Higher dimensional manifolds with S^1-category 2

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Abstract

A closed topological n-manifold M^n is of S^1-category 2 if it can be covered by two open subsets W_1,W_2 such that the inclusions $W_i \to M^n$ factor homotopically through maps $W_i \to S^1$. We show that for $n > 3$, if $cat_{S^1}(M^n) = 2$ then $M^n \approx S^n$ or $M^n \approx S^{n-1} \times S^1$ or the non-orientable S^{n-1}-bundle over S^1. [1] [2]

1 Introduction

For a fixed space A and a space X the A-*category* $cat_A X$ of X was defined by Clapp and Puppe [1]. In particular suppose $A = K$ a cell-complex and X a CW-complex. A subspace W of X is K-contractible (in X) if there exist maps $f : W \to K$ and $\alpha : K \to X$ such that the inclusion $\iota : W \to X$ is homotopic to $\alpha \cdot f$. Notice that a subset of a K-contractible set is also K-contractible. The K-*category* $cat_K X$ of X is the smallest number of sets, open and K-contractible (in X) needed to cover X. Note that in the case $K = P$, a point, $cat_P X = cat\, X$, the Lusternik-Schnirelmann category of X. We are interested here in the case $K = S^1$, $X = M^n$, a closed n-manifold, and consider the beginning case $cat_{S^1}(M^n) = 2$.

So suppose that M^n is a closed connected topological n-manifold with $cat_{S^1}(M^n) = 2$. Denoting by $S^{n-1}\tilde{\times}S^1$ either $S^{n-1} \times S^1$ or the non-orientable S^{n-1}-bundle

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over S^1 we obtain as obvious examples:

S^2, P^2, $S^1 \tilde{\times} S^1$ for $n = 2$.
S^3, lens spaces, $S^2 \tilde{\times} S^1$ for $n = 3$.
S^n, $S^{n-1} \tilde{\times} S^1$ for $n > 3$.

In [6] it is shown that for $n = 2, 3$ these are the only possibilities up to homotopy type, hence by Perelman [15] this is true up to homeomorphism type. For $n > 3$ it is shown in [7] that $\pi_1(M^n)$ is trivial or infinite cyclic. The main result of this paper is that in the latter case the above list is complete (Theorem 5).

Here is an outline of the proof. Assume that M^n is a closed orientable n-manifold, $n > 3$, and $cat_{S^1}(M^n) = 2$.

If $\pi_1(M^n) = 1$ then $cat_{S^1}(M^n) = cat(M^n) = 2$ implies $M \approx S^n$ (Fox, Smale, Freedman). Thus assume that $\pi_1(M^n) \cong \mathbb{Z}$ ([7]).

For the k-fold cyclic cover M_k of M^n ($k \geq 1$), $\pi_1(M_k) \cong \mathbb{Z}$ and $cat_{S^1}(M_k) = 2$. Then (by Poincaré Duality) $H^i(M_k) \approx \mathbb{Z}$ for $i = 0,\ 1,\ n-1,\ n$. Using a result of Clapp and Puppe (Proposition 2) this implies that $H_q(M_k) \cong H_q(S^{n-1} \times S^1)$ for all $q \geq 0$. Lift a 1-sphere $\Sigma^1 \subset M^n$ representing a generator of $\pi_1(M^n)$ to a 1-sphere $\Sigma^1_k \subset M^n_k$. We can asssume that Σ^1_k is locally flat and do surgery on Σ^1_k to obtain an n-sphere S^n_k with a locally flat $(n-2)$-sphere $\Sigma^{n-2}_k \subset S^n_k$ (Proposition 3) where $Y_k = M^n_k - \Sigma^1_k = S^n_k - \Sigma^{n-2}_k$ is a homology 1-sphere and $\pi_1(Y_k) = \mathbb{Z}$. Consider the action $t : \tilde{Y} \to \tilde{Y}$ on the universal covering space $\tilde{Y}$ of $Y = Y_1$ given by a generator t of $\pi_1(Y)$. Then $H_i(\tilde{Y})$ is a finitely generated $\mathbb{Z}(t)$-module, for each $i \geq 0$. Since $H_i(Y_k) = 0$ for $i > 1$ it follows that $t^m - 1 : H_i(\tilde{Y}) \to H_i(\tilde{Y})$ is an automorphism for each $m \geq 1$. An algebraic lemma (Lemma 4) then implies that $\tilde{H}_i(\tilde{Y}) = 0$ for all i and so $\tilde{Y}$ is contractible and Y is a homotopy S^1. By Stallings and Freedman Σ^{n-2}_1 is a topologically trivial knot in S^n_1 and it follows that $M^n = M^n_1 \approx S^{n-1} \times S^1$.

If M is non-orientable we give an argument using the 2-fold orientable cover of M.

2 Preliminaries

In this section we assume that M is a closed connected n-manifold. If $\mathrm{cat}_{S^1} M = 2$ then M is covered by two open sets W_0, W_1 such that for $i = 0, 1$, there are maps f_i and α_i such that the diagram $(*)$ is homotopy commutative.

$$(*) \qquad \begin{tikzcd} W_i \arrow[rr] \arrow[dr, "f_i"'] & & M \\ & S^1 \arrow[ur, "\alpha_i"'] & \end{tikzcd}$$

The following proposition, proved in [6], allows us to replace the open sets W_i by compact submanifolds that meet only along their boundaries.

Proposition 1. *If $\operatorname{cat}_{S^1} M = 2$ then M can be expressed as a union of two compact S^1-contractible n-submanifolds W_0, W_1 such that $W_0 \cap W_1 = \partial W_0 = \partial W_1$. Furthermore for $n > 2$, we may assume that α_i is an embedding and $\alpha_i(S^1)$ does not intersect $W_0 \cap W_1$.*

The main result of [7] is

Theorem 1. *If $\operatorname{cat}_{S^1} M = 2$ and $n > 3$ then $\pi_1(M) = 1$ or $\mathbb{Z}$ and the loops α_i, $i = 0, 1$, represent a generator of $\pi_1(M)$.*

Lemma 1. *Let $W_i \hookrightarrow M$ be S^1-contractible in M and let $p : \tilde{M} \to M$ be a finite sheeted covering map. If $p^{-1}(\alpha_i(S^1))$ is connected then $\tilde{W}_i := p^{-1}(W_i)$ is S^1-contractible in $\tilde{M}$.*

Proof. Let $W = W_0$ or W_1, $\tilde{W} = \tilde{W}_0$ or $\tilde{W}_1$. There is a homotopy $h_t : W \to M$ such that $h_0 = \iota$, $h_1 = \alpha_i \cdot f_i$. Let $\tilde{S} = p^{-1}(\alpha(S^1)) \cong S^1$ and define $\tilde{h}_0 : \tilde{W} \to \tilde{M}$ to be inclusion. By the homotopy lifting Theorem h_t lifts to a homotopy $\tilde{h}_t : \tilde{W} \to \tilde{M}$ such that $\tilde{h}_1(\tilde{W}) \subset p^{-1}(\alpha_i(f_i(W)) \subset \tilde{S}$. □

Lemma 2. *Suppose $\pi_1(M) = \mathbb{Z}$ and $p_k : M_k \to M$ is the k-fold cyclic cover ($k \geq 1$). If $cat_{S^1}(M) = 2$ then $cat_{S^1}(M_k) = 2$.*

Proof. There is a decomposition $M = W_0 \cup W_1$ as in Prop. 1. By Theorem 1 the loops α_i are homotopic to a loop γ representing a generator of $\pi_1(M) \cong \mathbb{Z}$. Since $p_k^{-1}(\gamma)$ is connected, $p_k^{-1}(\alpha_i(S^1))$ is connected. By Lemma 1 $M_k = \tilde{W}_0 \cup \tilde{W}_1$ where $\tilde{W}_i$ is S^1-contractible in M_k. □

Lemma 3. *Suppose M is non-orientable, $n > 2$, and $p : \tilde{M} \to M$ is the 2-fold orientable cover. If $cat_{S^1}(M) = 2$ then $cat_{S^1}(\tilde{M}) = 2$.*

Proof. The proof is the same as in the previous lemma, noting that for an orientation reversing loop ω in M, $\omega \simeq \gamma^m$ for some $m \neq 0$. Since $p^{-1}(\omega)$ is connected it follows that $p^{-1}(\gamma)$ and $p^{-1}(\alpha_i(S^1))$ are connected.

□

We will need the following algebraic lemma:

Lemma 4. *Let J be the infinite cyclic group with generator t and let A be a finitely generated $\mathbb{Z}(J)$- module. If for any $m \geq 1$ multiplication by $1 - t^m$ is an automorphism of A then $A = 0$.*

Proof. A is of type K in the sense of Levine [13] (p.8), i.e. A is a finitely generated $\mathbb{Z}(J)$- module such that $t - 1 : A \to A$, multiplication by $t - 1$, is an epimorphism (equivalently, an automorphism). A submodule or quotient module of type K is again of type K. A module A of type K is finite if every element of A has finite order ([16], proof on Lemma II.8, [13] p.8). Hence $Tor(A) := \{a \in A : a \text{ has finite order}\}$ is a finite group. Therefore the group of bijections from $Tor(A)$ to itself is finite, so the bijection $t : Tor(A) \to Tor(A)$ has finite order k and $t^k - 1 : Tor(A) \to Tor(A)$ is 0 and is an automorphism. Hence $Tor(A) = 0$.

Then [16] (Lemma II.12) A can be presented by a square matrix with entries in $\mathbb{Z}(J)$ whose determinant $\Delta(t)$ is normalized so that it is a polynomial in t with nonzero constant term. It generates the first elementary ideal of $A \otimes \mathbb{Q}$.

Following Weber [20] p.267, there are polynomials $\lambda_i, i = 1, \ldots, r$, such that for the direct sum of cyclic $\mathbb{Z}(J)$-modules $B = \oplus_{i=1}^r \mathbb{Z}(J)/(\lambda_i)$ one has a short exact sequence of $\mathbb{Z}(J)$-modules $0 \to B \to A \to A\,/\,b \cdot B \to 0$, where $B \to A$ is multiplication by a nonzero integer b, such that $A\,/\,b \cdot B$ is finite torsion group. Since now $t^m - 1 : A\,/\,b \cdot B \to A\,/\,b \cdot B$ is an epimorphism for each $m \geq 1$ it follows from the argument given above for $Tor(A)$ that $A\,/\,b \cdot B = 0$.

Hence $A \cong \oplus_{i=1}^r \mathbb{Z}(J)/(\lambda_i)$ and $\Delta(t) = \prod_{i=1}^r \lambda_i$ and it follows from the Proposition of section 2 of [20] and multiplicativity of resultants (Corollary p.262 of [20]) that $|Res(t^m - 1, \Delta(t))| =$ order of $A/(t^m - 1)A = 1$ for all $m \geq 1$.

This implies by [10] that $\Delta(t) = 1$ (This also follows from [9] (Thm 1) and Kronecker's Theorem [18]).

Now A can be presented by an $r \times r$ diagonal matrix S over $\mathbb{Z}(J)$ with $det(S) = \Delta(t)$. For each $a \in A$, $\Delta \cdot a = (detS) \cdot a = (\lambda_1 \lambda_2 \ldots \lambda_r) \cdot a = 0$. Hence $A = 1 \cdot A = \Delta \cdot A = 0$.

$\square$

3 Cat 2 and Homology($S^1 \times S^{n-1}$)

Let $\mathcal{D}_1$ denote the set of all cell-complexes of dimension ≤ 1. Following Clapp-Puppe [1] we let $cat_{\mathcal{D}_1} X$ be the smallest number m such that X can be covered by m open sets, each K-contractible in X, for some $K \in \mathcal{D}_1$.

Let G be any coefficient group.

Proposition 2. *Suppose that $cat_{\mathcal{D}_1} X \leq k$. Let $\alpha_1, \ldots, \alpha_k$ be cocycles such that $\alpha_i \in H^{j_i}(X; G)$, for some $j_i > 1$ ($i = 1, \ldots, k$). Then the cup product $\alpha_1 \cup \cdots \cup \alpha_k = 0$.*

Proof. This follows from [1]: Let $X = X_1 \cup \cdots \cup X_k$ be an open cover of X, where each X_i is K_i-contractible in X, for some 1-complex K_i. Then the inclusion induced homomorphism $\iota^* : H^{j_i}(X; G) \to H^{j_i}(X_i; G)$ factors through $H^{j_i}(K_i; G)$ which is 0 since $j_i > 1$. Hence in the exact cohomology sequence

$$\cdots \to H^{j_i}(X, X_i; G) \to H^{j_i}(X; G) \xrightarrow{\iota^*} H^{j_i}(X_i; G)$$

α_i pulls back to a relative cocycle $\hat{\alpha}_i$. Then the relative cup product $\hat{\alpha}_1 \cup \cdots \cup \hat{\alpha}_k \in H^q(X, \bigcup_{i=1}^k X_i; G) = H^q(X, X; G) = 0$, where $q = j_1 + \cdots + j_k$. It follows that $\alpha_1 \cup \cdots \cup \alpha_k = 0$. $\square$

Lemma 5. *If $\pi_1(X)$ is cyclic then $cat_{S^1} X = cat_{\mathcal{D}_1} X$.*

Proof. It suffices to show that if $W \subset X$ is K-contractible in X for some $K \in \mathcal{D}_1$ then W is S^1-contractible in X.

The inclusion $\iota : W \to X$ factors homotopically through $W \xrightarrow{f} K \xrightarrow{\alpha} X$. Since $\pi_1(X)$ is cyclic, α factors as $K \xrightarrow{f'} S^1 \xrightarrow{\alpha'} X$ and $\iota \simeq \alpha'(f'f)$. $\square$

Theorem 2. *Suppose M is a closed orientable n-manifold, $n > 3$, with $cat_{S^1} M = 2$. Then M is homeomorphic to S^n or M is a homology $S^1 \times S^{n-1}$.*

Proof. If $\pi_1(M) = 1$ then an S^1-contractible submanifold W is in fact contractible in M. Hence $catM = 2$ and it follows (see for example [5]) that M is a (homotopy) n-sphere.

Thus assume $\pi_1(M) \neq 1$, hence by Theorem 1 $\pi_1(M) = \mathbb{Z}$. By Poincarè Duality M has the same homology groups as $S^1 \times S^{n-1}$ if and only if M has the same cohomology groups as $S^1 \times S^{n-1}$. Assume that $H^*(M) \ncong H^*(S^1 \times S^{n-1})$. We show that this implies $cat_{\mathcal{D}_1} M > 2$. By Lemma 5 this would imply that $cat_{S^1} M > 2$.

Since $\pi_1(M) \cong \mathbb{Z}$ we have (by Poincarè Duality) $\mathbb{Z} \cong H^1(M) \cong H^{n-1}(M)$, $\mathbb{Z} \cong H^0(M) \cong H^n(M)$. By assumption there exists some i, $1 < i < n-1$ such that $H^i(M) \neq 0$. Then $H^i(M; \mathbb{Q}) \neq 0$ or $H^i(M; \mathbb{Z}_p) \neq 0$ for some prime p. Let $G = \mathbb{Q}$ if $H^i(M)$ is infinite and $G = \mathbb{Z}_p$ if $H^i(M)$ is finite. Then there exist $\alpha_1 \in H^i(M; G)$, $\alpha_2 \in H^{n-i}(M; G)$ such that $\alpha_1 \cup \alpha_2 = \mu \neq 0$, a generator of $H^n(M; G)$. By Proposition 2, $cat_{\mathcal{D}_1} M > 2$.

$\square$

4 Surgery on a Homology $S^1 \times S^{n-1}$

By the General Position Theorem of J. Dancis [2] any map from S^1 into a closed n-manifold M^n, $n > 3$, is homotopic to a locally flat embedding of S^1 into M^n. So if M^n is a homology $S^{n-1} \times S^1$ we can asssume that a regular neighborhood of a circle $\Sigma^1 \subset M$ that represents a generator of $H_1(M)$ is a tube $\Sigma^1 \times D^{n-1}$. We now show that we can do surgery on Σ^1 to obtain an n-sphere S^n.

Proposition 3. *Suppose the closed orientable n-manifold M, $n > 3$, is a homology $S^{n-1} \times S^1$. Denote by $\Sigma^1 \times D^{n-1}$ a regular neighborhood of a circle $\Sigma^1 \subset M$ that represents a generator of $H_1(M)$.*
Let $Y = M - int(\Sigma^1 \times D^{n-1})$ and $M_S = Y \cup (D^2 \times S^{n-2})$ where $\Sigma^1 \times S^{n-2}$ is identified with $S^1 \times S^{n-2}$.
Then Y is a homology 1-sphere and M_S is a homology n-sphere.
Furthermore, if $\pi_1(M) \cong \mathbb{Z}$ and $\Sigma^1 \subset M$ represents a generator of $\pi_1(M)$, then $\pi_1(Y) \cong \mathbb{Z}$ and $M_S \approx S^n$.

Proof. We first show that $H_j(Y) = 0$ for $1 < j < n-2$.

We identify Σ^1 with S^1. By excision and from the exact homology sequence of the pair $(S^1 \times D^{n-1}, S^1 \times S^{n-2})$ we obtain (using the fact that inclusion induces isomorphisms $H_i(S^1 \times S^{n-2}) \to H_i(S^1 \times D^{n-1})$ for $i = 0, 1$) that
$H_j(M, Y) \cong H_j(S^1 \times D^{n-1}, S^1 \times S^{n-2}) = \begin{cases} 0 & \text{for } j = 1, \dots, n-2 \\ \mathbb{Z} & \text{for } j = n-1,\, n \end{cases}$
This implies, using the exact homology sequence of the pair (M, Y) that
$H_j(Y) \cong H_j(M)$ for $1 \le j < n-2$.

We now show that M_S is a homology n-sphere.

Since M_S is a closed n-manifold we have $H_0(M_S) = \mathbb{Z} = H_n(M_S)$.
From the Mayer-Vietoris sequence of $M_S = Y \cup_{S^1 \times S^{n-2}} (D^2 \times S^{n-2})$ and using the fact that $H_j(Y) = 0$ for $1 < j < n-2$, we obtain $H_j(M_S) = 0$ for $2 < j < n-2$. For $j = 1, 2$ consider the exact sequence

$H_2(S^1 \times S^{n-2}) \xrightarrow{k_2} 0 \oplus H_2(D^2 \times S^{n-2}) \to H_2(M_S) \to H_1(S^1 \times S^{n-2}) \xrightarrow{k_1} H_1(Y) \oplus 0 \to H_1(M_S) \to H_0(S^1 \times S^{n-2}) \xrightarrow{k_0} H_0(Y) \oplus H_0(D^2 \times S^{n-2}) \to H_0(M_S) \to 0$

The homomorphism k_1 is induced by inclusion and is an isomorphism. The last three terms of the sequence show that k_0 is injective. It follows that $H_1(M_S) = 0$.
If $n > 4$ the first two terms of the sequence are 0 and it follows that $H_2(M_S) = 0$. For $n = 4$ these terms are $\mathbb{Z}$. Now k_2 is an isomorhphism (since a generator of $H_2(S^1 \times S^2)$ is represented by $\{point\} \times S^2 \xrightarrow{k_2} \{point\} \times S^2$) and it follows again that $H_2(M_S) = 0$.

Hence we have shown that $H_j(M_S) = 0$ for $j = 1, 2, \dots, n-3$. By Poincarè Duality, $H_{n-j}(M_S) = 0$ for $j = 1, 2, \dots, n-3$ and it follows (since for $n = 4$, $H_2(M_S) = 0$) that $H_j(M_S) = 0$ for $j = 1, 2, \dots, n-1$.

To complete the proof that Y is a homology S^1 we need to show that $H_q(Y) = 0$ for $n-2 \le q$.

Since $\partial Y \neq \emptyset$ we have $H_n(Y) = 0$.
Denote by S^{n-2} the $(n-2)$-sphere $\{P\} \times S^{n-2} \subset D^2 \times S^{n-2}$, where P is a point in ∂D^2. By Lefschetz Duality (see e.g. [19], Thm 6.17) $H_q(Y) = H_q(M_S - S^{n-2}) \cong H^{n-q}(M_S, S^{n-2})$. Since M_S is a homology sphere it follows (from the exact cohomology sequence of the pair (M_S, S^{n-2})) that this relative cohomology group is 0 for $q \ge 2$.

Finally, if $\pi_1(M) \cong \mathbb{Z}$ and $S^1 \subset M$ represents a generator of $\pi_1(M)$, then by Seifert-vanKampen we obtain $\pi_1(Y) \xrightarrow{\cong} \pi_1(M)$ and $\pi_1(M_S) = 1$. Since now M_S is a simply connected homology sphere it follows that M_S is the (homotopy)

n-sphere. $\square$

5 The Main Theorem

We first give a characterization of $S^1 \times S^{n-1}$ in terms of π_1 and the homology of coverings.

Theorem 3. *Suppose $\pi_1(M^n) = \mathbb{Z}$, with $n > 3$, and $H_*(M_k) = H_*(S^1 \times S^{n-1})$ for all $k \geq 1$, where M_k is the k-fold cover of M^n. Then M^n is homeomorphic to $S^1 \times S^{n-1}$.*

Proof. We perform surgery on M. Let $\Sigma^1 \times D^{n-1} \subset M$ be a tubular neighborhood of a 1-sphere representing a generator of $\pi_1(M)$, $Y = M - int(\Sigma^1 \times D^{n-1})$ and M_S the union of Y and $D^2 \times S^{n-2}$ pasted together along their boundaries as in Proposition 3. Then Y is a homology 1-sphere, $\pi_1(Y) = \mathbb{Z}$, $M_S \approx S^n$ and we have a locally flat knot (S^n, S^{n-2}) with exterior Y. By [11] (p.301 (III)), Y has the homotopy type of a finite CW-complex and for the following we may assume that Y is a finite CW-complex. Let $\Sigma^1_k \times D^{n-1}$ (resp. Y_k) be the lift of $\Sigma^1 \times D^{n-1}$ (resp. Y) to the k-fold cyclic cover M_k of M ($k \geq 1$). Then $\pi_1(M_k) \cong \mathbb{Z}$ with generator represented by Σ^1_k and by Lemma 2, Theorem 2 and Proposition 3, $cat_{S^1}(M_k) = 2$, $Y_k = M_k - int(\Sigma^1_k \times D^{n-1}) = S^n - int(S^{n-2}_k \times D^2)$ is a homology 1-sphere and $\pi_1(Y_k) = \mathbb{Z}$.

Consider the action $t : \tilde{Y} \to \tilde{Y}$ on the universal covering space $\tilde{Y}$ of $Y = Y_1$ given by a generator t of $\pi_1(Y)$. Since Y_k is a finite CW-complex, $H_i(\tilde{Y})$ is a finitely generated $\mathbb{Z}(t)$-module whose tensor product with $\mathbb{Q}$ is a finite dimensional vector space over $\mathbb{Q}$, for each $i \geq 0$. Since $H_i(Y_k) = 0$ for $i > 1$ it follows (see [14], p 118) that $t^m - 1 : H_i(\tilde{Y}) \to H_i(\tilde{Y})$ is an automorphism for each $m \geq 1$.

By Lemma 4 $H_i(\tilde{Y}) = 0$ for all $i \geq 1$ and so $\tilde{Y}$ is contractible and Y is a homotopy S^1. By Stallings (for $n > 4$) and Freedman (for $n = 4$) S^{n-2} is a topologically trivial knot in S^n, that is $Y = D^{n-1} \times S^1$. Hence M is the union of two copies of $D^{n-1} \times S^1$ pasted along their boundaries which implies $M \approx S^{n-1} \times S^1$. $\square$

Remark 1. Implicit in the proof of the previous theorem is a proof of the following Unknottedness Theorem.

Theorem 4. *Let (S^n, S^{n-2}), with $n > 3$, be a locally flat knot all of whose branched cyclic coverings are spheres and such that $\pi_1(S^n - S^{n-2}) = \mathbb{Z}$. Then (S^n, S^{n-2}) is unknotted.*

We now prove the Main Theorem.

Theorem 5. *Let M be a closed n-manifold with $n > 3$ and $cat_{S^1}(M) = 2$. Then $M = S^n$ or $S^{n-1} \times S^1$ or the non-orientable S^{n-1}-bundle over S^1.*

Proof. By Theorem 1 $\pi_1(M)$ is trivial or infinite cyclic. If $\pi_1(M)$ is trivial then $cat_{S^1}(M) = cat(M) = 2$ and so (see for example [5]) $M = S^n$.

If $\pi_1(M) = \mathbb{Z}$ and M is orientable let M_k be the k-fold cover of M. Then, for any $k \geq 1$, $cat_{S^1}(M_k) = 2$ by Lemma 2, and $H_*(M_k) = H_*(S^{n-1} \times S^1)$ by Theorem 2. It follows from Theorem 3 that M is homeomorphic to $S^{n-1} \times S^1$.

If M is non-orientable then by Lemma 3 the 2-fold orientable cover $\tilde{M}$ has $cat_{S^1}\tilde{M} = 2$ and hence $\tilde{M} \approx S^{n-1} \times S^1$. Now it follows from the classification of free involutions on $S^1 \times S^n$ (Thm 2.1 and (3.1) of [12]) that M is the non-orientable S^{n-1}-bundle over S^1. $\square$

References

[1] M. Clapp and D.Puppe, Invariants of the Lusternik-Schnirelmann type and the topology of critical sets, Trans. Amer. Math. Soc. 298 (1986), 603–620.

[2] J. Dancis, General Position Maps for Topological Manifolds in the 2/3rds Range, Trans. Amer. Math. Soc. 216 (1976), 249-266.

[3] R. H. Fox, On the Lusternik-Schnirelmann category, Ann. of Math. (2) 42, (1941), 333–370.

[4] H. Gluck, The Embedding of Two-Spheres in the Four-Sphere, Trans. Amer. Math. Soc. 104, (1962), 308-333.

[5] J. C. Gómez-Larrañaga and F. González-Acuña, Lusternik–Schnirelmann category of 3–manifolds, Topology 31 (1992), 791–800.

[6] J. C. Gómez-Larrañaga, F. González-Acuña and Wolfgang Heil, Fundamental groups of manifolds with S^1-category 2, Math. Z. 259 (2008), 419–432.

[7] J. C. Gómez-Larrañaga, F. González-Acuña and Wolfgang Heil, Manifolds with S^1-category 2 have cyclic fundamental groups, Math. Z. (2009) to appear (ISSN 1432-1823 (Online) DOI 10.1007/s00209-009-0598-4)

[8] C. McA. Gordon, Some aspects of classical knot theory, in Knot Theory Proceedings, Springer Lect. Notes Math., vol. 685 (1978), 1–60.

[9] F. González-Acuña, H. Short, Cyclic branched coverings of knots and homology spheres, Rev. Mat. Univ. Complut. Madrid 4 (1991), no. 1, 97–120.

[10] C. Hillar, Cyclic Resultants, J. Symbolic Comput. 39 (2005), no. 6, 653–669.

[11] R. Kirby, L. Siebenmann, Some basic Theorems about topological Manifolds, Foundational Essays on Topological Manifolds, Smoothings and Triangulations, Ann. Math. Studies Princeton Univ Press 88.

[12] B. Jahren, S. Kwasik, Free involutions on $S^1 \times S^n$, eprint arXiv:0802.2035 (02/2008).

[13] J. Levine, Knot modules. I. Trans. Amer. Math. Soc. 229 (1977), 1–50.

[14] J. W. Milnor, Infinite cyclic Coverings, Conference on the topology of manifolds, Prindle, Weber, Schmidt (1968), pp.115-133.

[15] John Morgan and Gang Tian, Ricci Flow and the Poincarè Conjecture, AMS Clay Math. Monographs, vol 3 (2007)

[16] M. A. Kervaire, Les nuds de dimensions suprieures. Bull. Soc. Math. France 93 (1965), 225–271.

[17] L.P. Neuwirth, Knot groups, Annals of Mathematics Studies 56, Princeton University Press (1965).

[18] H. Pollard, The Theory of Algebraic Numbers, Carus Math. Monographs, vol 9 (1975).

[19] E.H.Spanier, Algebraic Topology, McGraw Hill, N.Y. 1966.

[20] C. Weber, Sur une formule de R. H. Fox concernant l'homologie des revetements cycliques. Enseign. Math. (2) 25 (1979), no. 3-4, 261–272 (1980).