Quotient families of mapping classes

Eriko Hironaka[*]

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Abstract

Thurston's fibered face theory allows us to partition the set of pseudo-Anosov mapping classes on different compact oriented surfaces into subclasses with related dynamical behavior. This is done via a correspondence between the rational points on fibered faces in the first cohomology of a hyperbolic 3-manifold and the monodromies of fibrations of the 3-manifold over the circle. In this paper, we generalize examples of Penner, and define quotient families of mapping classes. We show that these mapping classes correspond to open linear sections of fibered faces. The construction gives a simple way to produce families of pseudo-Anosov mapping classes with bounded normalized dilatation and computable invariants, and gives concrete examples of mapping classes associated to sequences of points tending to interior and to the boundary of fibered faces. As an additional aid to calculations, we also develop the notion of Teichmüller polynomials for families of digraphs.

1   Introduction

Let S be a compact connected oriented surface of finite type. A mapping class $\phi : S \to S$ is a self-homeomorphism of S considered up to isotopy relative to the boundary of S. By the Nielsen-Thurston classification, mapping classes that are not periodic or reducible have the property that for any essential simple closed curve γ on S and any Riemannian metric ω, the growth of the sequence

$$\ell_\omega(\phi^n(\gamma))$$

is exponential, and furthermore the growth rate

$$\lambda = \lim_{n \to \infty} \ell_\omega(\phi^n(\gamma))^{\frac{1}{n}}$$

is independent of γ and ω (see [Thu2], [FM]). The growth rate $\lambda(\phi) = \lambda$ is called the *dilatation* of ϕ. Let $\mathcal{P}$ be the set of all pseudo-Anosov mapping classes.

In this paper, we investigate special families $\mathcal{Q}$ of intrinsically defined mapping classes with simple defining data, called quotient families. We show that quotient families correspond to the monodromy of linear segments on a fibered face of a 3-manifold, allowing us to describe the behavior of the *normalized dilatation*

$$L(S, \phi) = \lambda(\phi)^{|\chi(S)|},$$

for $(S, \phi) \in \mathcal{Q}$.

Quotient families. Quotient families are defined as follows. Let $\widetilde{S}$ be an oriented surface of infinite type with a properly discontinuous $\mathbb{Z}$-action generated by a map $\zeta : \widetilde{S} \to \widetilde{S}$ and a fundamental domain Σ that is of finite type, connected and compact. A mapping class (X, f) is *supported* on $Y \subset X$ if f is the identity outside of Y. Let $\widehat{\phi} : \widetilde{S} \to \widetilde{S}$ be a mapping class supported on Y, where, for some $m_0 \geq 1$,

$$Y \subset \Sigma \cup \zeta\Sigma \cup \cdots \cup \zeta^{m_0-1}\Sigma.$$

Let $J_{m_0} = (0, \frac{1}{m_0}) \cap \mathbb{Q}$ be the rational points in the open interval from 0 to $\frac{1}{m_0}$.

We will always write $\alpha \in J_{m_0}$ as a fraction in reduced form, i.e., $\alpha = \frac{k}{m}$, i.e., k, m are positive integers, and $(k, m) = 1$. Let (S_α, ϕ_α) be defined by

$$S_\alpha = \widetilde{S}/\zeta^m,$$

and

$$\phi_\alpha = r^a \eta,$$

where r is induced by ζ, η is induced by $\widehat{\phi}$, and $ak = 1 \pmod{m}$. We call the collection

$$\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi}) = \{(S_\alpha, \phi_\alpha) : \alpha \in J_{m_0}\}$$

the *quotient family associated to* $(\widetilde{S}, \zeta, \widehat{\phi})$, and J_{m_0} the *parameter set*.

We say $(\zeta\widehat{\phi})^m$ *stabilizes* if there is a mapping class $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ that commutes with ζ and satisfies

$$\widetilde{\phi}(s) = \zeta^{-m}(\zeta\widehat{\phi})^m(s)$$

for all $m \geq m_0$ and $s \in \widetilde{S}$. In this case, the restriction of $\widetilde{\phi}(s)$ to Σ determines a mapping class ϕ on the quotient surface $S = \widetilde{S}/(\zeta)$. In this case we say $\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$ is a Type I quotient family. Otherwise, it is of Type II.

Theorem 1.1 *If $\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$ is of Type I, then*

$$\lim_{\alpha \to 0} L(S_\alpha, \phi_\alpha) = L(S, \phi);$$

and if it is of Type II, then

$$\lim_{\alpha \to 0} L(S_\alpha, \phi_\alpha) = \infty.$$

In [Pen], Penner constructed a sequence of mapping classes $(S_g, \phi_g) \in \mathcal{P}$, $g \geq 2$, where S_g is a surface of genus g and two boundary components, and $L(S_g, \phi_g)$ is bounded. He uses these to show

that, the minimum dilatation δ_g of pseudo-Anosov mapping classes on closed surfaces S_g of genus g behaves asymptotically like

$$\delta_g \asymp \frac{1}{g}.$$

In [Val], Valdivia showed that Penner's sequence and some generalizations have the property that its normalized dilatations are bounded, and that furthermore their mapping tori are homeomorphic. The results in this paper show further that Penner's sequence (and its generalizations in [Bau, Val, Tsa]) can be thought of as a Cauchy sequence in a quotient family, and hence are Cauchy sequences on a one-dimensional linear section of a single fibered face. This information makes it possible to explicitly calculate defining polynomials (Alexander and Teichmüller polynomials) for the homological and geometric dilatations, as well as other properties (see Section 5.1).

Fibered faces. Thurston's theory of fibered faces [Thu1] gives a way to associate pseudo-Anosov mapping classes to rational points on open affine polyhedra F_α.

Let M be a fibered 3-manifold with first Betti number $b = b_1(M) \geq 2$. Choose an identification of $H^1(M; \mathbb{R}) = \mathbb{R}^b$ defined over the integers. An integral element $\alpha \in H^1(M; \mathbb{Z})$ is said to be *fibered* if it is dual to the fiber S_α of a fibration $\psi_\alpha : M \to S^1$. The monodromy of ψ_α is a mapping class denoted (S_α, ϕ_α). In [Thu1], Thurston defines a norm $|| \, ||$ on $\mathbb{R}^b$ so that if $\alpha \in H^1(M; \mathbb{Z})$ is fibered then

$$||\psi_\alpha|| = |\chi(S_\alpha)|,$$

where $\chi(S)$ is the topological Euler characteristic of the fiber surface S_α. The Thurston norm ball is a convex polyhedron, and the top dimensional faces F have the property that either all the integral points in the cone $F \cdot \mathbb{R}^+$ are fibered or none are. In the former case we say F is a *fibered face*. Given a rational point α on a fibered face F, there is a unique integral point, with relatively prime coefficients, on the ray emanating from the origin and passing through α. This point is dual to the fiber of a fibration of $\psi_\alpha : M \to S^1$, and it is characterized by the property that it is the unique integral element on the ray such that ψ has monodromy (S_α, ϕ_α) where S_α is connected. In this way, each fibered face defines a family of fibrations $\Psi(M, F)$ of M over the circle. Their monodromies $\mathcal{P}(M, F)$ can be identified with rational points on F so that the denominator of a rational point α on a fibered face equals $|\chi(S_\alpha)|$.

The restriction of the normalized dilatation L to $\mathcal{P}(M, F)$ defines a real-valued function on the rational points of F. By a theorem of Fried [Fri], the function L extends to a continuous convex function on F going to infinity toward the boundary of F (cf. [Mat, LO, McM1]). It follows that if F has dimension greater than or equal to 1 (or, equivalently, $b_1(M) \geq 2$) then compact subsets of F give rise to families of mapping classes with bounded normalized dilatation and unbounded topological Euler characteristic.

Theorem 1.2 *Let $\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$ be a quotient family. Then there is a 3-manifold M, and an embedding $\overline{A} : (0, \frac{1}{m_0}) \to T$, where T is a one-dimensional linear segment of a fibered face F, and the rational points of $(0, \frac{1}{m_0})$ map to the rational points of T. Furthermore, If $(\widetilde{S}, \zeta, \widehat{\phi})$ is of Type I, $\overline{A}$ extends over 0 to an interior points of F, if it is of Type II, it extends over 0 to a boundary point of F.*

Theorem 1.2 implies Theorem 1.1.

Polynomial invariants. One of the advantages of realizing a family of mapping classes as monodromies on a single fibered face is that one can compute invariants such as homological and geometric dilatations from multivariable Alexander and Teichmüller polynomials. Since Theorem 1.2 implies that each quotient family $\mathcal{Q}$ lies on a single fibered face, there is a single Alexander polynomial and a single Teichmüller polynomial computable from one element of $\mathcal{Q}$ from which all homological and geometric dilatations of mapping classes in $\mathcal{Q}$ can be computed (see Section 4 and Section 5). Thus, by studying the dynamics of a single member of the quotient family, one can determine whether the family is of Type I or Type II, and other properties such as which elements in the family have orientable stable and unstable foliations.

Outline of paper. This paper is organized as follows. In Section 2 we give properties of mapping classes on fibered faces. In Section 3, we prove Theorem 1.2. Applying the theory of fibered faces and Teichmüller polynomials, we use Theorem 1.2 to give a simple method for computing Alexander and Teichmüller polynomials for quotient families. We study examples of Type I and Type II quotient families in Section 5 including generalizations of Penner sequences, and in Section 6, we discuss a question of Farb, Leininger and Margalit concerning the structure of small dilatation pseudo-Anosov mapping classes.

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2 Fibered faces and linear deformations of pseudo-Anosov maps

In this section, we review properties of the monodromies of fibrations of a hyperbolic 3-manifold over the circle. We show that the set $\mathcal{P}$ of pseudo-Anosov mapping classes partitions into disjoint subfamilies $\mathcal{P}(M, F)$ corresponding to rational points on F, where M is a hyperbolic 3-manifold and F is a fibered face. We describe the mapping classes in $\mathcal{P}(M, F)$ using abelian unbranched coverings. This exposition expands on the discussion in [McM1] (Theorem 10.2).

Given (S, ϕ), let M be the mapping torus

$$M = S \times [0, 1]/(s, 1) \sim (\phi(s), 0).$$

Let $\psi \in H^1(M; \mathbb{Z})$ be the element induced by the corresponding fibration $M \to S^1$ over the circle. Let $M_\psi \to M$ be the cyclic covering over M defined by the epimorphism

$$\pi_1(M) \to \mathbb{Z}$$

induced by ψ.

Cut and paste description of M_ψ. The manifold M_ψ has the following cut and paste description. Let $C = S \times [0, 1]$. Let M_ψ is the union of copies C_j of C, for $j \in \mathbb{Z}$. Each C_j is glued to C_{j+1} by identifying $(s, 1)_j$ with $(\phi(s), 0)_{j+1}$. Define $\rho : M_\psi \to M$ to be the map that sends $(s, t)_j$ to the equivalence class of (s, t) in M.

Lemma 2.1 *The map $M_\psi \to M$ is a cyclic covering map, and the map $T : M_\psi \to M_\psi$ defined by*

$$T(s,t)_j = (s,t)_{j-1}$$

generates the group of covering automorphisms of M_ψ over M.

Proof. By the definition of mapping torus, M is the quotient of C by the identification

$$(s,1) \sim (\phi(s),0).$$

Since

$$\rho(s,1)_j = (s,1) = (\phi(s),0) = \rho(\phi(s),0)_{j+1},$$

ρ is well-defined. Since

$$T(s,1)_j = (s,1)_{j-1} = (\phi(s),0)_j = T(\phi(s),0)_{j+1},$$

the map T is well defined on M_ψ, and since

$$\rho T(s,t)_j = \rho(s,t)_{j-1} = (s,t) = \rho(s,t)_j,$$

T is a covering automorphism. □

A *flow* on a manifold X is a continuous map

$$f : X \times \mathbb{R} \to X,$$

with the property that, the maps $f_v : X \to X$ defined by $f_v(x) = f(x,v)$ satisfy

$$f_{u+v} = f_u \circ f_v.$$

Because of the additive property, to define f it suffices to define f_v for small $v > 0$.

Define a flow

$$f_v : M_\psi \to M_\psi,$$

such that for small $v > 0$

$$f_v(s,t)_j = \begin{cases} (s,t+v)_j & \text{if } 0 < t, \text{ and } t + v < 1 \\ (\phi^{-1}(s),v)_{j+1} & \text{if } t = 1 \text{ and } 0 < v < 1 \end{cases}$$

This kind of flow is called a *suspension flow*, and it induces a *circular suspension flow* on M with cross seciton S (see, for example, [Fri]).

Lemma 2.2 *The flow f_v has the property that*

$$f_1 T_\psi(s,t)_j = (\phi^{-1}(s),t)_j$$

for all $j \in \mathbb{Z}$.

Lifting to an infinite covering. Let $H = H_1(M; \mathbb{Z})/\mathrm{Tor} \simeq \mathbb{Z}^k$, and let

$$\mathfrak{h} : \pi_1(M) \to H$$

be the Hurewicz map composed with the quotient map $H_1(M; \mathbb{Z}) \to H_1(M; \mathbb{Z})/\mathrm{Tor}$. Then $\mathfrak{h}$ determines a regular $\mathbb{Z}^k$ covering

$$\widetilde{\rho} : \widetilde{M} \to M,$$

of M with group of covering automorphisms H.

The element $\psi \in H^1(M; \mathbb{Z})$ induces a homomorphism, $\psi : \pi_1(M) \to \mathbb{Z}$, which factors through $\mathfrak{h}$. Thus, we have an intermediate covering

$$
\begin{array}{ccc}
\widetilde{M} & & \\
\Big\downarrow{\scriptstyle \widetilde{\rho}} & \searrow & M_\psi \\
& \swarrow{\scriptstyle \rho_\psi} & \\
M. & &
\end{array}
$$

By an H-isotopy of $\widetilde{M}$, we mean a continuous map

$$I : \widetilde{M} \to [0, 1] \to \widetilde{M},$$

such that for fixed $t \in [0, 1]$, $I_t(x) = I(x, t)$ defines a homeomorphism commuting with H. As a covering space, $\widetilde{M}$ is well-defined up to H-isotopy.

Let $\widetilde{\phi}$ be a lift of ϕ to $\widetilde{S}$. The identification $M_\psi = \bigcup_j C_j$ lifts to an identification

$$\widetilde{M} = \bigcup_j \widetilde{C}_j,$$

where $\widetilde{C}_j = \widetilde{S} \times [0, 1]$ and $\widetilde{C}_j$ is glued to $\widetilde{C}_{j+1}$ by

$$(s, 1)_j = (\widetilde{\phi}(s), 0)_{j+1}.$$

The covering map T_ϕ lifts to

$$
\begin{array}{ccc}
\widetilde{T}_\phi : \widetilde{M} & \to & \widetilde{M} \\
(s, t)_j & \mapsto & (s, t)_{j-1}.
\end{array}
$$

The flow f_v also lifts to

$$\widetilde{f}_v : \widetilde{M} \to \widetilde{M},$$

so that

$$\widetilde{f}_1(s, t)_j = (\widetilde{\phi}^{-1}(s), t)_{j+1},$$

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Lemma 2.3 *The flow $\widetilde{f}_\nu$ satisifes*

$$\widetilde{f}_m \widetilde{T}_\phi^m(s,t)_j = (\widetilde{\phi}^{-m}(s),t)_j$$

for all $(s,t)_j \in \widetilde{M}$.

Deformations of ψ via coverings. All the elements of $\Psi(M,F)$ and their associated monodromy in $\mathcal{P}(M,F)$ can be described in terms of the covering $\widetilde{\rho}$ and the flow $\widetilde{f}_\nu$ as we will now explain.

Take any $\alpha \in \Psi(M,F)$, and let $K_\alpha \subset H$ be the kernel of the induced map

$$\alpha : H \to \mathbb{Z}.$$

Then α determines another free cyclic intermediate covering

where $M_\alpha = \widetilde{M}/K_\alpha$. Let $\widetilde{T}_\alpha \in H$ be a solution to $\alpha(\widetilde{T}_\alpha) = -1$, and let T_α be the induced map on M_α. If (S_α, ϕ_α) is the monodromy of α, then there is a corresponding flow $\widetilde{f}_v^{(\alpha)}$ on $\widetilde{M}$.

Theorem 2.4 ([Thu1, Thu2, Fri]) *The flows $\widetilde{f}_v^{(\alpha)}$ and $\widetilde{f}_v$ defined on $\widetilde{M}$ are H-isotopic. Furthermore, $\alpha \in H^1(M;\mathbb{Z})$ is in the same fibered cone as ψ if and only if the dual to α is a cross-section of the flow $\widetilde{f}_v$.*

Using Theorem 2.4, we can find ϕ_α as follows. Let $\widetilde{S}_\alpha$ be the preimage of a vertical section $S_\alpha \subset M$ in $\widetilde{M}$. Then $\widetilde{S}_\alpha$ has the property that K_α leaves $\widetilde{S}_\alpha$ invariant, and $S_\alpha = \widetilde{S}_\alpha/K_\alpha$.

Let $\widetilde{\phi}_\alpha$ be the map on $\widetilde{S}_\alpha$ defined as follows. Take any $x \in \widetilde{S}_\alpha$, and let $v(x) \geq 0$ be the smallest number so that $\widetilde{f}_{v(x)}(\widetilde{T}_\alpha(x)) \in \widetilde{S}_\alpha$. (The existence of $v(x)$ is a necessary and sufficient condition for α to lie in the fibered cone.) Let $\widetilde{\phi}_\alpha$ satisfy

$$\widetilde{\phi}_\alpha^{-1}(x) = \widetilde{f}_{v(x)}(\widetilde{T}_\alpha(x)). \tag{1}$$

Then $\widetilde{\phi}_\alpha$ commutes with the action of K_α on $\widetilde{S}_\alpha$, and defines a map ϕ_α on S_α.

To summarize, starting with a single mapping class $(S,\phi) \in \mathcal{P}(M,F)$, we can reconstruct the neighboring monodromy of M as follows.

   1. Define from (S,ϕ) a flow

$$\widetilde{f}_v : \widetilde{M} \to \widetilde{M}.$$

2. For each $\alpha \in \Psi(M, F)$, find a surface $\widetilde{S}_\alpha$ in $\widetilde{M}$ that is invariant under the action of the kernel K_α of

$$\alpha : H \to \mathbb{Z}.$$

3. The monodromy of α is (S_α, ϕ_α), where $S_\alpha = \widetilde{S}_\alpha / K_\alpha$ and ϕ_α can be found by solving $\alpha(\widetilde{T}_\alpha) = -1$ and applying the flow $\widetilde{f}_v$ as in Equation (1).

Linear deformations. By restricting attention to sections of a fibered face F by linear subspaces, we simplify the structure of the coverings $\widetilde{M}$ and $\widetilde{S}$.

Let k be an integer satisfying

$$0 < k \le b_1(M) - 1.$$

The k-dimensional linear subspaces of F are in one-to-one correspondence with $k+1$ dimensional subspaces of $H^1(M; \mathbb{R})$. For example, let

$$\{\psi, \beta_1, \dots, \beta_k\} \subset H^1(M; \mathbb{R}),$$

be a set of linearly independent integral elements, and let I be the k-dimensional linear section of F cut by the subspace of $H^1(M; \mathbb{R})$ generated by $\{\psi, \beta_1, \dots, \beta_k\}$. Then I is a linear section of F containing the projection of ψ.

The integral points in the cone in $H^1(M; \mathbb{R})$ over I can be analyzed via the $\mathbb{Z}^{k+1}$-covering of M determined by the natural map

$$\pi_1(M) \to H_{\psi, \beta_1, \dots, \beta_k},$$

where

$$H_{\psi, \beta_1, \dots, \beta_k} = \mathrm{Hom}(\langle \psi, \beta_1, \dots, \beta_k \rangle, \mathbb{Z}).$$

In particular, a *linear deformation* of a mapping class $(S, \phi) \in \mathcal{P}(M, F)$ is the intersection of rational elements in a neighborhood of the projection α_ψ of ψ to F and a one dimensional linear section of F passing through α_ψ.

3 Quotient families

In this section, we define quotient families $\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$, and prove Theorem 1.2.

We start with a triple $(\widetilde{S}, \zeta, \widehat{\phi})$ consisting of the following. The surface $\widetilde{S}$ is a connected, oriented, infinite type surface, which is locally of finite type, and $\widehat{\phi} : \widetilde{S} \to \widetilde{S}$ is a self-homeomorphism. There is a compact connected subsurface $\Sigma \subset S$ of finite type, and a finite union of boundary components and subarcs of boundary components $\tau^- \subset S$ with the property that

$$\zeta^{-1} \Sigma \cap \Sigma = \tau^-$$

and

$$\zeta^i \Sigma \cap \Sigma = \emptyset$$

if $i > 1$. Let $\tau^+ = \zeta\tau^-$, and $\tau_i^{\pm} = \zeta^i(\tau^{\pm})$. The subsurface Σ is called a *fundamental domain* for ζ. We can think of $\widetilde{S}$ as being obtained by gluing together i copies of Σ, namely $\Sigma_i = \zeta^i\Sigma$, by identifying τ_i^+ with τ_{i-1}^-, where $\tau_i^{\pm} = \zeta^i\tau^{\pm}$ (see Figure 1).

The mapping class $\widehat{\phi} : \widetilde{S} \to \widetilde{S}$ is supported on the set

$$Y = \Sigma \cup \zeta\Sigma \cup \cdots \cup \zeta^{m_0}\Sigma,$$
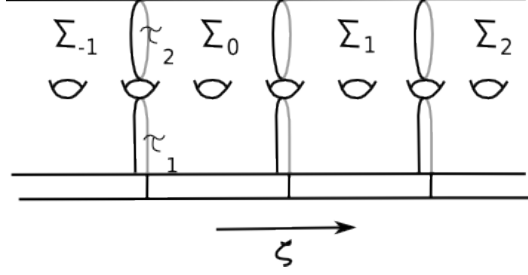
for some $m_0 > 1$.

Figure 1: Example of a $\mathbb{Z}$-surface with $\tau = \tau_1 \cup \tau_2$.

Let

$$\widehat{\phi}_i = \zeta^i\widehat{\phi}\zeta^{-i}.$$

For integers $m > m_0$, the maps $\{\widehat{\phi}_{im} \ : \ i \in \mathbb{Z}\}$ commute with each other. Define

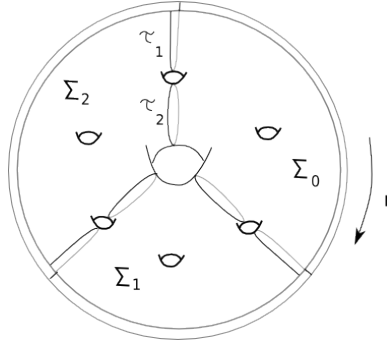$$\widehat{\phi}_{m\text{-rep}} = \circ_{i \in \mathbb{Z}}\widehat{\phi}_{im}.$$

Figure 2: Quotient surface S_m, where $m = 3$.

Let $J_{m_0} = \left(0, \frac{1}{m_0}\right) \cap \mathbb{Q}$. The associated quotient family $\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$ consists of mapping classes (S_α, ϕ_α), for $\alpha \in J_{m_0}$ defined as follows. We will always represent α as $\frac{k}{m}$, where $m > m_0$, $0 < k < m$, and $(m, k) = 1$. The surface S_α is the quotient

$$S_\alpha = \widetilde{S}/\zeta^m.$$

(See Figure 2.) Let r and η be the mapping classes on S_α induced by ζ and $\widehat{\phi}_0$. Since $m > m_0$, both these are well defined. Let $\phi_\alpha : S_\alpha \to S_\alpha$ be defined by

$$\phi_\alpha = r^a\eta,$$

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where a is the multiplicative inverse of k modulo m.

From the definitions we have the following.

Lemma 3.1 *The mapping class* $(\widetilde{S}, \widetilde{\phi}_\alpha)$ *lifts* (S, ϕ_α), *where*

$$\widetilde{\phi}_\alpha = \zeta^a \widehat{\phi}_{m\text{-rep}}.$$

Overview of proofs.

Starting with a triple $(\widetilde{S}, \zeta, \widehat{\phi})$ and $\alpha = \frac{k}{m} \in J_{m_0}$, we construct

(i) an unbranched covering $\widetilde{M}^{(\alpha)} \to M^{(\alpha)}$ of a 3-manifold $M^{(\alpha)}$;

(ii) generators $\widetilde{T}^{(\alpha)}, Z^{(\alpha)} : \widetilde{M}^{(\alpha)} \to \widetilde{M}^{(\alpha)}$ for the group of covering automorphisms $H^{(\alpha)}$ of $\widetilde{M}^{(\alpha)}$ over $M^{(\alpha)}$;

(iii) subsurfaces $\widetilde{X}_v^{(\alpha)} \subset \widetilde{M}^{(\alpha)}$, for $v \in \mathbb{R}$ (marked as the images of embeddings $q_v^{(\alpha)} : \widetilde{S} \to \widetilde{M}^{(\alpha)}$);

(iv) a flow $\widetilde{f}_v^{(\alpha)}$ on $\widetilde{M}^{(\alpha)}$, so that $\widetilde{f}_v^{(\alpha)}(\widetilde{X}_\eta^{(\alpha)}) = \widetilde{X}_{\eta+v}^{(\alpha)}$ for all $t \in \mathbb{R}$ that commutes with $\widetilde{T}^{(\alpha)}$ and $Z^{(\alpha)}$;

(v) a homeomorphism $R^{(\alpha)} : \widetilde{M}^{(\alpha)} \to \widetilde{M}^{(\alpha)}$ with the property that $R^{(\alpha)}$ restricts to a homeomorphism on each $\widetilde{X}_v^{(\alpha)}$, and, for $s \in \widetilde{S}$,

$$R^{(\alpha)}(q_v^{(\alpha)}(s)) = q_v^{(\alpha)}(\zeta(s)),$$

and

(vi) a map $\widehat{T}^{(\alpha)}$ with the property that $(\widehat{T}^{(\alpha)})^m = \widetilde{T}$, and $\widetilde{f}_{\frac{1}{m}}^{(\alpha)} \widehat{T}^{(\alpha)}$ restricts to a homeomorphism on each $\widetilde{X}_v^{(\alpha)}$ and for $v = 0$, it satisfies

$$\widetilde{f}_{\frac{1}{m}}^{(\alpha)} \widehat{T}^{(\alpha)}(q_0^{(\alpha)}(s)) = q_0^{(\alpha)}(\widehat{\phi}_{m\text{-rep}}^{-1}(s)).$$

We then show that $(\widetilde{M}^{(\alpha)}, \widetilde{f}^{(\alpha)}, \widetilde{T}^{(\alpha)}, Z^{(\alpha)})$ are independent of α; the monodromy of $A(\alpha)$ is the element $(S_\alpha, \phi_\alpha) \in \mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$; and describe an extension of A to $(-\frac{1}{m_0}, \frac{1}{m_0})$, whose image outside of 0 lies on fibered faces.

Cutting and pasting on a 3-manifold with a flow.

Consider a 3-manifold $\widetilde{N}$ (possibly infinite type) with a flow $t_v : \widetilde{N} \times \mathbb{R} \to \widetilde{N}$. We will construct a new 3-manifold and flow from a cross-section $\widetilde{Y} \subset \widetilde{N}$, a system of self-homeomorphisms of $\widetilde{Y}$, and a positive real number $\mu > 0$.

A *cross-section* $\widetilde{Y} \subset \widetilde{N}$ of t_v is a surface (possibly infinite type) with the property that

(i) for each $y \in \widetilde{Y}$, $t_v(y) \in \widetilde{Y}$ if only if $v = 0$, and

(ii) for all $z \in \widetilde{N}$, there is an $y \in \widetilde{Y}$ and $v \in \mathbb{R}$ such that $t_v(y) = z$.

Let $\widehat{\phi}_i : \widetilde{Y} \to \widetilde{Y}$ be any sequence of homeomorphisms. We define a new 3-manifold $\widetilde{P}$ and a flow $f_v^{(\mu)}$ on $\widetilde{P}$.

For $v \in \mathbb{R}$, define
$$\widetilde{Y}_v = \widetilde{t}_v(\widetilde{Y}),$$
and
$$
\begin{aligned}
q_v : \widetilde{Y} &\to \widetilde{Y}_v \\
y &\mapsto t_v(y).
\end{aligned}
$$
We use the translated surfaces $\widetilde{Y}_{i\mu}$, $i \in \mathbb{Z}$, as *cutting loci*.

Each $\widetilde{Y}_{i\mu}^{(\mu)}$ cuts $\widetilde{N}$ into two sides, $(\widetilde{N}_{i\mu}^{(\mu)})^{\pm}$, where
$$(\widetilde{N}_{i\mu}^{(\mu)})^+ = \{t_v(y) : v \geq 0, y \in \widetilde{Y}_{i\mu}^{(\mu)}\},$$
and
$$(\widetilde{N}_{i\mu}^{(\mu)})^- = \{t_v(y) : v \leq 0, y \in \widetilde{Y}_{i\mu}^{(\mu)}\}.$$
Let $(\widetilde{Y}_{i\mu}^{(\mu)})^{\pm} \subset (\widetilde{N}_{i\mu}^{(\mu)})^{\pm}$ be the two sides of $\widetilde{Y}_{i\mu}^{(\mu)}$ lying on the boundary of $(\widetilde{N}^{(\mu)})^{\pm}$. Let $q_{i\mu}^{\pm} : \widetilde{Y} \to (\widetilde{Y}_{i\mu}^{(\mu)})^{\pm}$ be the parameterizations defined by $q_{i\mu}$. Let $\widetilde{P}$ be obtained from $\widetilde{N}$ by cutting along $\widetilde{Y}_{i\mu}^{(i\mu)}$ and regluing so that
$$q_{i\mu}^-(s) = q_{i\mu}^+(\widehat{\phi}_i(s)).$$

For $\eta \notin \mathbb{Z}\mu$, the surfaces $\widetilde{Y}_\eta^{(\mu)}$ lie outside the cut locus in $\widetilde{N}$, and hence determine surfaces in $\widetilde{P}$, and $q_v : \widetilde{Y} \to \widetilde{Y}_v^{(\mu)}$ is well defined. For $i \in \mathbb{Z}$, the map $q_{i\mu}$ determines an identification $\widetilde{Y} \to (\widetilde{Y}_{i\mu}^{(\mu)})^{\pm}$, which we also denote by $q_{i\mu}^{\pm}$. Let
$$\widetilde{C}_i = \left(\widetilde{N}_{i\mu}^{(\mu)}\right)^+ \cap \left(\widetilde{N}_{(i+1)\mu}^{(\mu)}\right)^- \cup q_{i\mu}(\widetilde{Y}).$$

The flow t_v on $\widetilde{N}$ induces a flow $\widetilde{\ell}_v$ on $\widetilde{P}$ defined for small $v > 0$ by
$$\widetilde{\ell}_v(z) = \begin{cases} t_v(z) & \text{if } z, t_v(z) \in \widetilde{C}_i \\ t_v(q_{i\mu}(\widehat{\phi}_i^{-1}(s))) & \text{if } z = q_{i\mu}(s), \text{ for } s \in \widetilde{S}. \end{cases}$$

We say that $(\widetilde{P}, q_v, \widetilde{\ell}_v)$ is obtained from $(\widetilde{N}, t_v, \widetilde{Y}, \widehat{\phi}_i, \mu)$ by *cutting and pasting*.

Intermediate Construction.

Let $\widetilde{N} = \widetilde{S} \times \mathbb{R}$. Let
$$
\begin{aligned}
Z : \widetilde{N} &\to \widetilde{N} \\
(s, t) &\mapsto (\zeta(s), t).
\end{aligned}
$$

Let t_v be the flow on $\widetilde{N}$ defined by $t_v(s,t) = (s, t+v)$. Let $\widetilde{S}_v = \widetilde{S} \times \{v\}$ for $v \in \mathbb{R}$.

Let

$$p : \widetilde{S} \;\rightarrow\; \mathbb{R} \tag{2}$$

be a continuous map with the property that

(i) $p(\zeta(s)) = p(s) + 1$, for all $s \in \widetilde{S}$,

(ii) $p(\Sigma) \subset [0,1]$, and

(iii) $\tau^- = p^{-1}(0)$.

For $\alpha = \frac{k}{m} \in J_{m_0}$ define $\widetilde{Y}^{(\alpha)} \subset \widetilde{N}$ to be the image of $q^{(\alpha)}$, where

$$\begin{aligned}
q^{(\alpha)} : \widetilde{S} \;&\rightarrow\; \widetilde{N} \\
s \;&\mapsto\; (s, -\alpha p(s)).
\end{aligned} \tag{3}$$

Let $\widehat{\phi}_i = \zeta^i \widehat{\phi} \zeta^{-i}$ and define $(\widetilde{P}^{(\alpha)}, q_v^{(\alpha)}, \widetilde{\ell}_v^{(\alpha)})$ be obtained from $(\widetilde{N}, t_v, \widetilde{Y}^{(\alpha)}, \widehat{\phi}_i, \frac{1}{m})$ by cutting and pasting as above.

Lemma 3.2 *The restriction of Z on $\widetilde{Y}_v^{(\alpha)}$ is given by*

$$\begin{aligned}
Z : \widetilde{Y}_v^{(\alpha)} \;&\rightarrow\; \widetilde{Y}_{v+\alpha}^{(\alpha)} \\
q_v^{(\alpha)}(x) \;&\mapsto\; q_{v+\alpha}^{(\alpha)}(\zeta(x)).
\end{aligned}$$

Proof. Let $x \in \widetilde{Y}$, and $x' = \zeta(x)$. Then

$$\begin{aligned}
Z(q_0^{(\alpha)}(x)) \;&=\; Z(x, -\alpha p(x)) \\
&=\; (x', -\alpha p(\zeta^{-1}(x'))) \\
&=\; (x', -\alpha(p(x') - 1)) \\
&=\; \widetilde{\ell}_\alpha^{(\alpha)}(x', -\alpha p(x')) \\
&=\; \widetilde{\ell}_\alpha^{(\alpha)} q_0^{(\alpha)}(x') \\
&=\; q_\alpha^{(\alpha)}(x')
\end{aligned}$$

The statement follows since Z and t_v commute. $\qquad\square$

Lemma 3.3 *The map Z determines a homeomorphism $Z^{(\alpha)}$ of $\widetilde{P}^{(\alpha)}$ that commutes with the flow $\widetilde{\ell}_v^{(\alpha)}$.*

Proof. Any point in $\widetilde{P}^{(\alpha)}$ is of the form $q_u^{(\alpha)}(x)$, for some $u \in \mathbb{R}$ and $x \in \widetilde{Y}^{(\alpha)}$. Thus, by Lemma 3.2, for small v, and $u \notin \mathbb{Z}\alpha$, we have

$$
\begin{aligned}
\widetilde{\ell}_v^{(\alpha)}(Z(q_u^{(\alpha)}(x))) &= \widetilde{\ell}_v^{(\alpha)}(q_{u+\alpha}^{(\alpha)}(\zeta(x))) \\
&= q_{v+u+\alpha}^{(\alpha)}(\zeta(x)) \\
&= Z q_{v+u}^{(\alpha)}(x) \\
&= Z(\widetilde{\ell}_v(q_u^{(\alpha)}(x))).
\end{aligned}
$$

$\square$

We call $(\widetilde{P}^{(\alpha)}, Z, \widehat{T}^{(\alpha)}, \widetilde{\ell}_v^{(\alpha)})$ the *pre-covering* constructed from $(\widetilde{S}, \zeta, \widehat{\phi}, q^{(\alpha)})$.

Construction of $\widetilde{M}^{(\alpha)}$.

We now cut $\widetilde{P}^{(\alpha)}$ further to make a 3-manifold $\widetilde{M}^{(\alpha)}$ with a translation $\widetilde{T}^{(\alpha)}$, and a flow $\widetilde{f}_v^{(\alpha)}$ that is $\widetilde{T}^{(\alpha)}$-invariant.

Consider

$$
\widetilde{N}_1 = \{(s,v) \ : s \in \widetilde{S}, v \in [-1,0]\} \subset \widetilde{N}.
$$

Let $\widetilde{P}_1^{(\alpha)} \subset \widetilde{P}^{(\alpha)}$ be the submanifold obtained by restricting the cutting and gluing to $\widetilde{N}_1$.

Lemma 3.4 *The surface $\widetilde{S}_{-1} = \widetilde{S} \times \{-1\} \subset \widetilde{N}_1$ is not affected by the cutting and gluing, and the flow $\widetilde{\ell}_v$ equals the trivial flow t_v on $\widetilde{S}_{-1}$ for $v \in [0, 1-\alpha m_0]$.*

Proof. Recall that ϕ_0 is the identity on $\tau^- \subset \Sigma$, and the support of $\widehat{\phi}_0$ is contained in

$$
\Sigma \cup \zeta\Sigma \cup \cdots \cup \zeta^{m_0-1}\Sigma.
$$

It follows that $\widetilde{\ell}_v = t_v$ on $\widetilde{S}_{-1} \cap \widetilde{Y}_{i\alpha}^{(\alpha)}$ for all $v \in [0, 1-\alpha m_0]$. $\square$

Identify $\widetilde{S}_{-1}$ and $\widetilde{S}_0$ with their images in $\widetilde{P}_1^{(\alpha)}$ after cutting and pasting. Each $\widetilde{S}_i$, for $i = -1, 0$, is identified with $\widetilde{S}$ by the product structure of $\widetilde{N}$. These can be thought of as the *ends* of $\widetilde{P}^{(\alpha)}$. The map $Z^{(\alpha)}$ induces a discrete action on $\widetilde{N}$ and stabilizes the $\widetilde{S}_i$ for all i. Thus, it also induces a discrete action on $\widetilde{N}_1$ and $\widetilde{P}_1^{(\alpha)}$ that stabilizes the ends. For $i \in \mathbb{Z}$, let $\widetilde{P}_i^{(\alpha)}$ be a copy of $\widetilde{P}_1^{(\alpha)}$. Let $\widetilde{M}^{(\alpha)}$ be the 3-manifold obtained by gluing together the $\widetilde{P}_i^{(\alpha)}$ so that, for each $i \in \mathbb{Z}$, the positive end of $P_i^{(\alpha)}$ glues to the negative end of $\widetilde{P}_{i+1}^{(\alpha)}$. By the triviality of $\widetilde{\ell}_v^{(\alpha)}$ on $\widetilde{S}_{-1}$ for v small, the flows $\widetilde{\ell}_v^{(\alpha)}$ on each $\widetilde{P}_i^{(\alpha)}$ patch together to form a flow on $\widetilde{M}^{(\alpha)}$, which we will denote by $\widetilde{f}_v^{(\alpha)}$.

Figure 3 illustrates an example of $\widetilde{M}^{(\alpha)}$. The slanted lines are the images of $q_{i\alpha}^{(\alpha)}$, and the locus where $\widetilde{\ell}_v$ is not trivial is thickened.

Define

$$
\begin{aligned}
\widetilde{T}^{(\alpha)} : \widetilde{M}^{(\alpha)} &\to \widetilde{M}^{(\alpha)} \\
(x)_j &\mapsto (x)_{j-1}
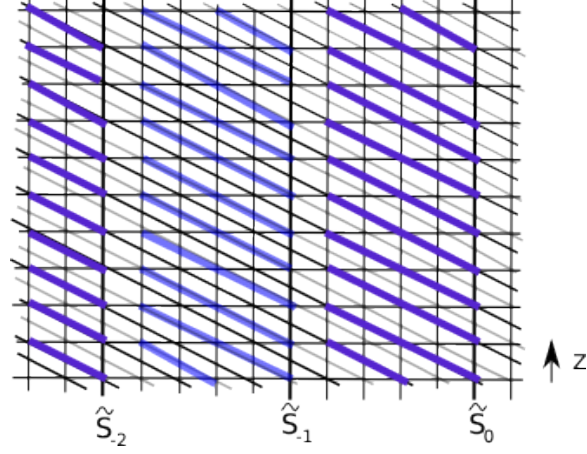\end{aligned}
$$

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Figure 3: The manifold $\widetilde{M}^{(\frac{2}{5})}$ and flow $f_v^{(\frac{2}{5})}$ with $m_0 = 2$.

for $(x)_j \in \widetilde{P}_j^{(\alpha)}$ and $j \in \mathbb{Z}$.

For $\alpha = \frac{k}{m} > 0$, define

$$\widehat{T}^{(\alpha)} : \widetilde{P} \to \widetilde{P}$$

so that for $0 < v < \frac{1}{m}$, and $s \in \widetilde{S}$,

$$\widehat{T}^{(\alpha)}(q_v^{(\alpha)}(s)) = q_{v-\frac{1}{m}}^{(\alpha)}(s),$$

and

$$\widehat{T}^{(\alpha)}((q_{i\frac{1}{m}}^{(\alpha)})^+(s)) = (q_{(i-1)\frac{1}{m}}^{(\alpha)})^+(s).$$

The map $\widehat{T}^{(\alpha)}$ is only right continuous with respect to v, and its locus of discontinuity lies in

$$\bigcup_{i \in \mathbb{Z}} \widetilde{Y}_{\frac{i}{m}}^{(\alpha)}.$$

Lemma 3.5 *The maps $Z^{(\alpha)}$ and $\widehat{T}^{(\alpha)}$ commute.*

Proof. For $0 \le v < \frac{1}{m}$, $s \in \widetilde{S}$, we have

$$
\begin{aligned}
Z^{(\alpha)}\widehat{T}^{(\alpha)}(\widetilde{f}_v q_{\frac{i}{m}}(s)) &= Z^{(\alpha)}\widetilde{f}_v q_{\frac{i-1}{m}}^{(\alpha)}(s) \\
&= \widetilde{f}_v Z^{(\alpha)} q_{\frac{i-1}{m}}^{(\alpha)}(s) \\
&= \widetilde{f}_v q_{\frac{i+k-1}{m}}^{(\alpha)}(\zeta(s)) \\
&= \widehat{T}^{(\alpha)}(\widetilde{f}_v q_{\frac{i+k}{m}}^{(\alpha)}(\zeta(s))) \\
&= \widehat{T}^{(\alpha)} Z^{(\alpha)}(\widetilde{f}_v q_{\frac{i}{m}}^{(\alpha)}(s)).
\end{aligned}
$$

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$\square$

The map $\widehat{T}^{(\alpha)}$ on $\widetilde{P}_1$ determines a unique map $\widehat{T}^{(\alpha)} : \widetilde{M}^{(\alpha)} \to \widetilde{M}^{(\alpha)}$ that commutes with $\widetilde{T}^{(\alpha)}$. Let $\widehat{\phi}_{m\text{-rep}} : \widetilde{S} \to \widetilde{S}$ be the composition

$$\widehat{\phi}_{m\text{-rep}} := \underset{i \in \mathbb{Z}}{\circ}\, \widehat{\phi}_{mi}.$$

This is well-defined since the ϕ_{mi} have disjoint supports for $i \in \mathbb{Z}$, and we have

$$\widehat{\phi}^{-1}_{m\text{-rep}} = \underset{i \in \mathbb{Z}}{\circ}\, \widehat{\phi}^{-1}_{mi}.$$

Lemma 3.6 *For small $0 \leq v < \frac{1}{m}$ we have*

$$\widetilde{f}^{(\alpha)}_{\frac{1}{m}} \widehat{T}^{(\alpha)}(q^{(\alpha)}_v(s)) = q^{(\alpha)}_v(\widehat{\phi}^{-1}_{m\text{-rep}}(s)).$$

Proof. On $\widetilde{P}_1$, we have, for $s \in \widetilde{S}$ and $0 \leq v < \frac{1}{m}$,

$$
\begin{aligned}
\widetilde{f}^{(\alpha)}_{\frac{1}{m}} \widehat{T}^{(\alpha)}(\widetilde{f}^{(\alpha)}_v q^{(\alpha)}_{i\frac{1}{m}}(s))
&= \widetilde{f}^{(\alpha)}_{\frac{1}{m}} \widehat{T}^{(\alpha)} \widetilde{q}^{(\alpha)}_{i\frac{1}{m}+v}(\widehat{\phi}^{-1}_i(s)) \\
&= \widetilde{f}^{(\alpha)}_{\frac{1}{m}} q^{(\alpha)}_{(i-1)\frac{1}{m}+v}(\widehat{\phi}^{-1}_i(s)) \\
&= \widetilde{f}^{(\alpha)}_v q^{(\alpha)}_{i\frac{1}{m}}(\widehat{\phi}^{-1}_i(s)).
\end{aligned}
$$

The rest follows by commutativity of $\widehat{T}^{(\alpha)}$ and $\widetilde{f}^{(\alpha)}_v$ with $\widetilde{T}^{(\alpha)}$. $\square$

Although $\widehat{T}^{(\alpha)}$ is not continuous, the m-th power of $\widehat{T}^{(\alpha)}$ is continuous as we see in the following.

Lemma 3.7 *For $\alpha = \frac{k}{m} \in J_{m_0}$,*
$$\widetilde{T}^{(\alpha)} = (\widehat{T}^{(\alpha)})^m.$$

Let $X^{(\alpha)}_v = \widetilde{Y}^{(\alpha)}_v \cap \widetilde{P}_1$. Since $\widetilde{T}^{(\alpha)}$ sends $X^{(\alpha)}_1 \cap \widetilde{S}_0$ to $X^{(\alpha)}_0 \cap \widetilde{S}_{-1}$,

$$\widetilde{X}^{(\alpha)}_0 = \bigcup_{i \in \mathbb{Z}} (\widetilde{T}^{(\alpha)})^i\, X^{(\alpha)}_i$$

is a subsurface of $\widetilde{M}^{(\alpha)}$, and for each $v \in \mathbb{R}$

$$\widetilde{X}^{(\alpha)}_v = \bigcup_{i \in \mathbb{Z}} (\widetilde{T}^{(\alpha)})^i\, X^{(\alpha)}_{v+i} = \widetilde{f}^{(\alpha)}_v \widetilde{X}^{(\alpha)}_0.$$

The following is a direct consequence of the definitions.

Lemma 3.8 *The map $\widetilde{T}^{(\alpha)}$ sends each $\widetilde{X}^{(\alpha)}_v$ to $\widetilde{X}^{(\alpha)}_{v-1}$.*

Corollary 3.9 *The surfaces $\widetilde{X}_v^{(\alpha)}$ are stabilized by the action of $(\widetilde{T}^{(\alpha)})^k (Z^{(\alpha)})^m$.*

Proof. By Lemma 3.2, the map $(Z^{(\alpha)})^m$ sends $\widetilde{X}_v^{(\alpha)}$ to $\widetilde{X}_{v+k}^{(\alpha)}$. The rest follows from Lemma 3.8. $\square$

Define

$$R^{(\alpha)} = Z^{(\alpha)} (\widehat{T}^{(\alpha)})^k.$$

Lemma 3.10 *The map $R^{(\alpha)}$ preserves each $\widetilde{X}_v^{(\alpha)}$ and we have, for $s \in \widetilde{S}$,*

$$R^{(\alpha)}(q_v^{(\alpha)}(s)) = q_v^{(\alpha)}(\zeta(s)).$$

Proof. Take $s \in \widetilde{S}$. By Lemma 3.2, we have

$$
\begin{aligned}
R^{(\alpha)}(q_v^{(\alpha)}(s)) &= Z^{(\alpha)}(\widehat{T}^{(\alpha)})^k (q_v^{(\alpha)}(s)) \\
&= Z^{(\alpha)}(q_{v-k}^{(\alpha)}(s)) \\
&= q_v^{(\alpha)}(\zeta(s)).
\end{aligned}
$$

$\square$

We call $(\widetilde{M}^{(\alpha)}, Z, \widehat{T}^{(\alpha)}, \widetilde{f}_v^{(\alpha)})$ the *covering manifold* constructed from $(\widetilde{S}, \zeta, \widehat{\phi}, q^{(\alpha)})$.

Independence of α.

We will show that $(\widetilde{M}^{(\alpha)}, Z, \widetilde{T}^{(\alpha)}, \widetilde{f}_v^{(\alpha)})$ are independent of $\alpha \in (0, \frac{1}{m_0})$, and hence define a single covering $\widetilde{M} \to M$ with covering automorphisms generated by $Z^{(\alpha)}$ and $\widetilde{T} = \widetilde{T}^{(\alpha)}$, and a flow $\widetilde{f}_v = \widetilde{f}_v^{(\alpha)}$.

Lemma 3.11 *For each $\alpha, \alpha' \in (0, \frac{1}{m_0})$, there is a homeomorphism*

$$h : \widetilde{M}^{(\alpha)} \to \widetilde{M}^{(\alpha')}$$

such that

(1) $h \circ Z = Z \circ h$,

(2) $h \circ \widetilde{f}_v^{(\alpha)} = \widetilde{f}_v^{(\alpha')} \circ h$, for $v \in \mathbb{R}$, and

(3) $h \circ \widetilde{T}^{(\alpha)} = \widetilde{T}^{(\alpha')} \circ h$.

Proof. Define a homeomorphism

$$h_1 : \widetilde{N}_1 \to \widetilde{N}_1$$

by

$$h_1(s,v) = \begin{cases} (s, (v+1)(\frac{1-\alpha' m_0}{1-\alpha m_0}) - 1) & \text{if } v \in [-1, -\alpha m_0] \\ (s, -\alpha' m_0 + \frac{\alpha'}{\alpha}(v + \alpha m_0)) & \text{if } v \in [-\alpha m_0, 0] \end{cases}$$

Then $h_1(\widetilde{X}^{(\alpha)}) = \widetilde{X}^{(\alpha')}$, and h_1 commutes with Z. Thus h_1 also defines a homeomorphism

$$h_1 : \widetilde{P}^{(\alpha)} \to \widetilde{P}^{(\alpha')}$$

that commutes with Z and satisfies

$$h_1 f_v^{(\alpha)} = f_v^{(\alpha')} h_1.$$

Since the map h_1 is the identity when restricted to $\widetilde{S}_i$, for $i = -1, 0$, the map extends to a homeomorphism $h : \widetilde{M}^{(\alpha)} \to \widetilde{M}^{(\alpha')}$ with the desired properties. $\qquad \square$

We will say $(\widetilde{M}, Z, \widetilde{T}, \widetilde{f}_v)$ is the *covering manifold* defined by $(\widetilde{S}, \zeta, \widehat{\phi}, q^{(\alpha)})$, for $\alpha \in J_{m_0}$.

Linear section of a fibered face of M.

Let H be the group of covering automorphisms of $\widetilde{M}$ over M. This is generated by $\widetilde{T}$ and Z. The duals ψ and β of $\widetilde{T}$ and Z generate a free abelian group of rank 2 in $H^1(M; \mathbb{Z})$. Let

$$\begin{aligned} A : J_{m_0} &\to H^1(M; \mathbb{Z}) \\ \alpha = \frac{k}{m} &\mapsto \psi_\alpha = m\psi + k\beta \end{aligned}$$

The map A determines a continuous map

$$\overline{A} : (0, \frac{1}{m_0}) \to H^1(M; \mathbb{R}),$$

whose images lie on fibered faces of the Thurston norm ball. That is $\overline{A}$ is defined by

$$\overline{A}(\alpha) = \frac{\psi_\alpha}{||\psi_\alpha||}.$$

We prove Theorem 1.2, in two steps. First we will show that for $\alpha \in J_{m_0}$, (S_α, ϕ_α) is the monodromy of ψ_α. Next we show that the map $\overline{A}$ extends continuously over 0, and $\overline{A}(0)$ is fibered if and only if $(\widetilde{S}, \zeta, \widehat{\phi})$ is of Type I.

Fix $\alpha = \frac{k}{m} \in J_{m_0}$. Since $\psi(\widetilde{T}) = -1$ and $\beta(Z) = 1$ by construction, $\widetilde{T}^k Z^m$ generates the kernel of α. Thus, by Corollary 3.9, the kernel of α stabilizes $\widetilde{X}_v^{(\alpha)}$ for all $v \in \mathbb{R}$.

Let $\Phi^{(\alpha)} \in H$ so that

(i) $\alpha(\Phi^{(\alpha)}) = -1$, and

(ii) $\widetilde{f}_{\frac{1}{m}}^{(\alpha)} \Phi^{(\alpha)}$ preserves $\widetilde{Y}_v^{(\alpha)}$ for all v.

Lemma 3.12 *The mth power $(R^{(\alpha)})^m$ generates the kernel of ψ_α restricted to H and preserves $\widetilde{X}_v \subset \widetilde{M}$ for all $v \in \mathbb{R}$.*

Proof. By Lemma 3.7

$$
\begin{aligned}
(R^{(\alpha)})^m &= (Z(\widehat{T}^{(\alpha)})^k)^m \\
&= Z^m(\widehat{T}^{(\alpha)})^{km} \\
&= Z^m\widetilde{T}^k
\end{aligned}
$$

$\square$

Let $a, b \in \mathbb{Z}$ be solutions to

$$ak + bm = 1.$$

and define

$$\Phi^{(\alpha)} = Z^{-a}\widetilde{T}^b \in H.$$

Then $\Phi^{(\alpha)}$ is a solution to

$$\alpha(\Phi^{(\alpha)}) = -1.$$

Lemma 3.13 *We have $\Phi^{(\alpha)} = \widehat{T}^{(\alpha)}(R^{(\alpha)})^{-a}$ and hence the surfaces $\widetilde{X}_v^{(\alpha)}$ are stabilized by $\widetilde{f}_{\frac{1}{m}}^{(\alpha)}\Phi^{(\alpha)}$.*

Proof. We have

$$
\begin{aligned}
\Phi^{(\alpha)} &= Z^{-a}\widetilde{T}^b \\
&= Z^{-a}(\widehat{T}^{(\alpha)})^{bm} \\
&= Z^{-a}(\widehat{T}^{(\alpha)})^{1-ak} \\
&= \widehat{T}^{(\alpha)}(R^{(\alpha)})^{-a}.
\end{aligned}
$$

Thus, for all $v \in \mathbb{R}$, $\Phi^{(\alpha)}$ maps $\widetilde{X}_v^{(\alpha)}$ to $\widehat{T}^{(\alpha)}(\widetilde{X}_v^{(\alpha)})$. Since $\widetilde{f}_{\frac{1}{m}}^{(\alpha)}\widehat{T}^{(\alpha)}$ preserves $\widetilde{X}_v^{(\alpha)}$ the claim follows.
$\square$

Let $\Phi^{(-\alpha)} = Z^a\widetilde{T}^b$. Then $\psi_{-\alpha}(\Phi^{(-\alpha)}) = -1$. Let $R^{(-\alpha)} = Z(\widehat{T}^{(-\alpha)})^k$. Then $R^{(-\alpha)}$ preserves $\widetilde{X}_v^{(-\alpha)}$ and

$$R^{(-\alpha)}q_v^{(-\alpha)}(s) = q_v^{(-\alpha)}(\zeta(s)).$$

Lemma 3.14 *We have $\Phi^{(-\alpha)} = (\widehat{T}^{(-\alpha)})^{-1}(R^{(-\alpha)})^a$ and hence the surfaces $\widetilde{X}_v^{(-\alpha)}$ are stabilized by $\widetilde{f}_{\frac{1}{m}}^{(-\alpha)}\Phi^{(-\alpha)}$.*

Proof. Analogously to Lemma 3.13, we have

$$
\begin{aligned}
\Phi^{(-\alpha)} &= Z^a \widetilde{T}^b \\
&= Z^a (\widehat{T}^{(-\alpha)})^{-bm} \\
&= Z^a (\widehat{T}^{(-\alpha)})^{ak-1} \\
&= (\widehat{T}^{(-\alpha)})^{-1} (R^{(-\alpha)})^a.
\end{aligned}
$$

Proof of Theorem 1.2. Take $s \in \widetilde{S}$. Then

$$
\begin{aligned}
\widetilde{f}_{\frac{1}{m}}^{(\alpha)} \Phi^{(\alpha)} q^{(\alpha)}(s) &= \widetilde{f}_{\frac{1}{m}}^{(\alpha)} \widehat{T}^{(\alpha)} (R^{(\alpha)})^{-a} q^{(\alpha)}(s) \\
&= \widetilde{f}_{\frac{1}{m}}^{(\alpha)} \widehat{T}^{(\alpha)} q^{(\alpha)} (\zeta^{-a}(s)) \\
&= q^{(\alpha)} (\widehat{\phi}^{-1} \zeta^{-a}(s)).
\end{aligned}
$$

It follows that ϕ_α lifts to $\zeta^a \circ \widehat{\phi}_m$-rep, and hence $\phi_\alpha = r^a \eta$, where r and η are the maps on S_α induced by R and $\widehat{\phi}_m$-rep.

We now show that we can complete the map $\overline{A}$ continuously over 0.

Suppose $(\widetilde{S}, \zeta, \widehat{\phi})$ is of finite type. Then

$$
(\zeta \widehat{\phi})^m = \zeta^{m - m_0} (\zeta \widehat{\phi})^{m_0},
$$

for all $m > m_0$. Let $\widehat{\phi}_i = \zeta^i \widehat{\phi} \zeta^{-i}$. Fix $m > m_0$.

Lemma 3.15 *The $\widehat{\phi}_i$ have the following property that*

$$
\widehat{\phi}_{i-m+1} \cdots \widehat{\phi}_i = \widehat{\phi}_{i-m_0+1} \cdots \widehat{\phi}_i
$$

for all $m \geq m_0$ and $i \in \mathbb{Z}$.

Proof.

$$
\begin{aligned}
\widehat{\phi}_{-m+1} \cdots \widehat{\phi}_0 &= \zeta^{-m+1} \widehat{\phi} \zeta^{m-1} \zeta^{-m+2} \widehat{\phi} \zeta^{m-2} \cdots \widehat{\phi} \\
&= \zeta^{-m} (\zeta \widehat{\phi})^m \\
&= \zeta^{-m_0} (\zeta \widehat{\phi})^{m_0} \\
&= \widehat{\phi}_{-m_0+1} \cdots \widehat{\phi}_0
\end{aligned}
$$

The rest follows by conjugating by ζ^i. $\square$

Let $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ be defined so that if $s \in \Sigma_i$, then

$$
\widetilde{\phi}(s) = \widehat{\phi}_{i-m+1} \cdots \widehat{\phi}_i(s).
$$

This is well-defined for $m > m_0$, since if $s \in \tau_i^+ = \tau_{i+1}^-$, then $\widehat{\phi}_{i+1}(s) = s$, and

$$
\begin{aligned}
\widehat{\phi}_{i-m+2} \cdots \widehat{\phi}_{i+1}(s) &= \widehat{\phi}_{i-m+2} \cdots \widehat{\phi}_i(s) \\
&= \widehat{\phi}_{i-m_0+1} \cdots \widehat{\phi}_i(s).
\end{aligned}
$$

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Lemma 3.16 *The maps ζ and $\widetilde{\phi}$ commute as functions on $\widetilde{S}$.*

Proof. Let $s \in \Sigma$. Then we have

$$
\begin{aligned}
\zeta\widetilde{\phi}(s) &= \zeta\widehat{\phi}_{-m_0+1}\widehat{\phi}_{-m_0+1}\cdots\widehat{\phi}_0(s) \\
&= \widehat{\phi}_{-m_0+2} \circ \cdots \circ \widehat{\phi}_0 \circ \widehat{\phi}_1(\zeta(s)) \\
&= \widetilde{\phi}(\zeta(s)).
\end{aligned}
$$

$\square$

Let $S = \widetilde{S}/\zeta$ and let $\phi : S \to S$ be the mapping class determined by $\widetilde{\phi}$.

Lemma 3.17 *The cohomology class ψ is a fibered element satisfying*

$$
\frac{\psi}{||\psi||} = \lim_{m \to \infty} \overline{A}(\frac{1}{m})
$$

on F, and (S, ϕ) is the associated monodromy.

Proof. The projection of ψ to F is the limit of the projections $A(\frac{1}{m})$ of

$$
m\psi + \beta, \qquad m > m_0
$$

on F and hence equals $\frac{\psi}{||\psi||}$. The element $\widetilde{T}$ satisfies $\psi(\widetilde{T}) = -1$, and sends $\widetilde{S}_i$ to $\widetilde{S}_{i-1}$. For $m > m_0$ and $s \in \widetilde{S}_0$,

$$
f_1^{(\frac{1}{m})}\widetilde{T}(s) = \widetilde{\phi}(s).
$$

Thus, (S, ϕ) is the monodromy associated to ψ, and the claim follows. $\square$

Remark 3.18 In the language of deformations in $\mathcal{P}$, if $(\widetilde{S}, \zeta, \widehat{\phi})$ is of finite type, and (S, ϕ) is pseudo-Anosov, then the mapping classes $(S_{\frac{1}{m}}, \phi_{\frac{1}{m}})$ converge to (S, ϕ) in $\mathcal{P}$. More generally, if α_n is a sequence in J_{m_0} converging to 0, then $(S_{\alpha_n}, \phi_{\alpha_n})$ converge to (S, ϕ) in $\mathcal{P}$.

Suppose $(\widetilde{S}, \zeta, \widehat{\phi})$ is of infinite type. In this case, there is no v such that $\widetilde{f}_v\widetilde{T}$ stabilizes $\widetilde{S}_i$. Thus, ψ does not lie in the cone over F, and $\frac{\psi}{||\psi||}$ lies on the boundary point of F. It follows that

$$
\lim_{\alpha \to 0} L(S_\alpha, \phi_\alpha) = \infty.
$$

This completes the proof of Theorem 1.2.

Remark 3.19 Our construction partially extends to the case when $\alpha \in (0, \frac{1}{m_0})$ is irrational. In this case, we can still define cross-sectional surfaces $\widetilde{Y}_v^{(\alpha)}$ and a transversal flow $\widetilde{f}_v$. These give a continuous interpolation of the monodromies of $\widetilde{M}$ over $\mathbb{R}$ obtained as lifts of (S_α, ϕ_α) for rational α (cf. [McM1]).

4 Teichmüller polynomial

In this section we briefly review an explicit method for computing dilatations for mapping classes on fibered faces using train tracks and Teichmuller polynomials. The results of Section 3 imply that mapping classes in quotient families have dilatations determined by a single Laurent polynomial. In Section 5 we will use Teichmüller polynomials to explicitly compute dilatations for examples.

4.1 Computing geometric dilatations using train tracks and the Teichmüller polynomial

The theory of train tracks was developed by W. Thurston (see [FLP, BH]). A simple algorithm for finding dilatations on fibered faces using Teichmüller polynomials is described in [McM1] (see also [LO]).

A *train track* t is a topological graph embedded on a surface S so that

i) each edge determines well-deifned tangent vectors at its endpoints, and

(ii) for each pair of edges meeting at a vertex v, the forward unit tangent vectors coincide up to sign.

If two edges meet at a vertex, and their tangent vectors agree, we say the edges meet at a *cusp*. Otherwise, they meet *smoothly*. A closed curve γ on t is *smooth* if whenever it approaches and leaves vertices along edges that meet smoothly. A simple closed curve γ on S is *carried* on t if γ is isotopic to a smooth closed curve on t.

By work of Thurston [FLP, BH], if (S, ϕ) is pseudo-Anosov, there is a train track t with the properties:

(i) the connected regions of the complement of t in S are homeomorphic to either disks or boundary parallel annuli;

(ii) if γ is carried on t, then $\phi(\gamma)$ is also carried on t;

(iii) for any essential closed curve $\gamma \subset S$, the image under iteration of ϕ, $\phi^n(\gamma)$ is eventually carried on t;

(iv) for large enough n, $\phi^n(\gamma)$ passes over every edge of t; and

(v) for every simple closed curve γ carried on t, $\phi(\gamma)$ is also carried on t.

If t satisfies the above conditions, we say that t is *compatible* with (S, ϕ).

Let t be compatible with (S, ϕ). If $\mathcal{B} = \{b_1, \dots, b_n\}$ are the branches (or edges) of t, let $V_\mathsf{t} = \mathbb{R}^\mathcal{B}$ be the vector space of real labels on $b_1, \dots, b_n$. Then each smooth curve γ on V_t determines a vector

v_γ with integer labels corresponding to the number of times γ passes over b_i. We call these labels the *geometric edge weights* of γ. Let $W_\mathfrak{t} \subset V_\mathfrak{t}$ be the subspace of weights spanned by vectors v_γ. We can think of $W_\mathfrak{t}$ as virtual curves on S. For a smooth closed curve γ on $\mathfrak{t}$, let $\phi_*(v_\gamma)$ be the vector corresponding to the isotopy class of $\phi(\gamma_v)$. This deifnes a linear map

$$\phi_* : W_\mathfrak{t} \to W_\mathfrak{t}.$$

Written with respect to any basis of $W_\mathfrak{t}$ consisting of vectors of the form v_γ, ϕ_* has non-negative entries. Since ϕ is pseudo-Anosov, some power of ϕ_* has strictly positive entries. That is, ϕ_* is a Perron-Frobenius matrix. The dilatation $\lambda(S, \phi)$ is the Perron-Frobenius eigenvalue of ϕ_*.

Let (S, ϕ) be pseudo-Anosv and suppose $\widetilde{S}$ is a regular covering of S with group of covering auto-morphism equal to G, such that ϕ lifts to $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$. Let $\mathfrak{t}$ be a train track compatible with (S, ϕ), and let $\widetilde{\mathfrak{t}}$ be a lift to $\widetilde{S}$. Then $\widetilde{\mathfrak{t}}$ is compatible with $(\widetilde{S}, \widetilde{\phi})$.

Let $\gamma_1, \ldots, \gamma_n$ closed curves on S such that $v_{\gamma_1}, \ldots, v_{\gamma_n}$ span $W_\mathfrak{t}$ and the set theoretic union of the γ_i is connected. Let $\widetilde{\gamma}_1, \ldots, \widetilde{\gamma}_n$ be a connected choice of path lifts to $\widetilde{S}$. These are smooth paths on $\widetilde{\mathfrak{t}}$, but typically not closed, and the set

$$\mathcal{G} = \{g\widetilde{\gamma}_i \; : \; i = 1, \ldots, n \quad \text{and} \quad g \in G\}$$

spans $W_{\widetilde{\mathfrak{t}}}$. Let $\widetilde{W}$ be the $\mathbb{R}G$ module generated by the elements in $\mathcal{G}$. Since $\widetilde{\phi}$ commutes with G, $\widetilde{\phi}_*$ defines an $\mathbb{R}G$-module homomorphism on $\widetilde{W}$. The characteristic polynomial $\Theta(\mathrm{u})$ is a polynomial with coefficients in $\mathbb{R}G$.

In particular, consider the case when G is free abelian. Let M be the mapping torus of (S, ϕ), and let $\widetilde{M} = \widetilde{S} \times \mathbb{R}$. Let $\widetilde{T}$ be the covering automorphism of $\widetilde{M}$ defined by

$$\widetilde{T}(x, t) = (\widetilde{\phi}(x), t - 1).$$

Then $\widetilde{T}$ and G generate a discrete properly discontinuous action H on $\widetilde{M}$, and $M = \widetilde{M}_H$. Let $F \subset H^1(M; \mathbb{Z})$ be the fibered face containing the homomorphisms $H_1(M; \mathbb{Z}) \to \mathbb{Z}$ defined by the corresponding fibration ψ. Then H defines a linear section F_H of F, and rational points on F_H are in one-to-one correspondence with epimomorphisms

$$\alpha : H \to \mathbb{Z}.$$

For each α, let (S_α, ϕ_α) be the corresponding monodromy.

Given an element $f \in \mathbb{Z}H$, and $\alpha : H \to \mathbb{Z}$, the *specialization* of

$$f = \sum_{g \in H} c_g g$$

at α is defined to be

$$f^{(\alpha)}(x) = \sum_{g \in H} c_g x^{\alpha(g)}.$$

The *house* of a polynomial $P(x)$ is defined by

$$|P| = \max\{|\mu| \; : \; \mu \in \mathbb{C}, P(\mu) = 0\}.$$

Theorem 4.1 (C. McMullen [McM1]) *For each α, the dilatation $\lambda(\phi_\alpha)$ satisfies*

$$\lambda(\phi_\alpha) = |\Theta^{(\alpha)}|.$$

Remark 4.2 The Θ defined above is a multiple of the Teichmüller polynomial defined in [McM1] by cyclotomic polynomials. These cyclotomic polynomials come from the action of ϕ_α on the vertices of the train track. Since the polynomials are cyclotomic and don't effect the house of the polynomial.

4.2 Homological dilatation

The homological dilatation of a mapping class (S, ϕ) is the largest eigenvalue of the linear map on the first homology of S

$$\phi_* : H_1(S; \mathbb{R}) \to H_1(S; \mathbb{R})$$

induced by ϕ. Let $\mathfrak{t}$ be a train track compatible with (S, ϕ). Choose $\gamma_1, \ldots, \gamma_n$ to be closed curves carried on $\mathfrak{t}$ whose homology classes generate $H_1(S; \mathbb{R})$. These can be obtained, for example, by iterating ϕ on any integral basis for $H_1(S; \mathbb{R})$.

Fix an orientation for each of the edges of $\mathfrak{t}$. For each $i = 1, \ldots, n$, let $w_i \in V_{\mathfrak{t}}$ be the sum of signed weights for γ_i on $\mathfrak{t}$, where a path traversing an edge in the opposite direction from the orientation contributes a negative one to the weight on that edge. We call the edge weights for γ_i the *algebraic edge weights*. The span W of $\{w_1, \ldots, w_n\}$ in $V_{\mathfrak{t}}$ is isomorphic to $H_1(S; \mathbb{R})$, and the action of ϕ_* on $H_1(S; \mathbb{R})$ defines a transformation on $V_{\mathfrak{t}}$ that preserves W.

The train track $\mathfrak{t}$ is *orientable* if one can choose orientations on the edges of $\mathfrak{t}$ so that

(i) for any pair of edges meeting smoothly at a vertex of $\mathfrak{t}$, the edges are oriented so that one points in and the other points out;

(ii) for any pair of edges meeting in a cusp at a vertex of $\mathfrak{t}$, the edges are oriented so that either both point into the vertex or both point out.

A pseudo-Anosov mapping class is *orientable* if its stable and unstable foliations are orientable.

Lemma 4.3 *A train track compatible with a pseudo-Anosov mapping class (S, ϕ) is orientable if and only if (S, ϕ) is orientable.*

Proof. If $\mathfrak{t}$ is orientable, then we may orient the generating curves $\gamma_1, \ldots, \gamma_n$ so that their corresponding algebraic weights are all non-negative, and equal the geometric weights. It follows that $\lambda_{\text{hom}}(\phi) = \lambda\text{geo}(\phi)$, and hence the (S, ϕ) is orientable. Conversely, if (S, ϕ) is orientable, then the orientation of the stable foliation determines an orientation on any compatible train track. $\qquad \square$

4.3 Families of digraphs and Teichmüller polynomials

A *digraph* is a finite directed graph Γ. It is *Perron-Frobenius*, or a *PF-digraph* if, for large enough n, its directed adjacency matrix T_Γ satisfies $(T_\Gamma)^n > 0$. For example, if (S, ϕ) is a pseudo-Anosov mapping class with associated train track $\mathfrak{t}$, the induced linear transformation ϕ_* defines a PF-digraph. Some useful references for digraphs are [Gan, Kit].

The following theorem is well-known.

Theorem 4.4 (Coefficient Theorem for Digraphs) *Let Γ be a digraph with m vertices. Let $b_{s,\ell}$ be the number of distinct linear subgraphs of Γ of size s and length ℓ. Then*

$$P_\Gamma(x) = x^m + \sum_{s,\ell}(-1)^s b_{s,\ell} x^{m-\ell}.$$

Let Γ be a digraph with m vertices such that each edge e is labeled by g_e for some $g_e \in G = \mathbb{Z}^k$. Assume that for all self-loops e, $g_e = (0, \ldots, 0)$ is the trivial element. From Γ we define a family of digraphs Γ_α parameterized by homomorphisms

$$\alpha : \mathbb{Z}^k \to \mathbb{Z}$$

as follows. For each edge e, let $g_e \in G$ be its label. Let Γ_α be the graph obtained from Γ by making $\alpha(g_e)$ subdivisions of the edge e, for each e. Then each edge e is replaced by $\alpha(g_e) + 1$ edges connected end-to-end.

Lemma 4.5 *The graphs Γ_α have the following properties.*

(i) *Every linear subgraph of Γ_α is of the form L_α for some linear subgraph L of Γ.*

(ii) *The total number of vertices of Γ_α is*

$$M = m + \sum_e \alpha(g_e),$$

where e ranges over the edges of L.

Let $T_\Gamma = [f_{i,j}]$ be the $m \times m$ matrix with entries

$$F_{i,j} = \sum_e g_e^{-1},$$

where the sum is taken over edges e from v_i to v_j. Let $\Theta_\Gamma(x)$ be the characteristic polynomial of T_Γ.

For any element $F = \sum_g a_g g \in \mathbb{Z}G$, the specialization of F at $\alpha : G \to \mathbb{Z}$ is defined by

$$F^{(\alpha)}(x) = \sum_\ell a_g x^{\alpha(g)}.$$

Theorem 4.6 *The PF-eigenvalue of Γ_α satisfies*

$$\lambda(\Gamma_\alpha) = |\Theta^{(\alpha)}|.$$

Proof. Let c be a cycle on Γ, let c_α be the corresponding cycle on Γ_α. Then if $e_1, \ldots, e_\ell$ are the edges in σ, the length of $c^{(\alpha)}$ is the sum

$$\ell(c^{(\alpha)}) = \ell + \alpha(g_{e_1}) + \alpha(g_{e_2}) + \cdots \alpha(g_{s_{e_\ell}}).$$

Define

$$g(c) = g_{e_1} + \cdots + g_{e_{s_\ell}}.$$

For each linear subgraph $L = [c_1, \ldots, c_{s_L}]$ of Γ, let

$$g(L) = \sum_{i=1}^{s_L} g(c).$$

Then

$$L_\alpha = [(c_1)_\alpha, \ldots, (c_{s_L})_\alpha],$$

and let

Then

$$\ell(L_\alpha) = \ell(L) + \sum \alpha(g(c)),$$

where we extend α to a linear map on $\mathbb{Z}G$.

By Lemma 4.5 and Theorem 4.4, we have

$$
\begin{aligned}
\lambda(\Gamma_\alpha) &= \left| \sum_L (-1)^{s(L_\alpha)} x^{M - \ell(L_\alpha)} \right| \\
&= \left| x^M \left(\sum_{L \in \mathcal{L}(\Gamma)} (-1)^{s(L)} x^{-\ell(L_\alpha)} \right) \right| \\
&= \left| x^M \left(\sum_{L \in \mathcal{L}(\Gamma)} (-1)^{s(L)} x^{-\ell(L)} x^{-\sum_i \alpha(g(c_i))} \right) \right| \\
&= \left| x^M \left(\sum_{L \in \mathcal{L}(\Gamma)} (-1)^{s(L)} (-1)^{\ell(L)} \prod_{i=1}^{s_L} (-g(c_i)) x^{-\ell(L)} \right)^{(\alpha)} \right| \\
&= \left| x^M \left(\sum_{L \in \mathcal{L}(\Gamma)} \mathfrak{s}(\sigma(L)) \prod_{i=1}^{s_L} (g(c_i)^{-1} x^{-\ell(L)})^{(\alpha)} \right| \\
&= \left| x^{M-m} \left(\sum_{L \in \mathcal{L}(\Gamma)} \mathfrak{s}(\sigma(L)) \prod_{i=1}^{s_L} (g(c_i)^{-1} x^{m-\ell(L)})^{(\alpha)} \right| \\
&= \left| x^{M-m} (\Theta_\Gamma^{(\alpha)}(x)) \right|.
\end{aligned}
$$

$\square$

5 Examples

In this section we study some explicit quotient families of Type I and Type II. Since quotient families are associated to linear sections of fibered faces, it is easy to compute Alexander and Teichmüller polynomials to compute homological and geometric dilatations.

5.1 Type I quotient familiy

The Type I quotient families of mapping classes described in Section 3 are generalizations of examples of Penner in [Pen] (see also, [Bau, Tsa, Val]). Penner showed such mapping classes are pseudo-Anosov and have bounded normalized dilatations by analyzing the transition matrices. In [Val], Valdivia proves that certain sequences of mapping classes generalizing Penner's examples are the monodromy of a single 3-manifold. Our results in Section 3 imply that we can obtain Penner's sequence, and its generalizations as the monodromy of $(A(\frac{1}{m}))$ for some triple $(\widetilde{S}, \zeta, \widehat{\phi})$ of finite type.

Since the quotient family $\mathcal{Q} = \mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$ is of finite type. Let (S, ϕ) be the associated minimal mapping class (see Lemma 3.17). Let $A : (-\frac{1}{m_0}, \frac{1}{m_0}) \to F$ be the associated parameterization of $\mathcal{Q}$. The Teichmüller polynomial Θ can be computed from a train track on (S, ϕ), and determines the dilatations of the mapping classes in $\mathcal{Q}$.

In [Pen] Penner defined an explicit sequence of mapping classes (see Figure 4) $(S_g, \phi_g) \in \mathcal{P}$ where S_g are closed surfaces of genus g and such that the normalized dilatations

$$\lambda(\phi_g)^g$$

is bounded, thus showing that

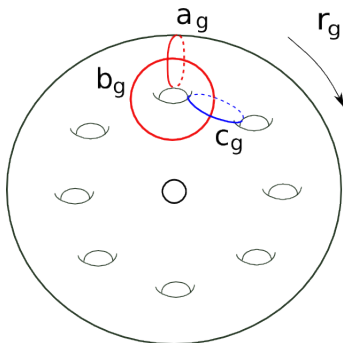$$\log(\lambda(\phi_g)) \asymp \frac{1}{g}.$$

Figure 4: Penner's original example.

The surface S_g has genus g and two boundary components. The mapping class ϕ_g is the composition of Dehn twists along the curves a_g, b_g and c_g with a rotation r_g of period g.
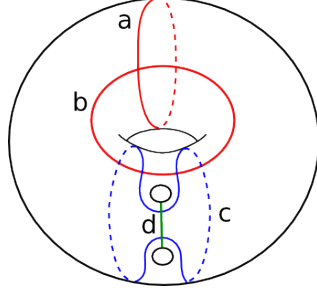
Figure 5: The minimal mapping class in the quotient family associated to Penner's sequence.

The surface (S, ϕ) is shown in Figure 5, where ϕ is the mapping class on the torus with two boundary components given by the product of Dehn twists $\delta_c \circ \delta_b^{-1} \circ \delta_a$ centered at the curves a, b and c, and d is the path connecting the two boundary components. By Theorem 1.2, we have the following.

Proposition 5.1 *Penner's sequence of mapping classes ϕ_g satisfies*

$$\lim_{g \to \infty} \overline{\lambda}(\phi_g) = \overline{\lambda}(\phi) \approx 46.9787.$$

The action of ϕ on the first homology $H^1(S, \mathbb{Z})$ is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence has a 1-dimensional invariant subspace. Thus, the mapping torus M has $b_1(M) = 2$. The cyclic covering $\widetilde{S} \to S$ defined by $\mathfrak{t}$ is drawn in Figure 6. Let ζ generate the group of covering automorphisms. Then $\zeta \times \{\text{id}\}$ and $T_{\widetilde{\phi}}$ define generators for $H_1(M; \mathbb{Z})$. Let μ be the dual of $\zeta \times \text{id}$, that is, the extension of the map $\pi_1(\widetilde{S}) \to \mathbb{Z}$ defined by $\mathfrak{t}$, and let ψ be the fibration map dual to ϕ.

Let $t, u \in H_1(M; \mathbb{Z})$ be duals to μ and $T_{\widetilde{\phi}}$ respectively. Let $\mathfrak{t}$ be the train track for ϕ given by smoothing the union of a, b and c at the intersections (see [Pen]). The Teichmüller polynomial is the characteristic polynomial for the action of the lift of ϕ on the cyclic covering of S defined by $\mathfrak{t}$ on the lift $\widetilde{\mathfrak{t}}$ of $\mathfrak{t}$, or more precisely on the space of allowable measurea on $\widetilde{\mathfrak{t}}$. Using the switch conditions, we can replace the space of allowable measures with the space of labels on the lifts of the curves a, b and c. Then the Teichmüller polynomial of the fibered face defined by ϕ is a factor of the characteristic polynomial of the *twisted transition matrix*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1+t \\ 1+t^{-1} & 2(1+t^{-1}) & 1+(1+t)(1+t^{-1}). \end{bmatrix},$$

and is given by

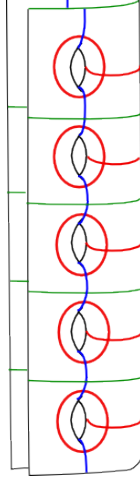$$\Theta(u, t) = u^2 - u(5 + t + t^{-1}) + 1.$$

Figure 6: The simultaneous cyclic covering of Penner's examples.

The Alexander polynomial Δ is the characteristic polynomial of the action of the lift $\widetilde{\phi}$ of ϕ on the first homology of $\widetilde{S}$. The lifts of a, b and c generate $H_1(\widetilde{S}; \mathbb{Z})$ as a $\mathbb{Z}[t, t^{-1}]$ module, and the action of $\widetilde{\phi}$ on these generators is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1-t \\ 1-t^{-1} & 2(1-t^{-1}) & 1+(1-t)(1-t^{-1}) \end{bmatrix}.$$

We thus have

$$\Delta(u,t) = \Theta(u,-t) = u^2 - u(5 - t - t^{-1}) + 1. \tag{4}$$

Normalized dilatations. By the relation between the Alexander and Thurston norms [McM2], it follows that the fibered cone C_ψ in $H^1(M; \mathbb{R})$ containing ψ is given by elements $a\psi + b\mu$, where

$$a > |b|,$$

and the Thurston norm is given by

$$||(a,b)||_T = \max\{2|a|, 2|b|\}.$$

The dilatation $\lambda(\phi_{(a,b)})$ corresponding to primitive integral points (a, b) in C_ψ is the largest solution of the polynomial equation

$$\Theta(x^a, x^b) = 0.$$

In particular, Penner's examples (S_g, ϕ_g) correspond to the points $(g, 1) \in C_\psi$, and we have the following.

Proposition 5.2 *The dilatation of ϕ_g is given by the largest root of the polynomial*

$$\Theta(x^g, x) = x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1.$$

The symmetry of Θ with respect to $x \mapsto -x$ and convexity of L on fibered faces implies the minimum normalized dilatation realized in $\mathcal{P}(M, F)$ must occur at $(a, b) = (1, 0)$. Thus, we have the following.

Proposition 5.3 *The minimum normalized dilatation for the monodromies in C_ψ is given by $\overline{\lambda}(\phi) \approx 46.9787$.*

Orientabilitiy: A pseudo-Anosov mapping class is *orientable* if it has orientable invariant foliations, or equivalently the geometric and homological dilatations are the same, and the spectral radius of the homological action is realized by a real (possibly negative) eigenvalue (see, for example, [LT] p. 5). Given a polynomial f, the largest complex norm amongst its roots is called the *house of f*, denoted $h(f)$. Thus, ϕ_g is orientable if and only if

$$h(\Delta(x^g, x)) = h(\Theta(x^g, x)). \tag{5}$$

Proposition 5.4 *The mapping classes (S_g, ϕ_g) are orientable if and only if g is even.*

Proof. By Equation (4), the homological dilatation of ϕ_g is the largest complex norm amongst roots of

$$\Delta(x^g, x) = x^{2g} + x^{g+1} - 5x^g + x^{g-1} + 1.$$

Let λ be the real root of $\Delta(x^g, x)$ with largest absolute value. Plugging λ into $\Theta(x^g, x)$ gives

$$\Theta(\lambda^g, \lambda) = -2\lambda^{g+1} - 2\lambda^{g-1} \neq 0.$$

while for $-\lambda$ we have

$$\Theta(-\lambda^g, -\lambda) = (-\lambda)^{g+1} - (\lambda)^{g+1} + (-\lambda)^{g-1} - (\lambda^{g-1}).$$

It follows that $h\Delta(x^g, x) = \lambda = h\Theta(x^g, x)$ if and only if g is even. $\square$

5.2 Quotient families of Type II.

We give two examples of quotient families of Type II.

The first example $\widehat{\phi}$ is defined by $T_{\widetilde{c_1}} T_{\widetilde{b}} T_{\widetilde{a}} T_r^{-1}$, where T_γ is a positive Dehn twist along γ, and a, b, c_1, r are the curves on $\widetilde{S}$ drawn in Figure 7. This example is originally due to C. Leininger, who described the associated generalized Penner sequence and its digraphs.

As before, let (S_m, ϕ_m) be the mapping classes associated to $A(\frac{1}{m})$. For example, when $m = 2$, S_m is a closed surface of genus 3, and $\phi_3 = \rho T_{c_0} T_{b_1} T_{a_0} T_{r_0}^{-1}$, where $a_0, b_0, c_0, r_0, a_1, b_1, c_1, r_1$ are as drawn in Figure 8.

We will show the following.

Proposition 5.5 *The mapping classes (S_m, ϕ_m) are associated to the image of $\frac{1}{m}$ under an embedding $A : (0, 1) \to F$, where F Is a fibered face. The corresponding quotient family is of Type II, and as m goes to infinity, $L(S_m, \phi_m)$ behaves asymptotically as*

$$L(S_m, \phi_m) \asymp \frac{1}{m}.$$

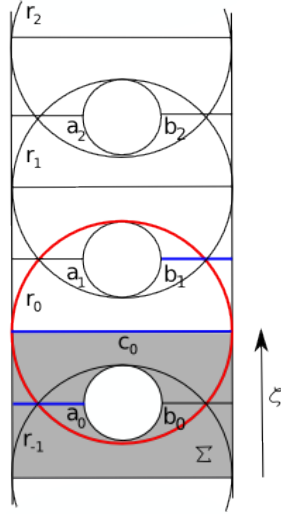From Penner's semi-group criterion, we have the following.

Figure 7: Infinite covering $\widetilde{S}$ and compactly supported map $\widehat{\phi}$.

Lemma 5.6 *The map ϕ_2 is pseudo-Anosov.*

Lemma 5.7 *A train track $\mathfrak{t}$ for ϕ_2 can be obtained by turning right on a, b, c, ta, tb, tc and turning left on r.*

Each $a_0, b_0, c_0, a_1, b_1, c_1, r_0, r_1$ determines a vector space of allowable weights on the edges of $\mathfrak{t}$. These span the space of all allowable weights.

The transition matrix with respect to these vectors can be written as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 & 0 & 5 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$
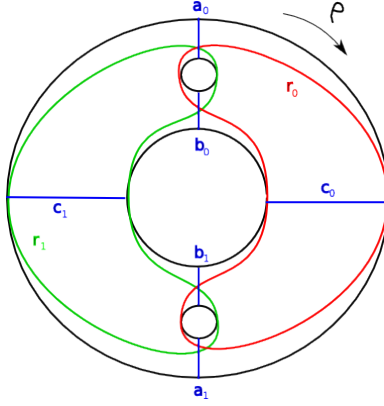

Figure 8: genus 3 case

The characteristic polynomial is

$$(u-1)^2(1+u)^2(1-2u-10u^2-2u^3+u^4),$$

and its house, or dilatation is approximately 4.37709.

We can compute the Teichmuller polynomial restricted to the corresponding linear section of the fibered face of the mapping torus of (S_2, ϕ_3) by considering the induced transition matrix on the lifted train track.

$$\begin{bmatrix} 0 & t & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & t & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2t & 0 & 0 & 2 & 0 & 5 & 2 \\ 0 & t & 0 & 0 & 1 & 0 & 2 & 1 \\ t & t & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 2t & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & t & t & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\Theta(u,t) = (t-u^2)^2(-t^2+(1+t)tu+10tu^2+(1+t)u^3-u^4).$$

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Let
$$\theta(x,y) = \frac{x^{10}\Theta(x^{-1}, x^{-2}y)}{(y-1)^2} = 1 - x - y(10 + x + x^{-1}) + (1 - x^{-1})y^2.$$

By Theorem 4.1, we have the following.

Proposition 5.8 *For all $m \geq 2$, the dilatation of ϕ_m is the house of the polynomial*
$$\theta(z, z^m) = 1 - z - z^m(10 + z + z^{-1}) + (-z^{-1} + 1)z^{2m}.$$

Proposition 5.9 *The minimum normalized dilatation for ϕ_m occurs at ϕ_2, and is approximately 367.064.*

Proof. Let $A : (0, \frac{1}{2}) \to F$ be the associated parameterization into a fibered face. One observes that for this example $\zeta\widehat{\phi}$ is conjugate to $\zeta^{-1}\widehat{\phi}$. Thus, the map A extends to $(0,1)$, and (S_α, ϕ_α) is conjugate to $(S_{1-\alpha}, \phi_{1-\alpha})$. It follows that $L(S_\alpha, \phi_\alpha) = L(S_{1-\alpha}, \phi_{1-\alpha})$, and by convexity of L, the minimum of L must occur at $\alpha = \frac{1}{2}$. $\qquad\square$

Remark 5.10 In this case, the minimum normalized dilatation for the linear section defined by A occurs at the mapping class with smallest topological Euler characteristic (in absolute value). This is not typical as we see in the variation below.

The following is a general property of PF-digraphs.

Proposition 5.11 *Let Γ be a PF-digraph, with m vertices, such that for some constants c and d, the digraph Γ has one self-loop and all other cycles have length greater than $\frac{m}{d} - c$, where m is large compared to k and c. Then the spectral radius $\lambda(\Gamma)$ of the PF matrix associated to Γ satisfies*
$$m^{\frac{1}{2m}} \leq \lambda(\Gamma) \leq m^{\frac{3}{m}}.$$

The digraph associated to (S_2, ϕ_2) is shown in Figure 9, where the dotted edge is considered as a solid edge. The digraphs associated to (S_m, ϕ_m) are gotten by subdividing each dotted edge using $m - 2$ vertices. One observes that there is a self-loop at a_1 independent of m.

Applying Proposition 5.11, it follows that as m approaches infinity, we have
$$\log L(S_{\frac{1}{m}}, \phi_{\frac{1}{m}}) \asymp \frac{\log m}{m}.$$

This completes the proof of Proposition 5.5.

Variation. The second example is the quotient family associated to $(\widetilde{S}, \zeta, \widehat{\phi})$, where $\widetilde{S}$ and ζ are as above, and
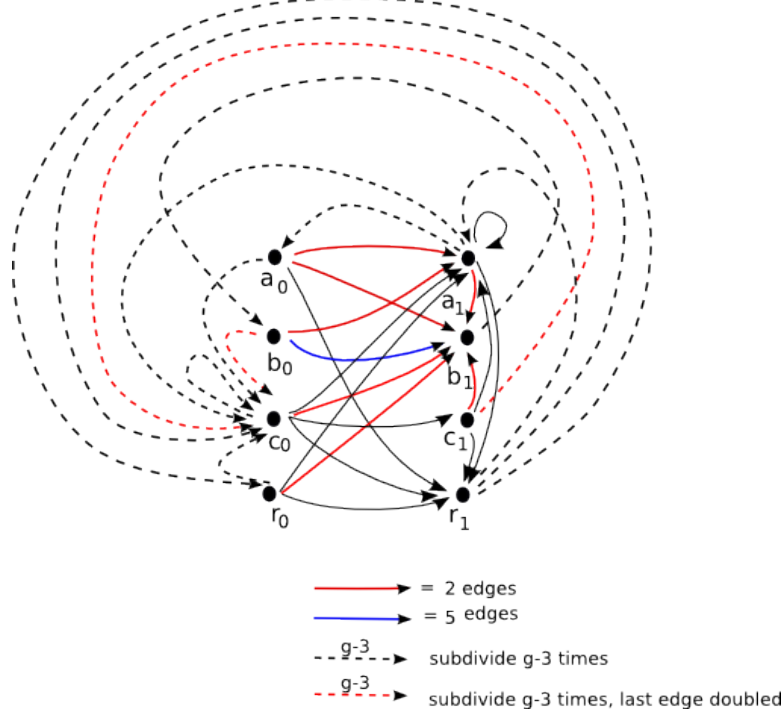$$\widehat{\phi} = T_{c_0}T_{b_1}T_{a_0}^2.$$

Figure 9: Digraph associated to (S_m, ϕ_m).

The mapping classes (S_m, ϕ_m) are the same as before, except that

$$\phi_m = \rho \circ T_{c_0} T_{b_1} T_{a_0}^2 T_{r_0}^{-1}.$$

In this case, the twisted transition matrix is

$$\begin{bmatrix} 0 & t & 0 & 0 & 3 & 0 & 2 & 1 \\ 0 & t & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2t & 0 & 0 & 4 & 0 & 5 & 2 \\ 0 & t & 0 & 0 & 2 & 0 & 2 & 1 \\ t & t & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 2t & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & t & 2t & 0 & 0 & 0 \end{bmatrix}.$$

and has characteristic polynomial

$$\Theta(u, t) = (t - u^2)^2 (t^2 - tu - 2t^2 u - 12tu^2 - 2u^3 - tu^3 + u^4).$$

Let

$$\theta(x, y) = \frac{x^4 \Theta(x^{-1}, x^{-2}y)}{(y-1)^2} = 1 - 2x - y(12 + x + x^{-1}) + (1 - 2x^{-1})y^2.$$

The map A extends to $A : (0, 1) \to F$, and as $\alpha \in (0, 1)$ approached 0 or 1, $A(\alpha)$ approaches the boundary of F, and the dilatation of $(S_{\frac{k}{m}}, \phi_{\frac{k}{m}})$ is given by $|\theta(z^k, z^m)|$.

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Lemma 5.12 *The behavior of $\lambda(A(\alpha))$ as α approaches 0 is given by*

$$\lim_{\alpha \to 0} \lambda(A(\alpha)) = 2.$$

Computation shows that the smallest normalized dilatation for rational points in $A(0,1)$ with denominator less than 70 occurs at $A(\frac{3}{5})$, and

$$\lambda(A\left(\frac{3}{5}\right)) \approx 1.93964.$$

In particular, for quotient families of Type II the minimum normalized dilatation at rational points (if it exists) need not occur at a point where the topological Euler characteristic has smallest absolute value. In this example, the minimum normalized dilatation occurs at some genus greater than or equal to 5, while the minimum genus is 2.

Question 5.13 *For quotient families of Type II, does the minimum normalized dilatation always occur at a rational point?*

6   Symmetry question for small dilatation mapping classes

Quotient families are examples of families that are strongly quasi-periodic. A mapping class is *quasi-periodic with support Y* if there are mapping classes $r, \eta : S \to S$ such that

(1) $\phi = r \circ \eta$,

(2) r is supported on a subsurface $X \subset S$ and is periodic relative to the boundary of X, and

(3) η is supported on Y.

A mapping class (S, ϕ) is *strongly quasi-periodic with support Y* if r is periodic on all of S, i.e., $X = S$.

A family of mapping classes $\mathcal{F} \subset \mathcal{P}$ is a *(strongly) quasi-periodic family* if for some κ,

$$\phi_\alpha = r_\alpha \circ \eta_\alpha$$

is (strongly) quasi-periodic with support Y_α, where

$$|\chi(Y_\alpha)| \leq \kappa.$$

Quotient families of mapping classes of Type I and Type II are strongly quasi-periodic.

The following question was posed by Farb, Leininger and Margalit in unpublished work.

Question 6.1 (Symmetry question) *Given $\ell > 1$, is the subset of $\mathcal{P}$ consisting of elements with normalized dilatation less than ℓ a strongly quasi-periodic family?*

Quasi-periodic families are found in [HK, Hir1, Hir2], including examples whose normalized dilatations converge to the smallest known accumulation point of L

$$L_0 = \left(\frac{3 + \sqrt{5}}{2} \right)^2,$$

which is the normalized dilatation of the simplest hyperbolic braid. It is not known whether or not these families are also strongly quasi-periodic.

Theorem 6.2 *Let (S, ϕ) be a mapping class. Then (S, ϕ) belongs to a quotient family if and only if $\phi = r \circ \eta$, where*

(i) r is periodic of order $m \geq 2$ with fundamental domain Σ with ends τ_- and $\tau_+ = \zeta \tau_-$,

(ii) η has support
$$Y \subset \Sigma \cup \zeta \Sigma \cup \cdots \cup \zeta^{m-2} \Sigma.$$

Proof. Let $\widetilde{S}$ and ζ by taking the cyclic covering of S corresponding to the map $H_1(S; \mathbb{Z}) \to \mathbb{Z}$ given by intersection number with τ_-. Let Σ' be a lift of Σ. Then η determines a map $\widehat{\phi}$ with support contained in
$$\Sigma' \cup \zeta \Sigma' \cup \cdots \cup \zeta^{m-2} \Sigma'.$$
Then (S, ϕ) lies in the quotient family defined by $(\widetilde{S}, \zeta, \widehat{\phi})$. $\qquad \square$

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