrm{Td}}_y(T^*_\theta) \cap \theta^* \widetilde{\mathrm{Td}}_{y,*}[f : Z \to Y] = \widetilde{\mathrm{Td}}_{y,*}[\theta^* f : Z \times_Y X \to X].$$

If one forgets the T-action, then the unnormalized equivariant Hirzebruch transformation above recovers the corresponding non-equivariant transformation $\widetilde{\mathcal{T}}_{y,*}$ from [BSY10] (either by its construction, or by the properties (a)-(c) from Theorem 6.1 and the corresponding results from [BSY10].)

Remark 6.2. Theorem 6.1 and its proof work more generally for a possibly singular, quasi-projective T-equivariant base variety X. Moreover, $\widetilde{\mathrm{Td}}_{y,*}$ commutes with exterior products:

$$\widetilde{\mathrm{Td}}_{y,*}[f \times f' : Z \times Z' \to X \times X'] = \widetilde{\mathrm{Td}}_{y,*}[f : Z \to X] \boxtimes \widetilde{\mathrm{Td}}_{y,*}[f' : Z' \to X'] \,.$$

This follows as in the non-equivariant context [BSY10, Corollary 3.1] from part (3) of Theorem 6.1 and the multiplicativity of the corresponding equivariant cohomological class for smooth and quasi-projective T-varieties X, X':

$$\widetilde{\mathrm{Td}}_y(T^*(X \times X')) = \widetilde{\mathrm{Td}}_y(T^*X) \boxtimes \widetilde{\mathrm{Td}}_y(T^*X') \in \widehat{H}^*_T(X \times X')[y] \,. \quad \lrcorner$$

One can also define a *normalized* equivariant Hirzebruch class transformation. With this aim, we introduce the following functorial **(co)homological Adams operations**:

$$\psi^*_{1+y} : \widehat{H}^*_T(X, \mathbb{Q}[y]) \to \widehat{H}^*_T(X, \mathbb{Q}[y]) \quad \text{and} \quad \psi_*^{1+y} : \widehat{H}^T_*(X, \mathbb{Q}[y]) \to \widehat{H}^T_*(X, \mathbb{Q}[y, (1+y)^{-1}])$$

given by multiplication with $(1+y)^i$ on $H^{2i}_T(X, \mathbb{Q}[y])$ respectively $(1+y)^{-j}$ on $H^T_{2j}(X, \mathbb{Q}[y])$, and satisfying the module and ring properties

$$\psi_*^{1+y}(- \cap -) = \psi^*_{1+y}(-) \cap \psi_*^{1+y}(-) \quad \text{and} \quad \psi^*_{1+y}(- \cup -) = \psi^*_{1+y}(-) \cup \psi^*_{1+y}(-) \,. \tag{19}$$

(In the Chow context, the (co)homological grading will not be doubled.)

Remark 6.3. These module and ring properties also hold for the functorial (co)homological duality transformations

$$\psi^*_{-1} : \widehat{H}^*_T(X, \mathbb{Q}) \to \widehat{H}^*_T(X, \mathbb{Q}) \quad \text{and} \quad \psi_*^{-1} : \widehat{H}^T_*(X, \mathbb{Q}) \to \widehat{H}^T_*(X, \mathbb{Q})$$

given by multiplication with $(-1)^i$ on $H^{2i}_T(X, \mathbb{Q})$ resp. $(-1)^j$ on $H^T_{2j}(X, \mathbb{Q})$. If X is smooth, these are consistent with the corresponding duality involutions in K-theory:

$$\mathrm{ch} \circ (-)^\vee = \psi^*_{-1} \circ \mathrm{ch}(-) : \mathrm{K}_T(X) \to \widehat{H}^*_T(X, \mathbb{Q}) \,,$$

since $c_1^T(\mathcal{L}^\vee) = -c_1^T(\mathcal{L})$ for a T-equivariant line bundle $\mathcal{L}$. Similarly, for the Grothendieck-Serre duality $\mathcal{D}$,

$$\mathrm{td}_* \circ \mathcal{D}(-) = \psi_*^{-1} \circ \mathrm{td}_*(-) : \mathrm{K}_T(X) \to \widehat{H}^T_*(X, \mathbb{Q}) \,,$$

since

$$\begin{aligned} \mathrm{td}_*(\omega_X) &= \mathrm{ch}(\omega_X)\,\mathrm{Td}(TX) \cap [X]_T = \mathrm{Td}(T^*X) \cap [X]_T \\ &= (-1)^{\dim X} \psi_*^{-1}(\mathrm{Td}(TX) \cap [X]_T) \,. \end{aligned} \quad \lrcorner$$

By definition, the (co)homological Adams operations satisfy

$$\psi^*_{1+y}(c_1^T(\mathcal{L})) = (1+y) c_1^T(\mathcal{L}); \quad \psi^*_{1+y}(\widetilde{\mathrm{Td}}_y(\mathcal{L})) = (1+y)\,\mathrm{Td}_y(\mathcal{L})$$

and $\psi_*^{1+y}([X]_T) = (1+y)^{-d}[X]_T$ for X of pure dimension d. It follows that

$$\psi_*^{1+y}(\widetilde{\mathrm{Td}}_y(TX) \cap [X]_T) = \psi^*_{1+y}(\widetilde{\mathrm{Td}}_y(TX)) \cap \psi_*^{1+y}([X]_T) = \mathrm{Td}_y(TX) \cap [X]_T \tag{20}$$

for X smooth and pure dimensional. In particular,

$$\psi_*^{1+y}(\widetilde{\mathrm{Td}}_y(TX)\cap[X]_T)\in\widehat{H}_*^T(X,\mathbb{Q}[y])\subseteq\widehat{H}_*^T(X,\mathbb{Q}[y,(1+y)^{-1}]). \tag{21}$$

This motivates the following:

Definition 6.4. *The* **normalized equivariant Hirzebruch transformation** $\mathrm{Td}_{y,*}$ *is defined by:*

$$\mathrm{Td}_{y,*}:=\psi_*^{1+y}\circ\widetilde{\mathrm{Td}}_{y,*}:\mathrm{K}_0^T(\mathrm{var}/X)\to\widehat{H}_*^T(X,\mathbb{Q}[y])\subseteq\widehat{H}_*^T(X,\mathbb{Q}[y,(1+y)^{-1}])\,. \tag{22}$$ ⌟

This transformation satisfies the same properties listed in Theorem 6.1 and Remark 6.2, with unnormalized classes replaced throughout by normalized classes. Furthermore, the normalized transformation has values in $\widehat{H}_*^T(X,\mathbb{Q}[y])$, by Theorem 6.1(c) and Equation (20) above, cf. (21).

Remark 6.5. The equivariant χ_y-genus of a T-variety Z may be calculated by

$$\chi_y(Z)=\widetilde{\mathrm{Td}}_y([Z\to\mathrm{pt}])\in\widehat{H}_T^*(\mathrm{pt})[y]\,.$$

By *rigidity* of the χ_y-genus (see [Web16, Theorem 7.2]), these quantities contain no information about the action of T, i.e., both are equal to the non-equivariant χ_y-genus under the embedding $\mathbb{Z}[y]\to\mathbb{Q}[y]\to\widehat{H}_T^*(\mathrm{pt})[y]$. (Cf. Remark 4.4, where the same conclusion is reached in K-theory.) In particular one also gets

$$\chi_y(Z)=\mathrm{Td}_y([Z\to\mathrm{pt}])\in\widehat{H}_T^*(\mathrm{pt})[y]\,.$$ ⌟

The definition of the Hirzebruch classes implies that

$$\mathrm{Td}_{y=-1}(TX)\cap[X]_T=c^T(TX)\cap[X]_T\quad\text{and}\quad\mathrm{Td}_{y=0}(TX)\cap[X]_T=\mathrm{Td}(TX)\cap[X]_T$$

for X smooth and pure dimensional. As in the non-equivariant context of [BSY10] (for the non-equivariant normalized Hirzebruch class $T_{y,*}$) this implies the following:

Corollary 6.6. *The equivariant Hirzebruch transformation* $\mathrm{Td}_{y,*}$ *fits into the following commutative diagram of natural transformations:*

$$\begin{array}{ccccc}
\mathcal{F}^T(X) & \xleftarrow{\;e\;} & \mathrm{K}_0^T(\mathrm{var}/X) & \xrightarrow{\mathrm{MC}_{y=0}} & \mathrm{K}_0(\mathfrak{coh}^T(\mathcal{O}_X)) \\
\big\downarrow{\scriptstyle c_*^T\otimes\mathbb{Q}} & & \big\downarrow{\scriptstyle \mathrm{Td}_{y,*}} & & \big\downarrow{\scriptstyle \mathrm{td}_*} \\
\widehat{H}_*^T(X) & \xleftarrow{\;y=-1\;} & \widehat{H}_*^T(X,\mathbb{Q}[y]) & \xrightarrow{\;y=0\;} & \widehat{H}_*^T(X)\,.
\end{array}$$

Here c_*^T is the *equivariant Chern class* transformation defined by Ohmoto [Ohm06], cf. §7.1. We note that c_*^T has values in the *integral* homology, and also the completion is not needed, i.e., $c_*^T:\mathcal{F}^T(X)\to H_*^T(X,\mathbb{Z})\subseteq\widehat{H}_*^T(X,\mathbb{Z})$. The rational coefficients are needed due to the use of the Chern character. Finally $e([\mathrm{id}_X])=1\!\!1_X$ even for X singular, so that the equivariant normalized Hirzebruch class $\mathrm{Td}_{y,*}(X):=\mathrm{Td}_{y,*}([\mathrm{id}_X])$ specializes for $y=-1$ also for a singular X to the (rationalized) equivariant Chern-Schwartz-MacPherson (CSM) class $c_*^T(X):=c_*^T(1\!\!1_X)$ of X, and similarly for a locally closed T-invariant subvariety $Z\subseteq X$:

$$\mathrm{Td}_{y=-1,*}[Z\hookrightarrow X]=c_*^T(1\!\!1_Z)\otimes\mathbb{Q}\,. \tag{23}$$

Note that if $H_*^T(X,\mathbb{Z})$ is torsion free, no information about $c_*^T(1\!\!1_Z)$ is lost by switching to rational coefficients; this is the case for the flag manifolds $X=G/P$. In §7 we will explain a more direct procedure yielding the equivariant CSM classes as the leading terms of motivic Chern classes of Schubert cells.

If $Z \subseteq X$ we will refer to the classes $\widetilde{\mathrm{Td}}_{y,*}[Z \hookrightarrow X]$ and $\mathrm{Td}_{y,*}[Z \hookrightarrow X]$ in $\widehat{H}^T_*(X, \mathbb{Q}[y])$ as the **unnormalized**, respectively the **normalized Hirzebruch class** of Z in X.

6.2. **Hirzebruch classes for flag manifolds.** We turn now to the study of the Hirzebruch transformation in the case when $X = G/B$. There is the following commutative diagram:

$$\begin{array}{ccccc} \mathrm{K}_T(X) \times \mathrm{K}_0(\mathfrak{coh}^T(\mathcal{O}_X)) & \xrightarrow{\otimes} & \mathrm{K}_0(\mathfrak{coh}^T(\mathcal{O}_X)) & \xrightarrow{\int_X} & \mathrm{K}_T(\mathrm{pt}) \\ \Big\downarrow{\scriptstyle \mathrm{ch} \otimes \mathrm{td}_*} & & \Big\downarrow{\scriptstyle \mathrm{td}_*} & & \Big\downarrow{\scriptstyle \mathrm{ch}=\mathrm{td}_*} \\ \widehat{H}^*_T(X) \times \widehat{H}^T_*(X) & \xrightarrow{\cap} & \widehat{H}^T_*(X) & \xrightarrow{\int_X} & \widehat{H}^*_T(\mathrm{pt})\,. \end{array}$$

Since X is smooth, from (15) and (16) we have that $\mathrm{td}_*(-) = \mathrm{ch}(-)\,\mathrm{Td}(TX) \cap [X]_T$. The functoriality of td_* gives the equivariant Grothendieck-Hirzebruch-Riemann-Roch theorem for an equivariant morphism of smooth T-varieties; cf. [EG00, Theorem 3.1]. In particular, the GHRR theorem and (1) imply that

$$\langle \mathrm{td}_*(\mathcal{O}^T_u), \mathrm{ch}(\mathcal{I}^{v,T}) \rangle = \delta_{u,v} = \langle \mathrm{td}_*(\mathcal{I}^T_u), \mathrm{ch}(\mathcal{O}^{v,T}) \rangle\,. \tag{24}$$

As a consequence of the fact that $\{[X(w)]_T\}_{w \in W}$ is a $H^*_T(\mathrm{pt})$-basis of $H^*_T(X)$, it is not difficult to show that the Todd classes $\mathrm{td}_*(X(w)) := \mathrm{td}_*(\mathcal{O}^T_w)$ of the Schubert varieties $X(w)$, respectively the Todd classes of the corresponding ideal sheaves $\mathrm{td}_*(\mathcal{I}_w)$, give two bases of $\widehat{H}^T_*(X)$ as a $\widehat{H}^*_T(\mathrm{pt})$-module. The key point is that the corresponding coefficient matrix is triangular with respect to the Bruhat ordering, with units on the diagonal; this follows from the functoriality of td_* for a closed inclusion $X(w) \hookrightarrow X$. The corresponding dual bases are given by the Chern characters $\mathrm{ch}(\mathcal{I}^{v,T})$, respectively $\mathrm{ch}(\mathcal{O}^{v,T})$ of the opposite Schubert varieties for $v \in W$. Specializing all the equivariant parameters to 0, one recovers the natural map $\widehat{H}^T_*(X) \to H_*(X)$ forgetting the T-action and mapping the equivariant Todd transformation to the ordinary one. In this case $\mathrm{td}_*(\mathcal{O}_w) = [X(w)] + \mathrm{l.o.t.}$ and $\mathrm{td}_*(\mathcal{I}_w) = [X(w)] + \mathrm{l.o.t.}$ (lower order terms).

Recall from Equation (8) that the Demazure and BGG operators are related by

$$\mathrm{ch}(\partial_i(-)) = \partial^H_i(\mathrm{Td}(T_{p_i})\,\mathrm{ch}(-)) : \mathrm{K}_T(X) \to \widehat{H}^*_T(X)\,. \tag{25}$$

Similarly, the equivariant Verdier-Riemann-Roch theorem (VRR) of [EG00, Theorem 3.1(d)] implies

$$\mathrm{td}_*(\partial_i(-)) = \mathrm{Td}(T_{p_i})\partial^H_i(\mathrm{td}_*(-)) : \mathrm{K}_0(\mathfrak{coh}^T(\mathcal{O}_X)) \to \widehat{H}^T_*(X)\,. \tag{26}$$

Then identities (25) and (26), combined with (3), translate into

$$\partial^H_i(\mathrm{Td}(T_{p_i})\,\mathrm{ch}(\mathcal{O}^T_w)) = \begin{cases} \mathrm{ch}(\mathcal{O}^T_{ws_i}) & \text{if } ws_i > w; \\ \mathrm{ch}(\mathcal{O}^T_w) & \text{otherwise.} \end{cases} \tag{27}$$

and

$$\mathrm{Td}(T_{p_i})\partial^H_i(\mathrm{td}_*(\mathcal{O}^T_w)) = \begin{cases} \mathrm{td}_*(\mathcal{O}^T_{ws_i}) & \text{if } ws_i > w; \\ \mathrm{td}_*(\mathcal{O}^T_w) & \text{otherwise} \end{cases}$$

Similarly, Lemma 3.4(a) translates into

$$(\partial^H_i\,\mathrm{Td}(T_{p_i}) - \mathrm{id})(\mathrm{ch}(\mathcal{I}^T_w)) = \begin{cases} \mathrm{ch}(\mathcal{I}^T_{ws_i}) & \text{if } ws_i > w; \\ -\,\mathrm{ch}(\mathcal{I}^T_w) & \text{otherwise.} \end{cases} \tag{28}$$

and

$$(\mathrm{Td}(T_{p_i})\partial_i^H - \mathrm{id})(\mathrm{td}_*(\mathcal{I}_w^T)) = \begin{cases} \mathrm{td}_*(\mathcal{I}_{ws_i}^T) & \text{if } ws_i > w; \\ -\,\mathrm{td}_*(\mathcal{I}_w^T) & \text{otherwise} \end{cases} \tag{29}$$

The specializations from Corollary 6.6 and the last two equations motivate the definition of the **unnormalized (ordinary and dual) Hirzebruch operators** $\widetilde{\mathcal{T}}_i^{\mathrm{Hir}}, \widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee} : \widehat{H}_T^*(X,\mathbb{Q})[y] \to \widehat{H}_T^*(X,\mathbb{Q})[y]$ by

$$\widetilde{\mathcal{T}}_i^{\mathrm{Hir}} := \widetilde{\mathrm{Td}}_y(T_{p_i})\partial_i^H - \mathrm{id}; \quad \widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee} = \partial_i^H \circ (\widetilde{\mathrm{Td}}_y(T_{p_i}) \cup (-)) - \mathrm{id}\,.$$

Similarly we define the **normalized Hirzebruch operators** $\mathcal{T}_i^{\mathrm{Hir}}, \mathcal{T}_i^{\mathrm{Hir},\vee} : \widehat{H}_T^*(X,\mathbb{Q})[y] \to \widehat{H}_T^*(X,\mathbb{Q})[y]$ by

$$\mathcal{T}_i^{\mathrm{Hir}} := \mathrm{Td}_y(T_{p_i})\partial_i^H - \mathrm{id}; \quad \mathcal{T}_i^{\mathrm{Hir},\vee} = \partial_i^H \circ (\mathrm{Td}_y(T_{p_i}) \cup (-)) - \mathrm{id}\,.$$

All these operators are $\widehat{H}_T^*(\mathrm{pt})$-linear. In the statements that follow we explain in detail their relation to the K-theoretic Demazure-Lusztig operators $\mathcal{T}_i, \mathcal{T}_i^\vee$.

Lemma 6.7. *The Hirzebruch operators satisfy the following commutativity relations:*

(a) As operators $\mathrm{K}_T(X)[y] \to \widehat{H}_T^*(X,\mathbb{Q})[y]$,

$$\mathrm{td}_*\,\mathcal{T}_i = \widetilde{\mathcal{T}}_i^{\mathrm{Hir}}\,\mathrm{td}_* \quad \textit{and} \quad \mathrm{ch}\,\mathcal{T}_i^\vee = \widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee}\,\mathrm{ch}\,. \tag{30}$$

(b) As operators $\mathrm{K}_T(X)[y] \to \widehat{H}_T^*(X,\mathbb{Q}[y,(1+y)^{-1}])$,

$$\psi_*^{1+y}\,\mathrm{td}_*\,\mathcal{T}_i = \mathcal{T}_i^{\mathrm{Hir}}\psi_*^{1+y}\,\mathrm{td}_* \quad \textit{and} \quad \psi_{1+y}^*\,\mathrm{ch}\,\mathcal{T}_i^\vee = \mathcal{T}_i^{\mathrm{Hir},\vee}\psi_{1+y}^*\,\mathrm{ch}\,. \tag{31}$$

Proof. The first commutativity relation in part (a) follows from the following sequence of equalities:

$$\begin{aligned} \widetilde{\mathcal{T}}_i^{\mathrm{Hir}}\,\mathrm{td}_* &= \widetilde{\mathrm{Td}}_y(T_{p_i})\partial_i^H\,\mathrm{td}_* - \mathrm{td}_* \\ &= \mathrm{ch}(\lambda_y(T_{p_i}^*))\,\mathrm{Td}(T_{p_i})\partial_i^H\,\mathrm{td}_* - \mathrm{td}_* \\ &= \mathrm{ch}(\lambda_y(T_{p_i}^*))\,\mathrm{td}_*\,\partial_i - \mathrm{td}_* \\ &= \mathrm{td}_*(\lambda_y(T_{p_i}^*)\partial_i - \mathrm{id}) \\ &= \mathrm{td}_*\,\mathcal{T}_i. \end{aligned}$$

Here the third equality follows from (26), and the fourth from the module property (16) of the Todd transformation. The second commutativity relation in (a) follows from

$$\begin{aligned} \widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee}\,\mathrm{ch} &= \partial_i^H(\widetilde{\mathrm{Td}}_y(T_{p_i})\,\mathrm{ch}) - \mathrm{ch} \\ &= \partial_i^H(\mathrm{Td}(T_{p_i})\,\mathrm{ch}(\lambda_y(T_{p_i}^*) \cup (-))) - \mathrm{ch} \\ &= \mathrm{ch}\,\partial_i(\lambda_y(T_{p_i}^*) \cup (-)) - \mathrm{ch} \\ &= \mathrm{ch}(\partial_i\lambda_y(T_{p_i}^*) - \mathrm{id}) \\ &= \mathrm{ch}\,\mathcal{T}_i^\vee. \end{aligned}$$

In this case, the second equality follows since ch is a ring homomorphism, the third from (25), and the rest from the definitions. Part (b) follows from (a) and the identities in Lemma 6.8 below. ☐

Lemma 6.8. *As operators* $\widehat{H}_T^*(X)[y] \to \widehat{H}_T^*(X,\mathbb{Q}[y,(1+y)^{-1}])$,

$$\psi_*^{1+y}\widetilde{\mathcal{T}}_i^{\mathrm{Hir}} = \mathcal{T}_i^{\mathrm{Hir}}\psi_*^{1+y} \quad \textit{and} \quad \psi_{1+y}^*\widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee} = \mathcal{T}_i^{\mathrm{Hir},\vee}\psi_{1+y}^*.$$

Proof. The first commutativity relation follows similarly to the proof of Lemma 6.7(a) above from the following sequence of equalities:

$$\begin{aligned}\psi_*^{1+y}\widetilde{\mathcal{T}}_i^{\mathrm{Hir}} &= \psi_*^{1+y}(\widetilde{\mathrm{Td}}_y(T_{p_i})\partial_i^H - \mathrm{id})\\ &= \psi_{1+y}^*(\widetilde{\mathrm{Td}}_y(T_{p_i}^*))\psi_*^{1+y}\partial_i^H - \psi_*^{1+y}\\ &= \frac{\psi_{1+y}^*(\widetilde{\mathrm{Td}}_y(T_{p_i}^*))}{1+y}\partial_i^H\psi_*^{1+y} - \psi_*^{1+y}\\ &= \mathrm{Td}_y(T_{p_i}^*))\partial_i^H\psi_*^{1+y} - \psi_*^{1+y}\\ &= \mathcal{T}_i^{\mathrm{Hir}}\psi_*^{1+y}.\end{aligned}$$

Here the second equality follows from the module property (19) of the Adams transformation, and the third uses the property

$$\psi_*^{1+y}\circ\partial_i^H = (1+y)^{-1}\partial_i^H\circ\psi_*^{1+y}\,,$$

since ∂_i^H shifts the complex homological degree by one. The second commutativity relation follows similarly from:

$$\begin{aligned}\psi_{1+y}^*\widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee} &= \psi_{1+y}^*(\partial_i^H\,\widetilde{\mathrm{Td}}_y(T_{p_i}) - \mathrm{id})\\ &= \partial_i^H\left(\frac{\psi_{1+y}^*}{1+y}(\widetilde{\mathrm{Td}}_y(T_{p_i})\cup(-))\right) - \psi_{1+y}^*\\ &= \partial_i^H\left(\frac{\psi_{1+y}^*\,\widetilde{\mathrm{Td}}_y(T_{p_i})}{1+y}\psi_{1+y}^*(-)\right) - \psi_{1+y}^*\\ &= \partial_i^H(\mathrm{Td}_y(T_{p_i}^*)\psi_{1+y}^*(-)) - \psi_{1+y}^*\\ &= \mathcal{T}_i^{\mathrm{Hir},\vee}\psi_{1+y}^*.\end{aligned}$$

In this case, the second equality uses the property

$$\psi_{1+y}^*\circ\partial_i^H = \partial_i^H\circ((1+y)^{-1}\psi_{1+y}^*)\,,$$

since ∂_i^H shifts the complex cohomological degree by minus one. The third equality follows since ψ_{1+y}^* is a ring homomorphism. □

Lemma 6.9. *(a) The ordinary (normalized/unnormalized) Hirzebruch operators are adjoint to the dual operators, i.e. for any* $a, b\in\widehat{H}_T^*(X,\mathbb{Q})[y]$,

$$\langle\mathcal{T}_i^{\mathrm{Hir}}(a), b\rangle = \langle a, \mathcal{T}_i^{\mathrm{Hir},\vee}(b)\rangle;\quad \langle\widetilde{\mathcal{T}}_i^{\mathrm{Hir}}(a), b\rangle = \langle a, \widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee}(b)\rangle,$$

where the pairing is extended by $\mathbb{Q}[y]$*-linearity.*[3]

(b) Each family of the Hirzebruch operators (ordinary / dual / (un)normalized) satisfies the same relations as the K-theoretic Demazure-Lusztig operators; cf. Proposition 3.2.

Proof. Part (a) can be proved by using the self-adjointness of ∂_i^H and the operators of multiplication by $\mathrm{Td}_y(T_{p_i})$, as follows. By definition,

$$\begin{aligned}\langle\mathcal{T}_i^{\mathrm{Hir}}(a), b\rangle &= \langle(\mathrm{Td}_y(T_{p_i})\partial_i^H - \mathrm{id})(a), b\rangle\\ &= \langle a, (\partial_i^H\,\mathrm{Td}_y(T_{p_i}) - \mathrm{id})b\rangle\\ &= \langle a, \mathcal{T}_i^{\mathrm{Hir},\vee}(b)\rangle.\end{aligned}$$

[3]Later in the context of Segre classes, we will tacitly extend this pairing by linearity from coefficients in $\mathbb{Q}[y]$ to $\mathbb{Q}[y,(1+y)^{-1}]$.

A similar proof works for the unnormalized operators.

We turn to the relations in (b). First, we deduce from (a) that it suffices to prove the statements for the ordinary operators. Then we use Lemma 6.7(a) again to show that for $a' \in \mathrm{K}_T(G/B)$,

$$\widetilde{\mathcal{T}}_{i_1}^{\mathrm{Hir}}\widetilde{\mathcal{T}}_{i_2}^{\mathrm{Hir}}\dots\widetilde{\mathcal{T}}_{i_k}^{\mathrm{Hir}}(\mathrm{td}_*(a')) = \mathrm{td}_*\,\mathcal{T}_{i_1}\mathcal{T}_{i_2}\dots\mathcal{T}_{i_k}(a').$$

Since td_* is surjective (after appropriately extending the coefficients via $\mathrm{ch} : \mathrm{K}_T(\mathrm{pt}) \to \widehat{H}_T^*(\mathrm{pt})$), this implies the claim for the unnormalized Hirzebruch operators. The same proof works for the normalized operators, using classes of the form $\psi_*^{1+y}\,\mathrm{td}_*(a')$, Lemma 6.7(b) and Lemma 6.8. We also note that in order to prove the statement for the dual operators, instead of adjointness, one may alternatively work with classes of type $\mathrm{ch}(a')$ and $\psi_{1+y}^*\,\mathrm{ch}(a')$. □

Remark 6.10. The results of Lemma 6.9(b) also follow from the following argument. Regard $\widehat{H}_T^*(\mathrm{pt})[y]$ as a $\mathrm{K}_T(\mathrm{pt})[y]$-algebra via the (injective) equivariant Chern character map. Then the transformations td_* and ch induce injective homomorphisms of $\mathrm{K}_T(\mathrm{pt})[y]$-algebras

$$\overline{\mathrm{td}_*}, \overline{\mathrm{ch}} : \mathrm{End}_{\mathrm{K}_T(\mathrm{pt})[y]}(\mathrm{K}_T(X)[y]) \to \mathrm{End}_{\widehat{H}_T^*(\mathrm{pt})[y]}(\widehat{H}_T^*(X)[y]).$$

The injectivity part follows by using suitable bases, such as images of Schubert classes $\mathcal{O}_w$ under the Todd or Chern character maps. Then Lemma 6.7 may be interpreted as giving the identities

$$\overline{\mathrm{td}_*}(\mathcal{T}_i) = \widetilde{\mathcal{T}}_i^{\mathrm{Hir}} \quad \text{and} \quad \overline{\mathrm{ch}}(\mathcal{T}_i^\vee) = \widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee}.$$

Since $\overline{\mathrm{td}_*}, \overline{\mathrm{ch}}$ are algebra homomorphisms, they will preserve relations satisfied by $\mathcal{T}_i, \mathcal{T}_i^\vee$, proving the claim for the unnormalized operators.

One may argue similarly in the case of normalized operators. Start by changing the coefficient ring using the algebra isomorphism

$$\widehat{H}_T^*(X)[y] \otimes_{\widehat{H}_T^*(\mathrm{pt})[y]} \widehat{H}_T^*(\mathrm{pt}, \mathbb{Q}[y,(1+y)^{-1}]) \xrightarrow{\ \sim\ } \widehat{H}_T^*(X, \mathbb{Q}[y,(1+y)^{-1}]).$$

Then the Adams operations

$$\psi_{1+y}^* : \widehat{H}_T^*(\mathrm{pt})[y] \to \widehat{H}_T^*(\mathrm{pt})[y] \text{ and } \psi_*^{1+y} : \widehat{H}_*^T(\mathrm{pt})[y] \to \widehat{H}_*^T(\mathrm{pt}, \mathbb{Q}[y,(1+y)^{-1}])$$

induce algebra isomorphisms

$$\overline{\psi_{1+y}^*} : \mathrm{End}_{\widehat{H}_T^*(\mathrm{pt},\mathbb{Q}[y,(1+y)^{-1}]}(\widehat{H}_T^*(X,\mathbb{Q}[y,(1+y)^{-1}])) \to \mathrm{End}_{\widehat{H}_T^*(\mathrm{pt},\mathbb{Q}[y,(1+y)^{-1}]}(\widehat{H}_T^*(X,\mathbb{Q}[y,(1+y)^{-1}]))$$

and

$$\overline{\psi_*^{1+y}} : \mathrm{End}_{\widehat{H}_T^*(\mathrm{pt},\mathbb{Q}[y,(1+y)^{-1}])}(\widehat{H}_*^T(X,\mathbb{Q}[y,(1+y)^{-1}])) \to \mathrm{End}_{\widehat{H}_T^*(\mathrm{pt},\mathbb{Q}[y,(1+y)^{-1}])}(\widehat{H}_*^T(X,\mathbb{Q}[y,(1+y)^{-1}]))$$

which by Lemma 6.8 satisfy

$$\overline{\psi_*^{1+y}}(\widetilde{\mathcal{T}}_i^{\mathrm{Hir}}) = \mathcal{T}_i^{\mathrm{Hir}} \quad \text{and} \quad \overline{\psi_{1+y}^*}(\widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee}) = \mathcal{T}_i^{\mathrm{Hir},\vee}.$$

Therefore the relations satisfied by the unnormalized operators are the same as those for the normalized ones. ⌟

As a consequence of Lemma 6.9 (b), we can set $\widetilde{\mathcal{T}}_w^{\mathrm{Hir}} = \widetilde{\mathcal{T}}_{i_1}^{\mathrm{Hir}} \cdots \widetilde{\mathcal{T}}_{i_k}^{\mathrm{Hir}}$, $\mathcal{T}_w^{\mathrm{Hir}} = \mathcal{T}_{i_1}^{\mathrm{Hir}} \cdots \mathcal{T}_{i_k}^{\mathrm{Hir}}$ for $w = s_{i_1} \cdots s_{i_k}$ a reduced decomposition in W. We now come to the main results about equivariant Hirzebruch classes of Schubert cells, starting with their recursion.

Theorem 6.11. *Let $w \in W$ and s_i a simple reflection such that $ws_i > w$.*

(a) The unnormalized Hirzebruch operators satisfy:

$$\widetilde{\mathcal{T}}_i^{\mathrm{Hir}}(\widetilde{\mathrm{Td}}_{y,*}(X(w)^\circ)) = \widetilde{\mathrm{Td}}_{y,*}(X(ws_i)^\circ);$$

(b) The normalized Hirzebruch operators satisfy:

$$\mathcal{T}_i^{\mathrm{Hir}}(\mathrm{Td}_{y,*}^T(X(w)^\circ)) = \mathrm{Td}_{y,*}^T(X(ws_i)^\circ).$$

(c) In particular, for $w \in W$, the Hirzebruch classes of Schubert cells are given by:

$$\widetilde{\mathrm{Td}}_{y,*}(X(w)^\circ) = \widetilde{\mathcal{T}}_{w^{-1}}^{\mathrm{Hir}}[e_{\mathrm{id}}]_T \quad \textit{and} \quad \mathrm{Td}_{y,*}^T(X(w)^\circ) = \mathcal{T}_{w^{-1}}^{\mathrm{Hir}}[e_{\mathrm{id}}]_T.$$

Proof. Part (a) follows from Theorem 4.5 (or more specifically [AMSS19, Corollary 5.2]) together with the commutation relations (30). The same argument applies to (b), using the definition of the normalized Hirzebruch transformation from (22), Theorem 4.5 again, and the commutation relations (31). Part (c) is a consequence of (a) and (b), taking into account that, by functoriality, $\widetilde{\mathrm{Td}}_{y,*}(e_{\mathrm{id}}) = \mathrm{Td}_{y,*}(e_{\mathrm{id}}) = [e_{\mathrm{id}}]_T$. □

Next we formulate the corresponding orthogonality results for the equivariant Hirzebruch classes of Schubert cells. In analogy with the definition of the operators $\mathfrak{L}_i$ giving the dual classes of the motivic Chern classes, define the operators $\mathfrak{L}_i^{\mathrm{Hir}}$ and $\widetilde{\mathfrak{L}}_i^{\mathrm{Hir}}$ by

$$\mathfrak{L}_i^{\mathrm{Hir}} := -y(\mathcal{T}_i^{\mathrm{Hir},\vee})^{-1} = \mathcal{T}_i^{\mathrm{Hir},\vee} + (1+y)\,\mathrm{id} \quad \text{and} \quad \widetilde{\mathfrak{L}}_i^{\mathrm{Hir}} := -y(\widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee})^{-1} = \widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee} + (1+y)\,\mathrm{id}\,.$$

Theorem 6.12. *For any $u, v \in W$,*

$$\langle \widetilde{\mathrm{Td}}_{y*}(X(u)^\circ), \widetilde{\mathfrak{L}}_{v^{-1}w_0}^{\mathrm{Hir}}([e_{w_0}]_T)\rangle_H = \delta_{u,v}\, \widetilde{\mathrm{Td}}_y(T_{w_0}X) \tag{32}$$

and

$$\langle \mathrm{Td}_{y*}(X(u)^\circ), \mathfrak{L}_{v^{-1}w_0}^{\mathrm{Hir}}([e_{w_0}]_T)\rangle_H = \delta_{u,v}\, \mathrm{Td}_y(T_{w_0}X)\,. \tag{33}$$

Proof. We apply the equivariant Todd class transformation to both sides in the expression from Theorem 4.6 to obtain

$$\begin{aligned}
\delta_{u,v}\,\mathrm{td}_*\left(\prod_{\alpha>0}(1+ye^{-\alpha})\right) &= \mathrm{td}_*\left(\langle \mathrm{MC}_y(X(u)^\circ), \widetilde{\mathrm{MC}}_y(Y(v)^\circ)\rangle_K\right) \\
&= \mathrm{td}_*\left(\int_X^K \mathrm{MC}_y(X(u)^\circ)\cdot \widetilde{\mathrm{MC}}_y(Y(v)^\circ)\right) \\
&= \int_X^{H^*} \mathrm{td}_*(\mathrm{MC}_y(X(u)^\circ))\cdot \mathrm{ch}(\widetilde{\mathrm{MC}}_y(Y(v)^\circ)) \\
&= \int_X^{H^*} \widetilde{\mathrm{Td}}_{y,*}(X(u)^\circ)\cdot \mathrm{ch}(\mathfrak{L}_{v^{-1}w_0}(\mathcal{O}^{w_0,T})) \\
&= \int_X^{H^*} \widetilde{\mathrm{Td}}_{y,*}(X(u)^\circ)\cdot \widetilde{\mathfrak{L}}_{v^{-1}w_0}^{\mathrm{Hir}}(\mathrm{ch}(\mathcal{O}^{w_0,T})) \\
&= \langle \widetilde{\mathrm{Td}}_{y,*}(X(u)^\circ), \widetilde{\mathfrak{L}}_{v^{-1}w_0}^{\mathrm{Hir}}(\mathrm{ch}(\mathcal{O}^{w_0,T}))\rangle_H.
\end{aligned}$$

Here we use K and H^* to indicate where the operation is taken, and the fifth equality follows from Lemma 6.7. Given this, the claim in (32) follows because

$$\mathrm{ch}(\mathcal{O}^{w_0,T}) = \frac{[e_{w_0}]_T}{\mathrm{Td}(TX)} = \frac{[e_{w_0}]_T}{\mathrm{Td}(T_{w_0}X])},$$

$$\widetilde{\mathrm{Td}}_y(T_{w_0}X) = \mathrm{ch}\left(\prod_{\alpha>0}(1+ye^{-\alpha})\right)\mathrm{Td}(T_{w_0}X),$$

and the fact that $\widetilde{\mathfrak{L}}_i^{\mathrm{Hir}}$ is $\widehat{H}_T^*(\mathrm{pt})[y]$-linear. The equality from (33) follows from (32) by application of the Adams operation ψ_*^{1+y}:

$$\begin{aligned}\delta_{u,v}\psi^*_{1+y}\left(\widetilde{\mathrm{Td}}_y(T_{w_0}X)\right) =&\psi_*^{1+y}\left(\langle\widetilde{\mathrm{Td}}_{y*}(X(u)^\circ),\widetilde{\mathfrak{L}}^{\mathrm{Hir}}_{v^{-1}w_0}([e_{w_0}]_T)\rangle_H\right)\\ =&\langle\psi_*^{1+y}\left(\widetilde{\mathrm{Td}}_{y*}(X(u)^\circ)\right),\psi^*_{1+y}\left(\widetilde{\mathfrak{L}}^{\mathrm{Hir}}_{v^{-1}w_0}([e_{w_0}]_T)\right)\rangle_H\\ =&\langle\mathrm{Td}_{y*}(X(u)^\circ),\mathfrak{L}^{\mathrm{Hir}}_{v^{-1}w_0}\psi^*_{1+y}([e_{w_0}]_T)\rangle_H\,.\end{aligned}$$

Then the claim follows because

$$\psi^*_{1+y}\left(\widetilde{\mathrm{Td}}_y(T_{e_{w_0}}X)\right)=(1+y)^{\dim X}\,\mathrm{Td}_y(T_{e_{w_0}}X)\text{ and }\psi^*_{1+y}([e_{w_0}]_T)=(1+y)^{\dim X}[e_{w_0}]_T,$$

with $[e_{w_0}]_T$ viewed as an equivariant cohomology class of complex degree $\dim X$ (by equivariant Poincaré duality). $\square$

We finish this section with the counterpart of Theorem 4.12, using now the (un)normalized Segre version of the Hirzebruch classes:

$$\frac{\widetilde{\mathrm{Td}}_{y*}(X(w)^\circ)}{\widetilde{\mathrm{Td}}_y(TX)}\quad\text{and}\quad\frac{\mathrm{Td}_{y*}(X(w)^\circ)}{\mathrm{Td}_y(TX)}.$$

Observe that for any smooth X, the class $\widetilde{\mathrm{Td}}_y(X)$ is invertible in the completed ring $\widehat{H}_T^*(X,\mathbb{Q}[y,(1+y)^{-1}])$, since its leading term is the invertible element $1+y$. Similarly $\mathrm{Td}_y(X)$ is invertible in $\widehat{H}_T^*(X,\mathbb{Q}[y])$, as its leading term is 1.

Theorem 6.13. *For any $w\in W$ one has in $\widehat{H}_T^*(X,\mathbb{Q}[y,(1+y)^{-1}])$, resp., $\widehat{H}_T^*(X,\mathbb{Q}[y])$:*

$$\frac{\widetilde{\mathrm{Td}}_{y*}(X(w)^\circ)}{\widetilde{\mathrm{Td}}_y(TX)}=\widetilde{\mathcal{T}}^{\mathrm{Hir},\vee}_{w^{-1}}\left(\frac{[e_{\mathrm{id}}]_T}{\widetilde{\mathrm{Td}}_y(T_{e_{\mathrm{id}}}X)}\right)\quad\textit{and}\quad\frac{\mathrm{Td}_{y*}(X(w)^\circ)}{\mathrm{Td}_y(TX)}=\mathcal{T}^{\mathrm{Hir},\vee}_{w^{-1}}\left(\frac{[e_{\mathrm{id}}]_T}{\mathrm{Td}_y(T_{e_{\mathrm{id}}}X)}\right),$$

as well as

$$\frac{\widetilde{\mathrm{Td}}_{y*}(Y(w)^\circ)}{\widetilde{\mathrm{Td}}_y(TX)}=\widetilde{\mathcal{T}}^{\mathrm{Hir},\vee}_{(w_0w)^{-1}}\left(\frac{[e_{w_0}]_T}{\widetilde{\mathrm{Td}}_y(T_{e_{w_0}}X)}\right)\quad\textit{and}\quad\frac{\mathrm{Td}_{y*}(Y(w)^\circ)}{\mathrm{Td}_y(TX)}=\mathcal{T}^{\mathrm{Hir},\vee}_{(w_0w)^{-1}}\left(\frac{[e_{w_0}]_T}{\mathrm{Td}_y(T_{e_{w_0}}X)}\right).$$

Proof. We only explain the proof for the opposite Schubert cells $Y(w)^\circ$, since the result for the Schubert cells $X(w)^\circ$ are shown in exactly the same way. (Alternatively, one may apply the automorphism w_0^L obtained by left multiplication by the longest element $w_0\in W$; see [AMSS17, §5.2] or [MNS22, §3.1].) We start with the unnormalized classes. The application of ch to (11) together with Lemma 6.7(a) imply:

$$\begin{aligned}\frac{\mathrm{ch}(\mathrm{MC}_y(Y(w)^\circ))}{\mathrm{ch}(\lambda_yT^*X)}=&\widetilde{\mathcal{T}}^{\mathrm{Hir},\vee}_{(w_0w)^{-1}}\left(\frac{\mathrm{ch}(\mathcal{O}^{w_0,T})}{\mathrm{ch}\left(\prod_{\alpha>0}(1+ye^{-\alpha})\right)}\right)\\ =&\widetilde{\mathcal{T}}^{\mathrm{Hir},\vee}_{(w_0w)^{-1}}\left(\frac{[e_{w_0}]_T}{\widetilde{\mathrm{Td}}_y(T_{e_{w_0}}X)}\right),\end{aligned}$$

with the last equality as in the proof of Theorem 6.12. Then the result follows from

$$\widetilde{\mathrm{Td}}_{y*}(Y(w)^\circ)=\mathrm{td}_*(\mathrm{MC}_y(Y(w)^\circ))=\mathrm{ch}(\mathrm{MC}_y(Y(w)^\circ))\,\mathrm{Td}(TX)$$

and

$$\widetilde{\mathrm{Td}}_y(TX)=\mathrm{ch}(\lambda_yT^*X)\,\mathrm{Td}(TX)\,.$$

To deduce the result for the normalized classes, we further apply the Adams transformation ψ^*_{1+y}, with $\mathrm{td}_*(\mathrm{MC}_y(Y(w)^\circ))$ and $[e_{e_0}]_T$ viewed as an equivariant cohomology class as before (by equivariant Poincaré duality, with $[e_{w_0}]_T$ of complex degree $\dim X$):

$$\begin{aligned}\psi^*_{1+y}\left(\frac{\widetilde{\mathrm{Td}}_{y,*}(Y(w)^\circ)}{\widetilde{\mathrm{Td}}_y(TX)}\right) &= \left(\psi^*_{1+y}\widetilde{\mathcal{T}}^{\mathrm{Hir},\vee}_{(w_0w)^{-1}}\right)\left(\frac{[e_{w_0}]_T}{\widetilde{\mathrm{Td}}_y(T_{e_{w_0}}X)}\right)\\ &= \left(\widetilde{\mathcal{T}}^{\mathrm{Hir},\vee}_{(w_0w)^{-1}}\psi^*_{1+y}\right)\left(\frac{[e_{w_0}]_T}{\widetilde{\mathrm{Td}}_y(T_{e_{w_0}}X)}\right)\\ &=\widetilde{\mathcal{T}}^{\mathrm{Hir},\vee}_{(w_0w)^{-1}}\left(\frac{(1+y)^{\dim X}[e_{w_0}]_T}{(1+y)^{\dim X}\,\mathrm{Td}_y(T_{e_{w_0}}X)}\right).\end{aligned}$$

Then the result follows from

$$\psi^*_{1+y}\left(\widetilde{\mathrm{Td}}_y(TX)\right) = (1+y)^{\dim X}\,\mathrm{Td}_y(TX)$$

and

$$\psi^*_{1+y}\left(\widetilde{\mathrm{Td}}_{y*}(Y(w)^\circ)\right) = (1+y)^{\dim X}\,\mathrm{Td}_{y*}(Y(w)^\circ)\,,$$

since

$$\psi^*_{1+y}(-)\cap[X]_T = \psi^*_{1+y}(-)\cap\psi^{1+y}_*((1+y)^{\dim X}[X]_T) = (1+y)^{\dim X}\psi^{1+y}_*(-\cap[X]_T)\,.$$

by the module property (19). □

6.3. **Specializations of (dual) Hirzebruch operators and Hirzebruch classes of Schubert cells.** We take this opportunity to record the specializations at $y=-1$ and $y=0$ for the (un)normalized Hirzebruch operators $\widetilde{\mathcal{T}}_i^{\mathrm{Hir}}$ and $\mathcal{T}_i^{\mathrm{Hir}}$ and their (shifted) dual operators $\widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee},\mathcal{T}_i^{\mathrm{Hir},\vee}$ and $\widetilde{\mathfrak{L}}_i^{\mathrm{Hir}},\mathfrak{L}_i^{\mathrm{Hir}}$. These follow from the definitions of these objects, using the corresponding specializations of the Hirzebruch classes from (17) and (18), and are stated in the next proposition.

Proposition 6.14. *The following hold:*

(a) The specializations at $y=0$ of the (un)normalized Hirzebruch operators are given by

$$(\widetilde{\mathcal{T}}_i^{\mathrm{Hir}})_{y=0} = (\mathcal{T}_i^{\mathrm{Hir}})_{y=0} = \mathrm{Td}(T_{p_i})\partial_i^H - \mathrm{id},$$

and for their duals by

$$(\widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee})_{y=0} = (\mathcal{T}_i^{\mathrm{Hir},\vee})_{y=0} = \partial_i^H\,\mathrm{Td}(T_{p_i}) - \mathrm{id}$$

so that

$$(\widetilde{\mathfrak{L}}_i^{\mathrm{Hir}})_{y=0} = (\mathfrak{L}_i^{\mathrm{Hir}})_{y=0} = \partial_i^H\,\mathrm{Td}(T_{p_i})\,.$$

(b) The specializations at $y=-1$ of the (un)normalized Hirzebruch operators is given by:

$$(\widetilde{\mathcal{T}}_i^{\mathrm{Hir}})_{y=-1} = -s_i; \quad (\mathcal{T}_i^{\mathrm{Hir}})_{y=-1} = \mathcal{T}_i^H,$$

and for their (shifted) duals by

$$(\widetilde{\mathcal{T}}_i^{\mathrm{Hir},\vee})_{y=-1} = (\widetilde{\mathfrak{L}}_i^{\mathrm{Hir}})_{y=-1} = -s_i^\vee = s_i; \quad (\mathcal{T}_i^{\mathrm{Hir},\vee})_{y=-1} = (\mathfrak{L}_i^{\mathrm{Hir}})_{y=-1} = \mathcal{T}_i^{H,\vee}\,,$$

where the operators s_i and $\mathcal{T}_i^H,\mathcal{T}_i^{H,\vee}$ are defined in (4)*, respectively* (5)*.*

Using these specializations of the (shifted dual) Hirzebruch operators, we can specialize in a similar way the corresponding results from the Theorems 6.11, 6.12 and 6.13 to $y = 0$ and $y = -1$. First we consider the case $y = 0$. We obtain:

$$\mathrm{Td}_{y=0,*}(X(w)^\circ) = \widetilde{\mathrm{Td}}_{y=0,*}(X(w)^\circ) = \mathrm{td}_*(\mathrm{MC}_0(X(w)^\circ)) = \mathrm{td}_*(\mathcal{I}_w^T) \tag{34}$$

by Theorem 5.1(b). Then Theorem 6.11 specializes for $y = 0$ to the recursion

$$(\mathrm{Td}(T_{p_i})\partial_i^H - \mathrm{id})(\mathrm{td}_*(\mathcal{I}_w^T)) = \mathrm{td}_*(\mathcal{I}_{ws_i}^T) \text{ for } ws_i > w$$

from (29) for $w \in W$ and s_i a simple reflection. Similarly Theorem 6.12 specializes for $y = 0$ to

$$\langle \mathrm{Td}_*(\mathcal{I}_u^T), (\mathfrak{L}^{\mathrm{Hir}}_{v^{-1}w_0})_{y=0}([e_{w_0}]_T)\rangle_H = \delta_{u,v}\,\mathrm{Td}(T_{e_{w_0}}X) \tag{35}$$

for $u, v \in W$. Since $\mathrm{ch}(\mathcal{O}^{w_0,T}) = \frac{[e_{w_0}]_T}{\mathrm{Td}(T_{e_{w_0}}X)}$, this translates into

$$\langle \mathrm{Td}_*(\mathcal{I}_u^T), (\mathfrak{L}^{\mathrm{Hir}}_{v^{-1}w_0})_{y=0}(\mathrm{ch}(\mathcal{O}^{w_0,T}))\rangle_H = \delta_{u,v}\ ,$$

from which we deduce by (24) (for $v \in W$) that:

$$\mathrm{ch}(\mathcal{O}^{v,T}) = (\mathfrak{L}^{\mathrm{Hir}}_{v^{-1}w_0})_{y=0}(\mathrm{ch}(\mathcal{O}^{w_0,T}))\ , \tag{36}$$

with $(\mathfrak{L}_i^{\mathrm{Hir}})_{y=0} = \partial_i^H\,\mathrm{Td}(T_{p_i})$. This recovers Equation (27).

Finally, since $\mathrm{td}_*(-) = \mathrm{ch}(-)\,\mathrm{Td}(TX)$ for $y = 0$, Theorem 6.13 specializes to

$$\mathrm{ch}(\mathcal{I}_w^T)) = \left(\mathcal{T}_{w^{-1}}^{\mathrm{Hir},\vee}\right)_{y=0}(\mathrm{ch}(\mathcal{I}_{\mathrm{id}}^T)) \quad \text{and} \quad \mathrm{ch}(\mathcal{I}^{w,T})) = \left(\mathcal{T}_{(w_0w)^{-1}}^{\mathrm{Hir},\vee}\right)_{y=0}(\mathrm{ch}(\mathcal{I}^{w_0,T}))$$

for $w \in W$, with $(\mathcal{T}_i^{\mathrm{Hir},\vee})_{y=0} = \partial_i^H\,\mathrm{Td}(T_{p_i}) - \mathrm{id}$, consistent with Equation (28).

Next we record the specializations of Theorems 6.11, 6.12 and 6.13 for $y = -1$. For simplicity we only consider the more interesting case of normalized classes and operators. Note that

$$\mathrm{Td}_{y=-1,*}(X(w)^\circ) = c_*^T(1\!\!1_{X(w)^\circ}) =: c_{\mathrm{SM}}^T(X(w)^\circ) \in H_*^T(G/B, \mathbb{Z})$$

by (23), since $H_*^T(G/B,\mathbb{Z})$ is torsion free. Recall that $c_{\mathrm{SM}}^T(X(w)^\circ)$ is *Chern-Schwartz-MacPherson* (CSM) class of the Schubert cell $X(w)^\circ$; this class is discussed in more detail in the next section. Theorem 6.11(b) for the normalized Hirzebruch classes *implies* for $y = -1$ the following recursion from [AM16, Theorem 6.4] (formulated in Equation (42) in the next section in terms of homogenized classes):

$$\mathcal{T}_i^H(c_{\mathrm{SM}}^T(X(w)^\circ)) = (c^T(T_{p_i})\partial_i^H - \mathrm{id})(c_{\mathrm{SM}}^T(X(w)^\circ)) = c_{\mathrm{SM}}^T(X(ws_i)^\circ) \tag{37}$$

for $w \in W$ and s_i a simple reflection, with $ws_i > w$.

Similarly, Theorem 6.12 for the normalized Hirzebruch classes *implies* for $y = -1$ the corresponding **Hecke orthogonality** of [AMSS17, Theorem 7.2] (with their equivariant parameter $\hbar \in H^2_{\mathbb{C}^*}(\mathrm{pt},\mathbb{Z})$ specialized here to $\hbar = 1$):

$$\langle c_{\mathrm{SM}}^T(X(u)^\circ), (\mathfrak{L}^{\mathrm{Hir}}_{v^{-1}w_0})_{y=-1}([e_{w_0}]_T)\rangle = \delta_{u,v} c^T(T_{e_{w_0}}X) = \delta_{u,v}\prod_{\alpha>0}(1+\alpha) \tag{38}$$

for $u, v \in W$. Here

$$(\mathfrak{L}^{\mathrm{Hir}}_{v^{-1}w_0})_{y=-1}([e_{w_0}]_T) = \mathcal{T}^{H,\vee}_{v^{-1}w_0}([e_{w_0}]_T) = c_{\mathrm{SM}}^{T,\vee}(Y(v)^\circ)$$

is the *dual Chern-Schwartz-MacPherson* class from [AMSS17, Equation (14)]. Note that, in terms of the duality operators from Remark 6.3,

$$c_{\rm SM}^{T,\vee}(Y(v)^\circ) = (-1)^{\dim X-\ell(v)}\psi_*^{-1}(c_{\rm SM}^T(Y(v)^\circ)\,,$$

since the homogenized operators satisfy

$$\mathcal{T}_i^{H,\vee,\hbar} = \hbar\partial_i^H + s_i = -(-\hbar\partial_i^H - s_i) = -\mathcal{T}_i^{H,\hbar}|_{h\mapsto -h}. \tag{39}$$

Finally Theorem 6.13 for the normalized Hirzebruch classes implies for $y=-1$:

$$\frac{c_{\rm SM}^T(X(w)^\circ)}{c^T(TX)} = \mathcal{T}_{w^{-1}}^{H,\vee}\left(\frac{[e_{\rm id}]_T}{\prod_{\alpha>0}(1-\alpha)}\right) = \frac{c_{\rm SM}^{T,\vee}(X(w)^\circ)}{\prod_{\alpha>0}(1-\alpha)} \tag{40}$$

and

$$\frac{c_{\rm SM}^T(Y(w)^\circ)}{c^T(TX)} = \mathcal{T}_{(w_0w)^{-1}}^{H,\vee}\left(\frac{[e_{w_0}]_T}{\prod_{\alpha>0}(1+\alpha)}\right) = \frac{c_{\rm SM}^{T,\vee}(Y(w)^\circ)}{\prod_{\alpha>0}(1+\alpha)}\,.$$

This recovers [AMSS17, Theorem 7.5], which is one of the main results of that paper (again with the equivariant parameter $\hbar\in H^2_{\mathbb{C}^*}(\mathrm{pt},\mathbb{Z})$ specialized to $\hbar=1$).

6.4. **Parabolic Hirzebruch classes.** We now consider the (generalized) partial flag manifold G/P, and we let $\pi: G/B\to G/P$ be the natural projection. The Schubert varieties $X(wW_P)^\circ$ in G/P are indexed by the elements in $w\in W^P$, with the image $\pi(X(w)^\circ) = X(wW_P)^\circ$ for $w\in W$. Applying td_* and $\psi_*^{1+y}\,\mathrm{td}_*$ to the equalities from Proposition 4.17 implies by functoriality the following counterpart for the Hirzebruch classes.

Proposition 6.15. *The following hold for $w\in W$:*

(a) For P parabolic:

$$\pi_*\,\widetilde{\mathrm{Td}}_{y*}(X(w)^\circ) = (-y)^{\ell(w)-\ell(wW_P)}\,\widetilde{\mathrm{Td}}_{y*}(X(wW_P)^\circ)\in \widehat{H}_*^T(G/P)[y]$$

and

$$\pi_*\,\mathrm{Td}_{y*}(X(w)^\circ) = (-y)^{\ell(w)-\ell(wW_P)}\,\mathrm{Td}_{y*}(X(wW_P)^\circ)\in \widehat{H}_*^T(G/P)[y]\,.$$

(b) More generally, let $P\subseteq Q$ be two standard parabolic subgroups, and $\pi': G/P\to G/Q$ the natural projection. Then

$$\pi'_*\,\widetilde{\mathrm{Td}}_{y*}(X(wW_P)^\circ) = (-y)^{\ell(wW_P)-\ell(wW_Q)}\pi_*\,\widetilde{\mathrm{Td}}_{y*}(X(wW_Q)^\circ)\in \widehat{H}_*^T(G/Q)[y]$$

and

$$\pi'_*\,\mathrm{Td}_{y*}(X(wW_P)^\circ) = (-y)^{\ell(wW_P)-\ell(wW_Q)}\pi_*\,\mathrm{Td}_{y*}(X(wW_Q)^\circ)\in \widehat{H}_*^T(G/Q)[y].$$

Specializing to $y=0$ (with the convention $0^0=1$), we get by Theorem 5.1(b):

$$\pi_*\,\mathrm{Td}_*(\mathcal{I}_w^T) = \begin{cases}\mathrm{Td}_*(\mathcal{I}_{wW_P}^T) & \text{if } \ell(w)=\ell(wW_P);\\ 0 & \text{otherwise.}\end{cases}$$

Similarly, specializing the normalized classes to $y=-1$, we get for $w\in W$:

$$\pi_*c_{\rm SM}^T(X(w)^\circ) = c_{\rm SM}^T(X(wW_P)^\circ) \text{ and } \pi'_*c_{\rm SM}^T(X(wW_P)^\circ) = c_{\rm SM}^T(X(wW_Q)^\circ). \tag{41}$$

These equalities hold in $H_*^T(G/P,\mathbb{Z})$ and $H_*^T(G/Q,\mathbb{Z})$, since these are torsion free.

7. The Chern-Schwartz-MacPherson classes as leading terms

We have seen in the Corollary 6.6 that the Chern-Schwartz-MacPherson (CSM) classes may be recovered from the Hirzebruch transformation by specializing at $y = -1$. In this section we take a different route, and recover the CSM classes directly, by identifying them as the leading terms of the motivic Chern classes; cf. Theorem 7.1. We will illustrate this process for complete flag varieties, since these are the main object of study in this paper.

7.1. **Chern-Schwartz-MacPherson classes.** We first recall the context leading to the definition of CSM classes. According to a conjecture attributed to Deligne and Grothendieck, there is a unique natural transformation $c_* : \mathcal{F}(-) \to H_*(-, \mathbb{Z})$ from the functor of constructible functions on a complex algebraic variety X to homology (i.e., even degree Borel-Moore homology, or Chow groups), where all morphisms are proper, such that if X is smooth then $c_*(1\!\!1_X) = c(TX) \cap [X]$. This conjecture was proved by MacPherson [Mac74]; the class $c_*(1\!\!1_X)$ for possibly singular varieties X was shown to coincide via Alexander duality with a class defined earlier by M.-H. Schwartz [Sch65a, Sch65b, BS81]. For any constructible subset $W \subseteq X$, the class $c_{\mathrm{SM}}(W) := c_*(1\!\!1_W) \in H_*(X, \mathbb{Z})$ is called the *Chern-Schwartz-MacPherson* (CSM) class of W in X. If X is a T-variety, an equivariant version of the group of constructible functions $\mathcal{F}^T(X)$ and a Chern class transformation $c_*^T : \mathcal{F}^T(X) \to H_*(X; \mathbb{Z})$ were defined by Ohmoto [Ohm06]; see [AMSS17, §3.2] for a summary of Ohmoto's definition and a discussion of alternative (equivalent) definitions. This is the notion we consider in this paper.

7.2. **The homogenized CSM class via the motivic Chern class.** We now consider the case of flag varieties, so $X = G/B$. If

$$c_{\mathrm{SM}}^T(X(w)^\circ) = \sum_i c_{\mathrm{SM}}^T(X(w)^\circ)_i \in H_*^T(G/B, \mathbb{Z}),$$

where $c_{\mathrm{SM}}^T(X(w)^\circ)_i \in H_{2i}^T(G/B, \mathbb{Z})$, the homogenized CSM class is defined to be

$$c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) := \sum_i \hbar^i c_{\mathrm{SM}}^T(X(w)^\circ)_i \in H_0^{T\times\mathbb{C}^*}(G/B, \mathbb{Z}).$$

Here $\mathbb{C}^*$ acts trivially on G/B and $\hbar \in H^2_{\mathbb{C}^*}(\mathrm{pt}, \mathbb{Z})$ is a generator. Consider the Schubert expansion of the homogenized CSM class:

$$c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) = \sum_{u\le w} c'_{u,w}(\hbar, t)[X(u)]_T \in H_0^{T\times\mathbb{C}^*}(X),$$

where $c'_{u,w}(\hbar, t) \in H^*_{T\times\mathbb{C}^*}(\mathrm{pt}, \mathbb{Z})$ is a homogeneous polynomial of degree $\ell(u)$. As usual $t = (t_1, \dots, t_s)$ stands for a sequence of variables corresponding to a basis of the character group of T; see §2.1. We also recall that α_i denote the simple roots, regarded as elements of $H^2_T(\mathrm{pt})$. With $\mathcal{T}_i^{H,\hbar}$ as in (6), we have

$$\mathcal{T}_i^{H,\hbar}(c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ)) = c_{\mathrm{SM}}^{T,\hbar}(X(ws_i)^\circ)\,; \tag{42}$$

this is the homogenized version of (37) above. We will now verify that, combined with Proposition 3.5, (42) implies that the CSM class of the Schubert cell is the 'initial term' of the motivic Chern class $\mathrm{MC}_y(X(w)^\circ)$, where $y = -e^{-\hbar}$.

Theorem 7.1. *Let $w \in W$ and consider the Schubert expansions*

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u\le w} c_{u,w}(y, e^t)\mathcal{O}_u^T \in \mathrm{K}_T(G/B)[y]$$

and

$$c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) = \sum_{u\le w} c'_{u,w}(\hbar,t)[X(u)]_T \in H_0^{T\times\mathbb{C}^*}(G/B,\mathbb{Z}),$$

where $c'_{u,w}(\hbar,t) \in H^{2\ell(u)}_{T\times\mathbb{C}^*}(\mathrm{pt})$. *With* $y=-e^{-\hbar}$ *(and hence* $\mathrm{ch}_{\mathbb{C}^*}(y) = -e^{-\hbar}$ *cf. §2.3), the following hold:*

(a) The image $\mathrm{ch}_{T\times\mathbb{C}^*}(c_{u,w}(y,e^t))$ *of the coefficient* $c_{u,w}(y,e^t)$ *under the Chern character belongs to* $\prod_{i\ge\ell(u)} H^{2i}_{T\times\mathbb{C}^*}(\mathrm{pt})$.

(b) The coefficient $c'_{u,w}(\hbar,t)$ *equals the term of degree* $\ell(u)$ *in* $c_{u,w}(y,e^t)$*, i.e.,*

$$c'_{u,w}(\hbar,t) = (\mathrm{ch}_{T\times\mathbb{C}^*}(c_{u,w}(y,e^t)))_{\ell(u)}.$$

Equivalently,

$$c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ) = \textit{degree } 0 \textit{ component of } \mathrm{ch}(\mathrm{MC}_y(X(w)^\circ)).$$

Proof. Both parts follow by induction on $\ell(w)$, using the recursion calculating $\mathrm{MC}_y(X(w)^\circ)$ from Theorem 4.5 combined with Proposition 3.5; in part (b) we utilize the recursion for $c_{\mathrm{SM}}^{T,\hbar}(X(w)^\circ)$ from (42). □

Example 7.2. Consider the equivariant motivic Chern class in $\mathrm{K}_T(\mathbb{P}^1)[y]$:

$$\mathrm{MC}_y(X(s_1)^\circ) = (1+e^{-\alpha_1}y)\mathcal{O}^T_{s_1} - (1+(1+e^{-\alpha_1})y)\mathcal{O}^T_{\mathrm{id}}.$$

The specialization $y=-e^{-\hbar}$ in the coefficient $c_{s_1,s_1}(y,e^t)$ gives:

$$c_{s_1,s_1}(-e^{-\hbar},e^t) = 1-e^{-\alpha_1-\hbar} = \hbar+\alpha_1+\text{h.o.t.}$$

(higher order terms). The term of degree 1 is $c'_{s_1,s_1} = \hbar+\alpha_1$. Similarly, the specialization of $c_{\mathrm{id},s_1}(y,e^t)$ gives

$$c_{\mathrm{id},s_1}(y,e^t) = -(1-(1+e^{-\alpha_1})e^{-\hbar}) = -1+e^{-\hbar}+e^{-\hbar-\alpha_1} = -1+2+\text{h.o.t.}$$

By Theorem 7.1,

$$c_{\mathrm{SM}}^{T,\hbar}(X(s_1)^\circ) = (\hbar+\alpha_1)[X(s_1)]_T + [X(\mathrm{id})]_T,$$

consistent with a positivity property from Conjecture 1 below. ⌟

Consider now the non-equivariant case, i.e., in the expansions from Theorem 7.1 we set $\alpha=0$, so $e^\alpha\mapsto 1$. In this case we denote the coefficients in the two expansion by $c_{u,w}(y)$ respectively $c'_{u,w}(\hbar)$. Note that $c_{u,w}(y)\in\mathbb{Z}[y]$ and $c'_{u,w}(\hbar)\in\mathbb{Z}[\hbar]$. Furthermore, by homogeneity,

$$c'_{u,w}(\hbar) = \overline{c}_{u,w}\hbar^{\ell(u)},$$

where $\overline{c}_{u,w}\in\mathbb{Z}$ is an integer. Next we give a more direct relation between these coefficients, using that the polynomial $c_{u,w}(y)$ is divisible by $(1+y)^{\ell(u)}$ by Proposition 4.15.

Corollary 7.3. *The coefficient* $\overline{c}_{u,w}$ *equals the specialization at* $y=-1$ *of* $\frac{c_{u,w}(y)}{(1+y)^{\ell(u)}}$*:*

$$\overline{c}_{u,w} = \left(\frac{c_{u,w}(y)}{(1+y)^{\ell(u)}}\right)_{y\mapsto -1}.$$

Proof. Let $Q_{u,w}(y) := \sum a_i y^i$ in $\mathbb{Z}[y]$ be the quotient $\frac{c_{u,w}(y)}{(1+y)^{\ell(u)}}$. By Theorem 7.1, the coefficient $\overline{c}_{u,w}$ equals the term of degree 0 in the specialization $Q_{u,w}(-e^{-\hbar})$. Since $-e^{-\hbar} = -1+$ higher order terms, $\overline{c}_{u,w} = Q_{u,w}(-1)$ as claimed. □

Example 7.4. Consider the non-equivariant version of Example 7.2 :

$$\mathrm{MC}_y(X(s_1)^\circ) = (1+y)\mathcal{O}_{s_1} - (1+2y)\mathcal{O}_{\mathrm{id}}.$$

According to Corollary 7.3, we need to divide each coefficient $c_{u,s_1}(y)$ by $(1+y)^{\ell(u)}$ and then specialize at $y=-1$. We obtain:

$$c^{\hbar}_{\mathrm{SM}}(X(s_1)^\circ) = \hbar[X(s_1)] + [X(\mathrm{id})].$$

The non-homogenized class is obtained by setting $\hbar$ to 1. ⌟

Example 7.5. Consider the motivic Chern class $\mathrm{MC}_y(X(s_1s_2)^\circ) \in \mathrm{K}(\mathrm{Fl}(3))[y]$:

$$\mathrm{MC}_y(X(s_1s_2)^\circ) = (1+y)^2\mathcal{O}_{s_1s_2} - (1+y)(1+2y)\mathcal{O}_{s_1} - (1+y)(1+3y)\mathcal{O}_{s_2} + (5y^2+5y+1)\mathcal{O}_{\mathrm{id}}.$$

As before, we need to divide each coefficient $c_{u,s_1s_2}(y)$ by $(1+y)^{\ell(u)}$ and then specialize at $y=-1$. We obtain:

$$c_{\mathrm{SM}}(X(s_1s_2)^\circ) = [X(s_1s_2)] + [X(s_1)] + 2[X(s_2)] + [X(\mathrm{id})].$$ ⌟

8. Positivity, unimodality, and log-concavity conjectures

In this section we record several conjectures involving Schubert expansions of the motivic Chern and CSM classes and about the structure constants of the CSM classes. Some of these conjectures have been made by other authors; our goal is to collect all these statements in a single place.

We start with the CSM classes, since this is the case when we have the most (partial) results.

8.1. **Positivity of Schubert expansions of CSM classes.**

Consider the (non-equivariant) CSM class of a Schubert cells in a generalized flag manifolds G/P:

$$c_{\mathrm{SM}}(X(wW_P)^\circ) = \sum_{vW_P \le wW_P} c_{v,w}[X(vW_P)],$$

with $c_{v,w} \in \mathbb{Z}$. For $G/P = \mathrm{Gr}(k;n)$, it was conjectured in [AM09] that the coefficients $c_{v;w}$ are nonnegative; this was proved in some special cases in *loc.cit.* and in [Mih15, Jon10, Str11], and in full generality (for Grassmannians) by J. Huh [Huh16]. The recursive algorithm from [AM16] yielded calculations of CSM classes of Schubert cells in any G/P, and provided supporting evidence that the CSM classes of Schubert cells in all flag manifolds are effective. This conjecture was proved in [AMSS17, Corollary 1.4].

Equivariantly, the numerical evidence supports the following conjecture.

Conjecture 1 (Equivariant Positivity). *Let $X(wW_P)^\circ \subseteq G/P$ be any Schubert cell and consider the Schubert expansion of the equivariant CSM class:*

$$c^T_{\mathrm{SM}}(X(wW_P)^\circ) = \sum_{v\le w} c_{v,w}(\alpha)[X(vW_P)]_T \quad \in H^T_*(G/P).$$

Then $c_{v;w}(\alpha)$ is a polynomial in positive roots α with non-negative coefficients.

In the non-equivariant case, while we have proved that $c_{v,w} \ge 0$ in [AMSS17], the evidence suggests a stronger result.

Conjecture 2 (Strong positivity). *Let $X(wW_P)^\circ \subseteq G/P$ be any Schubert cell and consider the Schubert expansion:*

$$c_{\mathrm{SM}}(X(wW_P)^\circ) = \sum_{v\le w} c_{v,w}[X(vW_P)] \quad \in H_*(G/P;\mathbb{Z}).$$

Then $c_{v,w} > 0$ *for all* $v \leq w$.

Huh's result for Grassmannians in [Huh16] shows that each homogeneous component $c_{\mathrm{SM}}(X(wW_P)^\circ)_k$ of the CSM class is represented by a non-empty irreducible variety. This is slightly weaker than the requirement in Conjecture 2. On the other hand, if this variety may be chosen to be T-stable, then Huh's result and the positivity results of Graham [Gra01] would imply Conjecture 1 for Grassmannians.

For any parabolic subgroup P, let $\pi : G/B \to G/P$ be the natural projection. Since $\pi_*(c_{\mathrm{SM}}(X(w)^\circ)) = c_{\mathrm{SM}}(X(wW_P)^\circ)$ (see e.g., (41)) it follows that if Conjecture 1 or Conjecture 2 holds for cells in G/B, then it also holds in G/P.

8.2. **Positivity of CSM/SM structure constants.** The CSM classes $c_{\mathrm{SM}}(X(w)^\circ)$ may be viewed as deformations of the fundamental classes $[X(w)]$; we consider analogues of the Littlewood-Richardson coefficients in the context of these classes. Since $c_{\mathrm{SM}}(X(w)^\circ)$ is naturally a homology class, we focus on structure constants for their Poincaré duals, the Segre-MacPherson (SM) classes. **In this section we only consider the non-equivariant context.** In general, if Z is a subvariety of a nonsingular variety X, we set

$$s_{\mathrm{SM}}(Z,X) = \frac{c_{\mathrm{SM}}(Z)}{c(T(X))}$$

in the homology or Chow group of Z; see e.g., [AMSS22]. We will implicitly push-forward this class to $H_*(X)$.

Since $c(T(G/B)) \cdot c(T^*(G/B)) = 1$, as proved in [AMSS17, Lemma 8.2], in G/B we have

$$s_{\mathrm{SM}}(Y(w)^\circ, G/B) = \frac{c_{\mathrm{SM}}(Y(w)^\circ)}{c(T(G/B))} = c(T^*(G/B)) \cap c_{\mathrm{SM}}(Y(w)^\circ)\,.$$

We also proved that

$$s_{\mathrm{SM}}(X(ws_i)^\circ, G/B) = \mathcal{T}_i^{H,\vee} s_{\mathrm{SM}}(X(w)^\circ, G/B)$$

(cf. (40) above for the equivariant version of this equality). Poincaré duality states that for any parabolic $P \supset B$,

$$\langle s_{\mathrm{SM}}(Y(vW_P)^\circ, G/P), c_{\mathrm{SM}}(X(wW_P)^\circ)\rangle = \delta_{v,w}, \tag{43}$$

cf. [AMSS17, Theorem 7.1]. This can be proved using a transversality formula due to Schürmann [Sch17], extended equivariantly in [AMSS17, Corollary 10.3].

As a consequence of (40) (cf. [AMSS17, Eq. (36)]), the Schubert expansions of $c_{\mathrm{SM}}(X(w)^\circ)$ and $s_{\mathrm{SM}}(X(w)^\circ, G/B)$ in G/B are related by changing signs. More precisely, if

$$s_{\mathrm{SM}}(X(w)^\circ, G/B) = \sum f_{v;w}[X(v)] \quad \in H_*(G/B),$$

then with notation as above $f_{v;w} = (-1)^{\ell(w)-\ell(v)} c_{v;w}$. This follows because the homogenized operator $\mathcal{T}_i^{H,\hbar} = \hbar\partial_i - s_i$, giving CSM classes, and its adjoint $\mathcal{T}_i^{H,\vee,\hbar} = \hbar\partial_i + s_i$, giving SM classes, differ by a sign; see also (39) for the more general equivariant statement.

Consider now the structure constants

$$s_{\mathrm{SM}}(Y(u)^\circ, G/B) \cdot s_{\mathrm{SM}}(Y(v)^\circ, G/B) = \sum e_{u,v}^w s_{\mathrm{SM}}(Y(w)^\circ, G/B). \tag{44}$$

Schürmann's transversality theorem [Sch17] shows that

$$e_{u,v}^w = \chi(g_1 Y(u)^\circ \cap g_2 Y(v)^\circ \cap g_3 X(w)^\circ),$$

the topological Euler characteristic of the intersection of three Schubert cells translated in general position via $g_1, g_2, g_3 \in G$. This interpretation of the structure constants holds for any G/P, although the relation between the Schubert expansions of CSM and SM classes does not extend beyond G/B. (Still, the SM classes are known to be Schubert alternating; see [AMSS22].)

Due to its statement involving only 'classical' objects, perhaps the most remarkable positivity conjecture in this paper is the next.

Conjecture 3 (Alternation of Euler characteristic)**.** *The Euler characteristic of the intersection of three Schubert cells in general position in G/P is alternating, i.e., for any $u, v, w \in W^P$,*

$$(-1)^{\ell(u)+\ell(v)+\ell(w)}\chi(g_1 Y(uW_P)^\circ \cap g_2 Y(vW_P)^\circ \cap g_3 X(wW_P)^\circ) \geq 0.$$

Utilizing deep connections between SM classes to the theory of integrable systems, this was proved by Knutson and Zinn-Justin for d-step flag manifolds with $d \leq 3$, and it was conjectured to hold for $d = 4$; cf. [KZJ21, p. 43]. Independently, and based on multiplications of SM classes from [AMSS17], the authors of this paper stated this conjecture in several conference and seminar talks, for partial flag manifolds G/P in arbitrary Lie type[4].

S. Kumar [Kum22] conjectured that the CSM class of the Richardson cells are Schubert positive, that is, if

$$c_{\text{SM}}(Y(u)^\circ \cap X(v)^\circ) = \sum_w f_{u,v}^w [Y(w)],$$

then $f_{u,v}^w \geq 0$. (We also learned about this conjecture independently from Rui Xiong, and it is now stated in [FGX22, Conjecture 9.2].) It is shown in [Kum22] that this implies Conjecture 3. Note that the *Segre* class of the Richardson cell

$$s_{\text{SM}}(R_v^{u,\circ}, G/B) = s_{\text{SM}}(Y(u)^\circ \cap X(v)^\circ, G/B)$$

is Schubert alternating by [AMSS22, Theorem 1.1], since the inclusion of $Y(u)^\circ \cap X(v)^\circ$ is an affine morphism.

For G/B, the absolute value of the structure constants $e_{u,v}^w$ from (44) give the structure constants to multiply CSM classes of Schubert cells. This generalizes the positivity in ordinary Schubert Calculus: if $\ell(uW_P) + \ell(vW_P) = \ell(wW_P)$, then the intersection in question is 0 dimensional and reduced, and the Euler characteristic counts the number of points in the intersection. A different algorithm to calculate the SM structure constants is given in [Su21].

We end this section by proving a property of the sum of the coefficients $e_{u,v}^w$.

Proposition 8.1. *Consider the multiplication*

$$s_{\text{SM}}(Y(uW_P)^\circ, G/P) \cdot s_{\text{SM}}(Y(vW_P)^\circ, G/P) = \sum_w e_{u,v}^w s_{\text{SM}}(Y(wW_P)^\circ, G/P).$$

Then $\sum_w e_{u,v}^w = \delta_{w_0 uW_P, vW_P}$.

Proof. By the transversality formula from [Sch17], and omitting the ambient G/P for short,

$$\begin{aligned} s_{\text{SM}}(Y(uW_P)^\circ) \cdot s_{\text{SM}}(Y(vW_P)^\circ) &= s_{\text{SM}}(X(w_0 uW_P)^\circ) \cdot s_{\text{SM}}(Y(vW_P)^\circ) \\ &= s_{\text{SM}}(X(w_0 uW_P)^\circ \cap Y(vW_P)^\circ). \end{aligned}$$

[4]After this paper was submitted, a proof of the conjecture was given in [SSW23].

By Poincaré duality (43) and the equality $c(T(G/P)) = \sum_w c_{\text{SM}}(X(wW_P)^\circ)$, the sum of the coefficients $e_{u,v}^w$ equals

$$\int_{G/P} s_{\text{SM}}(Y(uW_P)^\circ) \cdot s_{\text{SM}}(Y(vW_P)^\circ) \cdot c(T(G/P))$$
$$= \int_{G/P} s_{\text{SM}}(X(w_0uW_P)^\circ \cap Y(vW_P)^\circ) \cdot c(T(G/P))$$
$$= \int_{G/P} c_{\text{SM}}(X(w_0uW_P)^\circ \cap Y(vW_P)^\circ) = \delta_{w_0uW_P, vW_P}.$$

Here the last equality follows because the Richardson cell $X(w_0uW_P)^\circ \cap Y(vW_P)^\circ$ is torus-stable, therefore its Euler characteristic equals the Euler characteristic of the fixed locus; see [BB73, Corollary 2], applied for a general $\mathbb{C}^* \subseteq T$. In this case the fixed locus is empty or one point, giving $\delta_{w_0uW_P, vW_P}$. □

Example 8.2. Take G is Lie type G_2 and write $s_{\text{SM}}(-)$ for $s_{\text{SM}}(-, G/B)$. Then

$$\begin{aligned} s_{\text{SM}}(Y(\text{id})^\circ) \cdot s_{\text{SM}}(Y(\text{id})^\circ) &= s_{\text{SM}}(Y(\text{id})^\circ) - s_{\text{SM}}(Y(s_1)^\circ) - s_{\text{SM}}(Y(s_2)^\circ) \\ &+ 2s_{\text{SM}}(Y(s_2s_1)^\circ) + 4s_{\text{SM}}(Y(s_1s_2)^\circ) \\ &- 9s_{\text{SM}}(Y(s_1s_2s_1)^\circ) - 11s_{\text{SM}}(Y(s_2s_1s_2)^\circ) \\ &+ 22s_{\text{SM}}(Y(s_2s_1s_2s_1)^\circ) + 34s_{\text{SM}}(Y(s_1s_2s_1s_2)^\circ) \\ &- 57s_{\text{SM}}(Y(s_1s_2s_1s_2s_1)^\circ) - 51s_{\text{SM}}(Y(s_2s_1s_2s_1s_2)^\circ) \\ &+ 67s_{\text{SM}}(Y(s_2s_1s_2s_1s_2s_1)^\circ). \end{aligned}$$

Observe that these structure constants are alternating, and add up to 0, confirming Conjecture 3 and Proposition 8.1 in this case. ⌟

8.3. **Unimodality and log-concavity for CSM polynomials.** Following [Sta89], a sequence $a_0, \ldots, a_n$ is **unimodal** if there exists i_0 such that

$$a_0 \leq a_1 \leq \ldots \leq a_{i_0} \geq a_{i_0+1} \geq \ldots \geq a_n.$$

The sequence is **log-concave** if for any $1 \leq i \leq n-1$,

$$a_i^2 \geq a_{i-1}a_{i+1}.$$

A log-concave sequence of nonnegative integers with no internal zeros is unimodal. A polynomial $P(x) = \sum a_i x^i$ is unimodal, resp., log-concave, if its sequence of coefficients satisfies the corresponding property.

Consider any class $\kappa = \sum c_w[X(wW_P)]$ in $H_*(G/P)$. We define the H**-polynomial** associated to κ by

$$H(\kappa) := \sum c_w x^{\ell(w)} = \sum c_i x^i,$$

determining the coefficients c_i. For $w \in W^P$ we denote by $H_w(x) := H(c_{\text{SM}}(X(wW_P)))$ the H-polynomial of the CSM class of the corresponding Schubert *variety*.

Conjecture 4 (Unimodality and log-concavity)**.** *Let $X(w) \subseteq G/P$ be any Schubert variety. Then the following hold:*

(a) The polynomial H_w is unimodal with no internal zeros.

(b) If G is of Lie type A (i.e., $G/P = \mathrm{Fl}(i_1, \ldots, i_k; n)$ is a partial flag manifold) then H_w is log-concave.

A similar conjecture for Mather classes can be found in [MS20].

Example 8.3. Consider the Grassmannian $\mathrm{Gr}(3,6)$ and the Schubert variety $X_{(2,1)}$ of dimension 3. Then

$$c_{\mathrm{SM}}(X_{(2,1)}) = [X_{(2,1)}] + 3[X_2] + 3[X_{1,1}] + 8[X_1] + 5[X_0].$$

Its H-polynomial is

$$x^3 + 6x^2 + 8x + 5,$$

which is log-concave.

For the 5-dimensional quadric Q^5, the H-polynomial of $c(TQ^5)$ is

$$x^5 + 5x^4 + 11x^3 + 26x^2 + 18x + 6$$

which is unimodal, but not log-concave. ⌟

8.4. **Conjectures about the motivic Chern classes.** Given the (proved and conjectural) positivity properties of the CSM classes of Schubert cells, it is natural to conjecture analogous properties for the motivic Chern classes of Schubert cells. The following conjecture was stated by Fehér, Rimányi, and Weber [FRW17] in type A, and in [AMSS19, Conjecture 1] for arbitrary Lie type.

Conjecture 5 (Positivity of MC classes)**.** *Consider the Schubert expansion:*

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u \le w} c_{u,w}(y, e^t)\mathcal{O}_u^T \in K_T(G/B)[y].$$

Then for any $u \le w \in W$, we have

$$(-1)^{\ell(w)-\ell(u)} c_{w,u}(y, e^t) \in \mathbb{Z}_{\ge 0}[y][e^{-\alpha_1}, \dots, e^{-\alpha_r}] \subseteq \mathrm{K}_T(\mathrm{pt})[y],$$

i.e., the coefficients $(-1)^{\ell(w)-\ell(u)} c_{u,w}(y, e^t)$ are polynomials in the variables y and characters $e^{-\alpha_1}, \dots, e^{-\alpha_r}$ in simple roots with non-negative coefficients.

The conjecture implies that the coefficients of the non-equivariant motivic Chern classes of Schubert cells are sign-alternating: $(-1)^{\ell(w)-\ell(u)} c_{u,w}(y) \in \mathbb{Z}_{\ge 0}[y]$. Conjecture 5 holds after specializing $y = 0$, as a consequence of Theorem 5.1(b) and the fact that the ideal sheaves are alternating in Schubert classes; see (2) above, and e.g., [Bri05, Proposition 4.3.2]. More evidence for Conjecture 5 is available, since some particular coefficients $c_{u,w}(y, e^t)$ are known to be positive. For instance, the coefficient

$$c_{w,w}(y, e^t) = \prod_{\alpha > 0, w(\alpha) < 0} (1 + y e^{w\alpha})$$

calculated in Lemma 4.11 is positive. The specialization at $y = -1$ gives the equivariant class ι_w of the structure sheaf of the fixed point of w as in Theorem 5.1 above (see Proposition 5.1). This is consistent with the conjecture; the class ι_w is known to be Schubert alternating, by e.g., the positivity in equivariant K-theory proved by Anderson, Griffeth and Miller [AGM11]. Finally, we verified Conjecture 5 for flag manifolds of type A_n for $n \le 5$ (i.e., up to $\mathrm{Fl}(5)$), and for the Lie types B_2, C_2, D_3 and G_2, by means of a computer calculation.

Fehér, Rimányi, and Weber [FRW17] also observed a conjectural log-concavity property for motivic Chern classes in Lie type A. We confirmed their observations in many examples, and also checked additional examples in other Lie types.

Conjecture 6 (Log concavity for MC classes)**.** *Let $X(w)^\circ \subseteq G/B$ and consider the Schubert expansion of the non-equivariant motivic Chern class:*

$$\mathrm{MC}(X(w)^\circ) = \sum_{v \le w} c_{v;w}(y)\mathcal{O}_v.$$

Then $c_{v;w}(y)$ is log-concave.

Example 8.4. Consider G of Lie type G_2. The coefficient of $\mathcal{O}_{\mathrm{id}}$ in the expansion of $\mathrm{MC}_y(X(w_0)^\circ)$ is

$$64y^6+141y^5+125y^4+69y^3+29y^2+8y+1.$$

This is a log-concave polynomial. Its specialization at $y=-1$ gives 1, reflecting the fact that it calculates the Euler characteristic of the big cell in G_2/B. ⌟

Remark 8.5. To simplify the notation, we only stated Conjecture 6 in the non-equivariant case. As pointed out in [FRW17], the conjecture has a natural extension to the equivariant setting. ⌟

8.5. **An interpretation in the Hecke algebra.** We can use Hecke algebra to give a combinatorial interpretation of the coefficients $c_{u,w}(y)$ in Conjecture 5. For the cohomology case, see [Lee18, Theorem 6.2].

Recall the K-theoretic BGG operator ∂_i satisfies $\partial_i^2=\partial_i$ and the braid relations. The operators $\mathcal{T}_i$ (and $\mathcal{T}_i^\vee$) satisfy the finite Hecke algebra relation, see Proposition 3.2. Besides, we also have:

Lemma 8.6. *For any simple root α and torus weight λ,*

$$\partial_{s_\alpha}\mathcal{L}_\lambda-\mathcal{L}_{s_\alpha\lambda}\partial_{s_\alpha}=\frac{\mathcal{L}_\lambda-\mathcal{L}_{s_\alpha\lambda}}{1-\mathcal{L}_\alpha}\in \mathrm{End}_{K_T(\mathrm{pt})[y]}\mathrm{K}_T(G/B)[y],$$

$$\mathcal{T}_{s_\alpha}\mathcal{L}_\lambda-\mathcal{L}_{s_\alpha\lambda}\mathcal{T}_{s_\alpha}=(1+y)\frac{\mathcal{L}_{s_\alpha\lambda}-\mathcal{L}_\lambda}{1-\mathcal{L}_{-\alpha}}\in \mathrm{End}_{K_T(\mathrm{pt})[y]}\mathrm{K}_T(G/B)[y]$$

and

$$\mathcal{T}^\vee_{s_\alpha}\mathcal{L}_\lambda-\mathcal{L}_{s_\alpha\lambda}\mathcal{T}^\vee_{s_\alpha}=(1+y)\frac{\mathcal{L}_{s_\alpha\lambda}-\mathcal{L}_\lambda}{1-\mathcal{L}_{-\alpha}}\in \mathrm{End}_{\mathrm{K}_T(\mathrm{pt})[y]}\mathrm{K}_T(G/B)[y].$$

Here $\frac{\mathcal{L}_\lambda-\mathcal{L}_{s_\alpha\lambda}}{1-\mathcal{L}_\alpha}$ is defined as follows. Suppose $\frac{e^\lambda-e^{s_\alpha\lambda}}{1-e^\alpha}=\sum_\mu e^\mu$, then $\frac{\mathcal{L}_\lambda-\mathcal{L}_{s_\alpha\lambda}}{1-\mathcal{L}_\alpha}:=\sum_\mu\mathcal{L}_\mu$.

Proof. We can check the equalities on the fixed point basis. Then all of them follow from Lemma 3.3 and the equality $\mathcal{L}_\lambda\otimes\iota_w=e^{w\lambda}\iota_w$. □

Let us recall the definition of the K-theoretic Kostant-Kumar Hecke algebra [KK90] and the affine Hecke algebra. Let P denote the weight lattice of G.

Definition 8.7. (1) *The Kostant-Kumar Hecke algebra $\mathcal{H}$ is a free $\mathbb{Z}$ module with basis $\{D_we^\lambda|w\in W,\lambda\in P\}$, such that*
- *For any $\lambda,\mu\in P$, $e^\lambda e^\mu=e^{\lambda+\mu}$.*
- *For any simple root α, $D_{s_\alpha}^2=D_{s_\alpha}$.*
- *For any $w,y\in W$ such that $\ell(wy)=\ell(w)+\ell(y)$, $D_wD_y=D_{wy}$*
- *For any simple root α and $\lambda\in P$,*

$$e^\lambda D_i-D_ie^{s_i\lambda}=\frac{e^\lambda-e^{s_i\lambda}}{1-e^{\alpha_i}}.$$

(2) *The affine Hecke algebra $\mathbb{H}$ is a free $\mathbb{Z}[q,q^{-1}]$ module with basis $\{T_we^\lambda|w\in W,\lambda\in P\}$, such that*
- *For any $\lambda,\mu\in P$, $e^\lambda e^\mu=e^{\lambda+\mu}$.*
- *For any simple root α, $(T_{s_\alpha}+1)(T_{s_\alpha}-q)=0$.*
- *For any $w,y\in W$ such that $\ell(wy)=\ell(w)+\ell(y)$, $T_wT_y=T_{wy}$*
- *For any simple root α and $\lambda\in P$,*

$$T_\alpha e^{s_\alpha\lambda}-e^\lambda T_\alpha=(1-q)\frac{e^\lambda-e^{s_\alpha\lambda}}{1-e^{-\alpha}}.$$

⌟

It follows from Lemma 8.6 that the Kostant-Kumar Hecke algebra $\mathcal{H}$ acts on $\mathrm{K}_T(G/B)$ by sending D_i to ∂_i and e^λ to $\mathcal{L}_\lambda$, see [KK90]; the affine Hecke algebra $\mathbb{H}$ acts on $\mathrm{K}_T(G/B)[y,y^{-1}]$ by sending q to $-y$, T_i to $\mathcal{T}_i$ (or $\mathcal{T}_i^\vee$), and e^λ to $\mathcal{L}_\lambda$, see [Lus85]. In the rest of the section, we always identify q with $-y$.

For any simple root α_i, let

$$T_i := (1+ye^{\alpha_i})D_i - 1 = D_i(1+ye^{-\alpha_i}) - (1+y+ye^{-\alpha_i}) \in \mathcal{H}[y].$$

Then these T_i and e^λ satisfies the relations in the affine Hecke algebra $\mathbb{H}$. Therefore, T_w is well-defined for all $w \in W$.

For any w, we can expand $T_{w^{-1}}$ as a linear combination of terms $D_{u^{-1}}$,

$$T_{w^{-1}} := \sum_{u\leq w} D_{u^{-1}} a_{u,w}(y;e^t), \tag{45}$$

for some $a_{u,w}(y,e^t) \in \mathbb{C}[T][y]$. It is easy to compute $a_{w,w}(y;e^t) = \prod_{\alpha>0, w\alpha<0}(1+ye^{w\alpha})$, which equals the coefficient $c_{w,w}(y;e^t)$ by Lemma 4.11. In fact, we have the following more general relation.

Proposition 8.8. *For any $u \leq w \in W$, we have*

$$a_{u,w}(y;e^t) = c_{u,w}(y;e^t).$$

Proof. Under the action of the Kostant-Kumar Hecke algebra $\mathcal{H}$ on $\mathrm{K}_T(G/B)$, T_i is sent to the DL operator $\mathcal{T}_i$ from (7). By Theorem 4.5, Equation (3) and the equality $\mathcal{L}_\lambda \otimes \mathcal{O}^T_{\mathrm{id}} = e^\lambda \mathcal{O}^T_{\mathrm{id}}$, applying (45) to $\iota_{\mathrm{id}} \in \mathrm{K}_T(G/B)$ we get

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u\leq w} a_{u,w}(y,e^t)\mathcal{O}^T_u.$$

Therefore, $a_{u,w}(y;e^t) = c_{u,w}(y;e^t)$. ∎

Proposition 8.8 provides a purely combinatorial way to compute the coefficients $c_{u,w}(y;e^t)$. In particular, we can check Conjecture 5 in the case when $\ell(w) \leq 2$ as follows:

(1) $\ell(w) = 1$. From $T_i = D_i(1+ye^{-\alpha_i}) - (1+y+ye^{-\alpha_i})$, we get $c_{\mathrm{id},s_i} = -(1+y+ye^{-\alpha_i})$ and $c_{s_i,s_i} = 1+ye^{-\alpha_i}$. (This is consistent with Example 4.8.)

(2) $\ell(w) = 2$. Pick two simple roots α_i, α_j. We calculate

$$\begin{aligned} T_iT_j &= \left(D_i(1+ye^{-\alpha_i}) - (1+y+ye^{-\alpha_i})\right)\left(D_j(1+ye^{-\alpha_j}) - (1+y+ye^{-\alpha_j})\right) \\ &= D_iD_j(1+ye^{-\alpha_j})(1+ye^{-s_j\alpha_i}) - D_j(1+ye^{-\alpha_j})(1+y+ye^{-s_j\alpha_i}) \\ &- D_i\left((1+ye^{-\alpha_i})(1+y+ye^{-\alpha_j}) - y(1+ye^{-\alpha_j})\frac{e^{-\alpha_i}-e^{-s_j\alpha_i}}{1-e^{\alpha_j}}\right) \\ &+ (1+y+ye^{-\alpha_j})(1+y+ye^{-\alpha_i}) - y(1+ye^{-\alpha_j})\frac{e^{-\alpha_i}-e^{-s_j\alpha_i}}{1-e^{\alpha_j}}, \end{aligned}$$

where

$$\frac{e^{-\alpha_i}-e^{-s_j\alpha_i}}{1-e^{\alpha_j}} = -e^{-s_j\alpha_i} - e^{-s_j\alpha_i+\alpha_j} - \cdots - e^{-\alpha_i-\alpha_j}.$$

This verifies Conjecture 5 when $\ell(w) = 2$.

To illustrate this, consider $G = SL(3,\mathbb{C})$, $i = 2$, and $j = 1$. Then

$$\begin{aligned}
T_2T_1 &= D_2D_1(1+ye^{-\alpha_1})(1+ye^{-s_1\alpha_2}) - D_1(1+ye^{-\alpha_1})(1+y+ye^{-s_1\alpha_2})\\
&- D_2\left((1+ye^{-\alpha_2})(1+y+ye^{-\alpha_1}) - y(1+ye^{-\alpha_1})\frac{e^{-\alpha_2}-e^{-s_1\alpha_2}}{1-e^{\alpha_1}}\right)\\
&+ (1+y+ye^{-\alpha_1})(1+y+ye^{-\alpha_2}) - y(1+ye^{-\alpha_1})\frac{e^{-\alpha_2}-e^{-s_1\alpha_2}}{1-e^{\alpha_1}}\\
&= D_2D_1(1+ye^{-\alpha_1})(1+ye^{-\alpha_2-\alpha_1}) - D_1(1+ye^{-\alpha_1})(1+y+ye^{-\alpha_2-\alpha_1})\\
&- D_2\left((1+ye^{-\alpha_2})(1+y+ye^{-\alpha_1}) + y(1+ye^{-\alpha_1})e^{-\alpha_1-\alpha_2}\right)\\
&+ (1+y+ye^{-\alpha_1})(1+y+ye^{-\alpha_2}) + y(1+ye^{-\alpha_1})e^{-\alpha_1-\alpha_2}.
\end{aligned}$$

Under the above 'dictionary' between the Hecke algebra elements and operators, this recovers Example 4.9.

9. Star duality

Recall that by 'star duality' we mean the involution $\star : \mathrm{K}_T(X) \to \mathrm{K}_T(X)$ which sends (the class of) a vector bundle $[E]$ to $[E^\vee] = [\mathrm{Hom}_{\mathcal{O}_X}(E,\mathcal{O}_X)]$. This is not an involution of $\mathrm{K}_T(\mathrm{pt})$-algebras, but it satisfies $\star(\mathbb{C}_\lambda) = \mathbb{C}_{-\lambda}$, where $\mathbb{C}_\lambda$ denotes the trivial line bundle with weight λ. We extend $\star$ to $\mathrm{K}_T(X)[y,y^{-1}]$ by linearity, requiring $\star(y^i) = y^i$ for $i \in \mathbb{Z}$.

The goal of this section is to study the effect of this duality on the motivic Chern classes for Schubert cells in $X = G/B$. We are motivated by a result of Brion [Bri05, Proposition 4.3.4] who proved that in $\mathrm{K}(G/B)$,

$$\mathcal{L}_{-\rho}\otimes\mathcal{I}_w = (-1)^{\operatorname{codim} X(w)}\star(\mathcal{O}_w).$$

where (recall) ρ denotes the half sum of the positive roots. (See also Proposition 9.2 below.) We will upgrade this to the (equivariant) motivic Chern classes $\mathrm{MC}_y(X(w)^\circ)$ and the (equivariant, normalized) classes $\widetilde{\mathrm{MC}}_y(X(w)^\circ)$. The reason behind this choice is that (non-equivariantly) the specialization $y = 0$ gives the ideal sheaves $\mathcal{I}_w$, respectively the structure sheaves $\mathcal{O}_w$; cf. Theorem 5.1 and Theorem 4.6. (By Theorem 4.6, the opposite classes $\widetilde{\mathrm{MC}}_y(Y(w)^\circ)$ are orthogonal to the motivic classes.)

Recall the DL operators $\mathcal{T}_i$ and $\mathfrak{L}_i$ from (7) respectively (9). These determine recursively the motivic Chern classes $\mathrm{MC}_y(X(w)^\circ) = \mathcal{T}_{w^{-1}}(\mathcal{O}^T_{\mathrm{id}})$ (Theorem 4.5) and the (normalized) classes $\widetilde{\mathrm{MC}}_y(X(w)^\circ) = \mathfrak{L}_{w^{-1}}(\mathcal{O}^T_{\mathrm{id}})$.

We state next the main result in this section.

Theorem 9.1. *Let $w \in W$. Then the following hold:*

(a) $\mathbb{C}_{-\rho}\otimes\mathcal{L}_{-\rho}\otimes\mathrm{MC}_y(X(w)^\circ) = (-1)^{\operatorname{codim} X(w)}\star(\widetilde{\mathrm{MC}}_y(X(w)^\circ))$.

(b) Consider the Schubert expansions

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u\le w} c_{u,w}(y;e^t)\mathcal{O}^T_u;\quad \widetilde{\mathrm{MC}}_y(X(w)^\circ) = \sum_{u\le w} d_{u,w}(y;e^t)\mathcal{I}^T_u.$$

Then $c_{u,w}(y;e^t) = (-1)^{\ell(u)-\ell(w)}\star(d_{u,w}(y;e^t))$, *or, equivalently,*

$$\langle \mathrm{MC}_y(X(w)^\circ), \mathcal{I}^u\rangle = (-1)^{\ell(w)-\ell(u)}\star\langle \widetilde{\mathrm{MC}}_y(X(w)^\circ), \mathcal{O}^{u,T}\rangle.$$

(c) Consider the Schubert expansions

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u\le w} a_{u,w}(y;e^t)\mathcal{I}^T_u;\quad \widetilde{\mathrm{MC}}_y(X(w)^\circ) = \sum_{u\le w} b_{u,w}(y;e^t)\mathcal{O}^T_u.$$

Then $a_{u,w}(y;e^t) = (-1)^{\ell(u)-\ell(w)} b_{u,w}(y;e^t)$*, or, equivalently,*

$$\langle \mathrm{MC}_y(X(w)^\circ), \mathcal{O}^{u,T}\rangle = (-1)^{\ell(w)-\ell(u)} \langle \widetilde{\mathrm{MC}}_y(X(w)^\circ), \mathcal{I}^{u,T}\rangle.$$

Before we give the proof of this theorem, we recall that the $y=0$ specialization recovers a known relation between the ideal sheaves and structure sheaves; see Proposition 9.2 below and compare to [Bri05, Proposition 4.3.4]. Brion proves the result in the non-equivariant case, and for completeness we sketch a proof for the equivariant generalization. Aside from the intrinsic interest, we also note that we use this result in the proof of Theorem 9.1.

Proposition 9.2 (Brion)**.** *Let* $w \in W$*. Then the following holds in* $\mathrm{K}_T(X)$*:*

$$\mathbb{C}_{-\rho} \otimes \mathcal{L}_{-\rho} \otimes \mathcal{I}^T_w = (-1)^{\operatorname{codim} X(w)} \star (\mathcal{O}^T_w).$$

Proof. Following Brion's proof, as equivariant sheaves,

$$\star(\mathcal{O}^T_w) = (-1)^{\operatorname{codim} X(w)} \omega_{X(w)} \cdot \omega_X^{-1};$$

this uses the fact that Schubert varieties are irreducible and Cohen-Macaulay - see [Bri05, §3.3]. The difference in the equivariant case is that the canonical sheaf of $X(w)$ needs to be twisted by a trivial bundle:

$$\omega_{X(w)} = \mathcal{O}_{X(w)}(-\partial X(w)) \otimes \mathcal{L}_\rho \otimes \mathbb{C}_{-\rho}.$$

This follows from [BK05, Proposition 2.2.2] to which one applies the projection formula. (Note that our convention defining $\mathcal{L}_\lambda$ is opposite to the one from [BK05].) If $X(w) = G/B$, then $\omega_X = \mathcal{L}_{2\rho}$ (see e.g., [BK05, (8) in §2.1]). The result follows from this. □

Define the $\mathbb{Z}$-linear endomorphism

$$\Psi : \mathrm{K}_T(X) \to \mathrm{K}_T(X); \quad [E]_T \mapsto \mathbb{C}_\rho \otimes \mathcal{L}_\rho \otimes \star[E]_T.$$

The previous proposition shows that

$$\Psi(\mathcal{I}^T_w) = (-1)^{\operatorname{codim} X(w)} \mathcal{O}^T_w. \tag{46}$$

We need the following lemma.

Lemma 9.3. *Let* $w \in W$*. Then*

$$\Psi(\iota_w) = \frac{(-1)^{\dim G/B}}{e^{w(\rho)-\rho}} \iota_w.$$

Proof. We start by observing that

$$(\star(\iota_w))|_u = \star((\iota_w)|_u) = \delta_{u,w} \lambda_{-1} T_w(G/B).$$

Then

$$\begin{aligned}
\Psi(\iota_w)|_u &= (\mathbb{C}_\rho \otimes \mathcal{L}_\rho \otimes \star(\iota_w))|_u \\
&= \delta_{w,u} e^{\rho+w(\rho)} \lambda_{-1}(T_w(G/B)) \\
&= \delta_{w,u} e^{\rho+w(\rho)} \prod_{\alpha>0} (1-e^{-w\alpha}) \\
&= \delta_{w,u} \frac{(-1)^{\dim G/B}}{e^{w(\rho)-\rho}} \prod_{\alpha>0} (1-e^{w\alpha}) \\
&= \frac{(-1)^{\dim G/B}}{e^{w(\rho)-\rho}} \iota_w|_u.
\end{aligned}$$

The claim follows from the injectivity of the localization map. □

The map Ψ intertwines with the Hecke algebra action in the following way.

Theorem 9.4. *For any $a \in \mathrm{K}_T(X)$,*

$$\Psi(\mathcal{T}_i(a)) = -\mathfrak{L}_i(\Psi(a)).$$

In particular, if $w \in W$, then

$$\Psi((\partial_i - \mathrm{id})(\mathcal{I}_w^T)) = (-1)^{\operatorname{codim} X(w)+1}\partial_i(\mathcal{O}_w^T).$$

Proof. The last statement follows from the first after specializing at $y = 0$ and using (46) and Lemma 3.4. Therefore it suffices to prove the first statement. By localization, it suffices to prove this for the fixed point basis elements $a := \iota_w$. Let $n = \dim G/B$. We use the formulae from Lemma 3.3 and Lemma 9.3 to calculate:

$$\Psi(\mathcal{T}_i(\iota_w)) = (-1)^n \frac{-(1+y)}{e^{w(\rho)-\rho}(1-e^{w(\alpha_i)})}\iota_w + (-1)^n \frac{1+ye^{w(\alpha_i)}}{e^{ws_i(\rho)-\rho}(1-e^{w(\alpha_i)})}\iota_{ws_i}. \tag{47}$$

By definiton of $\mathfrak{L}_i$,

$$-\mathfrak{L}_i\Psi(\iota_w) = -(\mathcal{T}_i^\vee + (1+y)\,\mathrm{id})\Psi(\iota(w)) = \frac{(-1)^{n+1}}{e^{w(\rho)-\rho}}(\mathcal{T}_i^\vee(\iota_w) + (1+y)\iota_w).$$

Using now the action of $\mathcal{T}_i^\vee$ from Lemma 3.3 we calculate the last term as

$$\frac{(-1)^n}{e^{w(\rho)-\rho}}\left((1+y)\left(\frac{1}{1-e^{-w(\alpha_i)}} - 1\right)\iota_w - \frac{1+ye^{w(\alpha_i)}}{1-e^{-w(\alpha_i)}}\iota_{ws_i}\right). \tag{48}$$

A simple algebra calculation shows that the coefficients of ι_w in both (47) and (48) are equal. The equality of the coefficients of ι_{ws_i} is proved similarly, using in addition that $s_i(\rho) = \rho - \alpha_i$. ☐

Remark 9.5. Theorem 9.4 has a particularly natural interpretation in terms of the Kostant-Kumar Hecke algebra $\mathcal{H}$. We keep the notation from §8.5. There is a Hecke algebra automorphism $A : \mathcal{H} \to \mathcal{H}$ sending $D_i \mapsto 1 - D_i$ and $e^\lambda \mapsto e^{-\lambda}$. Let $L_i := D_i(1 + ye^{\alpha_i}) + y$. Then it follows from the definition that

$$A(T_i) = A((1 + ye^{\alpha_i})D_i - 1) = -L_i.$$

Therefore, Theorem 9.4 shows that $\Psi : \mathrm{K}_T(X) \to \mathrm{K}_T(X)$ commutes with the Hecke automorphism A. ⌟

Proof of Theorem 9.1. Observe first that $\mathcal{O}_{\mathrm{id}}^T = \mathcal{I}_{\mathrm{id}}^T = \iota_{\mathrm{id}}$, and that $\Psi(\iota_{\mathrm{id}}) = (-1)^{\dim G/B}\iota_{\mathrm{id}}$. Then, by Theorem 9.4,

$$\begin{aligned}\Psi(\mathcal{T}_{w^{-1}}(\mathcal{O}_{\mathrm{id}}^T)) &= (-1)^{\ell(w^{-1})}\mathfrak{L}_{w^{-1}}\Psi(\iota_{\mathrm{id}}) = (-1)^{\operatorname{codim} X(w)}\mathfrak{L}_{w^{-1}}(\iota_{\mathrm{id}}) \\ &= (-1)^{\operatorname{codim} X(w)}\widetilde{\mathrm{MC}}_y(X(w)^\circ).\end{aligned}$$

Since $\mathcal{T}_{w^{-1}}(\mathcal{O}_{\mathrm{id}}^T) = \mathrm{MC}_y(X(w)^\circ)$ by Theorem 4.5, this proves (a).

The equality

$$c_{u,w}(y; e^t) = (-1)^{\ell(u)-\ell(w)} \star (d_{u,w}(y; e^t)),$$

which, by (1), is equivalent to

$$\langle \mathrm{MC}_y(X(w)^\circ), \mathcal{I}^{u,T}\rangle = (-1)^{\ell(w)-\ell(u)} * \langle \widetilde{\mathrm{MC}}_y(X(w)^\circ), \mathcal{O}^{u,T}\rangle,$$

follows by applying Ψ to both sides of

$$\mathrm{MC}_y(X(w)^\circ) = \sum_{u \leq w} c_{u,w}(y; e^t)\mathcal{O}_u^T$$

and using Proposition 9.2. Finally, for part (c) we use that $\mathcal{O}^{u,T} = \sum_{v \geq u} \mathcal{I}^{v,T}$ (proved by Möbius inversion). Then:

$$\begin{aligned}\langle \mathrm{MC}_y(X(w)^\circ), \mathcal{O}^{u,T} \rangle =& \langle \mathrm{MC}_y(X(w)^\circ), \sum_{v \geq u} \mathcal{I}^{v,T} \rangle \\ =& \sum_{v \geq u} (-1)^{\ell(w)-\ell(v)} \star \langle \widetilde{\mathrm{MC}}_y(X(w)^\circ), \mathcal{O}^{v,T} \rangle \\ =& (-1)^{\ell(w)-\ell(u)} \langle \widetilde{\mathrm{MC}}_y(X(w)^\circ), \mathcal{I}^{u,T} \rangle.\end{aligned}$$

This finishes the proof. □

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Mathematics Department, Florida State University, Tallahassee FL 32306
E-mail address: aluffi@math.fsu.edu

Department of Mathematics, Virginia Tech University, Blacksburg, VA 24061
E-mail address: lmihalce@vt.edu

Mathematisches Institut, Universität Münster, Germany
E-mail address: jschuerm@uni-muenster.de

Yau Mathematical Sciences Center, Tsinghua University, 100084, Beijing, China
E-mail address: changjiansu@mail.tsinghua.edu.cn