The Manin constant

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Abstract.

The Manin constant of an elliptic curve is an invariant that arises in connection with the conjecture of Birch and Swinnerton-Dyer. One conjectures that this constant is 1; it is known to be an integer. After surveying what is known about the Manin constant, we establish a new sufficient condition that ensures that the Manin constant is an *odd* integer. Next, we generalize the notion of the Manin constant to certain abelian variety quotients of the Jacobians of modular curves; these quotients are attached to ideals of Hecke algebras. We also generalize many of the results for elliptic curves to quotients of the new part of $J_0(N)$, and conjecture that the generalized Manin constant is 1 for newform quotients.

1 Introduction

Let E be an elliptic curve over $\mathbf{Q}$, and and let N be the conductor of E. By [BCDT01], we may view E as a quotient of the modular Jacobian $J_0(N)$. After possibly replacing E by an isogenous curve, we may assume that the kernel of the map $J_0(N) \to E$ is connected, i.e., that E is an *optimal* quotient of $J_0(N)$.

Let ω be the unique (up to sign) minimal differential on a minimal Weierstrass model of E. The pullback of ω is a rational multiple c of the differential associated to the normalized new cuspidal eigenform $f_E \in S_2(\Gamma_0(N))$ associated to E. The absolute value of c is the Manin constant c_E of E. The Manin constant plays a role in the conjecture of Birch and Swinnerton-Dyer (see, e.g., [GZ86, p. 130]) and in work on modular parametrizations (see [Ste89, SW04, Vat05]). It is known that the Manin constant is an integer; this fact is important to Cremona's computations of elliptic curves (see

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[Cre97, pg. 45]), and algorithms for computing special values of elliptic curve L-functions. Manin conjectured that $c_E = 1$. In Section 2, we summarize known results about c_E, and give the new result that $2 \nmid c_E$ if if 2 is not a congruence prime and $4 \nmid N$.

In Section 3, we generalize the definition of the Manin constant and many of the results mentioned so far to optimal quotients of $J_0(N)$ and $J_1(N)$ of any dimension associated to ideals of the Hecke algebra. The generalized Manin constant comes up naturally in studying the conjecture of Birch and Swinnerton-Dyer for such quotients (see [AS05, §4]), which is our motivation for studying the generalization. We state what we can prove about the generalized Manin constant, and make a conjecture that the constant is also 1 for quotients associated to newforms. The proofs of the theorems stated in Section 3 are in Section 4. Section 5 is an appendix written by J. Cremona about computational verification that the Manin constant is 1 for many elliptic curves.

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2 OPTIMAL ELLIPTIC CURVE QUOTIENTS

Let N be a positive integer and let $X_0(N)$ be the modular curve over $\mathbf{Q}$ that classifies isomorphism classes of elliptic curves with a cyclic subgroup of order N. The Hecke algebra $\mathbf{T}$ of level N is the subring of the ring of endomorphisms of $J_0(N) = \mathrm{Jac}(X_0(N))$ generated by the Hecke operators T_n for all $n \geq 1$. Suppose f is a newform of weight 2 for $\Gamma_0(N)$ with integer Fourier coefficients, and let I_f be kernel of the homomorphism $\mathbf{T} \to \mathbf{Z}[\ldots, a_n(f), \ldots]$ that sends T_n to $a_n(f)$. Then the quotient $E = J_0(N)/I_f J_0(N)$ is an elliptic curve over $\mathbf{Q}$. We call E the *optimal quotient* associated to f. Composing the embedding $X_0(N) \hookrightarrow J_0(N)$ that sends ∞ to 0 with the quotient map $J_0(N) \to E$, we obtain a surjective morphism of curves $\phi_E : X_0(N) \to E$.

DEFINITION 2.1 (MODULAR DEGREE). The *modular degree* m_E of E is the degree of ϕ_E.

Let $E_{\mathbf{Z}}$ denote the Néron model of E over $\mathbf{Z}$ (see, e.g., [Sil92, App. C, §15], [Sil94] and [BLR90]). Let ω be a generator for the rank 1 $\mathbf{Z}$-module of invariant differential 1 forms on $E_{\mathbf{Z}}$. The pullback of ω to $X_0(N)$ is a differential $\phi_E^*\omega$ on $X_0(N)$. The newform f defines another differential $2\pi i f(z)dz = f(q)dq/q$ on $X_0(N)$. Because the action of Hecke operators is compatible with the map $X_0(N) \to E$, the differential $\phi_E^*\omega$ is a $\mathbf{T}$-eigenvector with the same eigenvalues as $f(z)$, so by [AL70] we have $\phi_E^*\omega = c \cdot 2\pi i f(z)dz$ for some $c \in \mathbf{Q}^*$ (see also [Man72, §5]).

DEFINITION 2.2 (MANIN CONSTANT). The *Manin constant* c_E of E is the absolute value of the rational number c defined above.

The following conjecture is implicit in [Man72, §5].

CONJECTURE 2.3 (MANIN). *We have* $c_E = 1$.

Significant progress has been made towards this conjecture. In the following list of theorems, p denotes a prime and N denotes the conductor of E.

THEOREM 2.4 (EDIXHOVEN [EDI91, PROP. 2]). *The constant* c_E *is an integer.*

Edixhoven proved this using an integral q-expansion map, whose existence and properties follow from results in [KM85]. We generalize his theorem to quotients of arbitrary dimension in Section 3.

THEOREM 2.5 (MAZUR, [MAZ78, COR. 4.1]). *If* $p \mid c_E$*, then* $p^2 \mid 4N$.

Mazur proved this by applying theorems of Raynaud about exactness of sequences of differentials, then using the "q-expansion principle" in characteristic p and a property of the Atkin-Lehner involution. We generalize Mazur's theorem in Section 3.

The following two results refine the above results at $p = 2$.

THEOREM 2.6 (RAYNAUD [AU96, PROP. 3.1]). *If* $4 \mid c_E$*, then* $4 \mid N$.

THEOREM 2.7 (ABBES-ULLMO [AU96, THM. A]). *If* $p \mid c_E$*, then* $p \mid N$.

We generalize Theorem 2.6 in Section 3. However, it is not clear if Theorem 2.7 generalizes to dimension greater than 1. It would be fantastic if the theorem could be generalized. It would imply that the Manin constant is 1 for newform quotients A_f of $J_0(N)$, with N odd and square free, which be useful for computations regarding the conjecture of Birch and Swinnerton-Dyer.

B. Edixhoven also has unpublished results (see [Edi89]) which assert that the only primes that can divide c_E are 2, 3, 5, and 7; he also gives bounds that are independent of E on the valuations of c_E at 2, 3, 5, and 7. His arguments rely on the construction of certain stable integral models for $X_0(p^2)$.

See the appendix (Section 5) for a discussion of the following computational theorem:

THEOREM 2.8 (CREMONA). *If E is an optimal elliptic curve over* $\mathbf{Q}$ *with conductor at most* 60000 *(or with conductor* < 130000*, except possibly for 14 exceptions), then* $c_E = 1$.

To the above list of theorems we add the following:

THEOREM 2.9. *If* $p \mid c_E$ *then* $p^2 \mid N$ *or* $p \mid m_E$.

This theorem is a special case of Theorem 3.9 below. In view of Theorem 2.5, our new contribution is that if m_E is odd and $\mathrm{ord}_2(N) = 1$, then c_E is odd. This hypothesis is *very stringent*—of the optimal elliptic curve quotients of conductor ≤ 120000, only 56 of them satisfy the hypothesis. In the notation of [Cre], they are

14a, 46a, 142c, 206a, 302b, 398a, 974c, 1006b, 1454a, 1646a, 1934a, 2606a, 2638b, 3118b, 3214b, 3758d, 4078a, 7054a, 7246c, 11182b, 12398b, 12686c, 13646b, 13934b, 14702c, 16334b, 18254a, 21134a, 21326a, 22318a, 26126a, 31214c, 38158a, 39086a, 40366a, 41774a, 42638a, 45134a, 48878a, 50894b, 53678a, 54286a, 56558f, 58574b, 59918a, 61454b, 63086a, 63694a, 64366b, 64654b, 65294a, 65774b, 71182b, 80942a, 83822a, 93614a

Each of the curves in this list has conductor $2p$ with $p \equiv 3 \pmod 4$ prime. The situation is similar to that of [SW04, Conj. 4.2], which suggests there are infinitely many such curves. See also [CE05] for a classification of elliptic curves with odd modular degree.

3 Quotients of arbitrary dimension

For $N \geq 4$, let Γ be either $\Gamma_0(N)$ or $\Gamma_1(N)$, let X be the modular curve over $\mathbf{Q}$ associated to Γ, and let J be the Jacobian of X. Let I be a *saturated* ideal of the corresponding Hecke algebra $\mathbf{T}$, so $\mathbf{T}/I$ is torsion free. Then $A = A_I = J/IJ$ is an optimal quotient of J since IJ is an abelian subvariety.

For a newform $f = \sum a_n(f)q^n \in S_2(\Gamma)$, consider the ring homomorphism $\mathbf{T} \to \mathbf{Z}[\ldots, a_n(f), \ldots]$ that sends T_n to $a_n(f)$. The kernel $I_f \subset \mathbf{T}$ of this homomorphism is a prime ideal of $\mathbf{T}$.

Definition 3.1 (Newform quotient). The *newform quotient* A_f associated to f is the quotient $J/I_f J$.

Shimura introduced A_f in [Shi73] where he proved that A_f is an abelian variety over $\mathbf{Q}$ of dimension equal to the degree of the field $\mathbf{Q}(\ldots, a_n(f), \ldots)$. He also observed that there is a natural map $\mathbf{T} \to \mathrm{End}(A_f)$ with kernel I_f.

For the rest of this section, fix a quotient A associated to a saturated ideal I of $\mathbf{T}$; note that A may or may not be attached to a newform.

3.1 Generalization to quotients of arbitrary dimension

If R is a subring of $\mathbf{C}$, let $S_2(R) = S_2(\Gamma; R)$ denote the $\mathbf{T}$-submodule of $S_2(\Gamma; \mathbf{C})$ consisting of cuspforms whose Fourier expansions at ∞ have coefficients in R. Note that $S_2(R) \cong S_2(\mathbf{Z}) \otimes R$. If B is an abelian variety over $\mathbf{Q}$ and n is a positive integer, we denote by $B_{\mathbf{Z}[1/n]}$ the Néron model of B over $\mathbf{Z}[1/n]$.

The inclusion $X \hookrightarrow J$ that sends the cusp ∞ to 0 induces an isomorphism

$$H^0(X, \Omega_{X/\mathbf{Q}}) \cong H^0(J, \Omega_{J/\mathbf{Q}}).$$

Let ϕ_2 denote our fixed choice of optimal quotient map $J \to A$. Then ϕ_2^* induces an inclusion $\psi : H^0(A_{\mathbf{Z}}, \Omega_{A/\mathbf{Z}}) \hookrightarrow H^0(J, \Omega_{J/\mathbf{Q}})[I] \cong S_2(\mathbf{Q})[I]$, and we have the following commutative diagram:

$$\begin{tikzcd} H^0(A, \Omega_{A/\mathbf{Q}}) \arrow[r, hook] & H^0(J, \Omega_{J/\mathbf{Q}})[I] \arrow[r, "\cong"] & S_2(\mathbf{Q})[I] \\ H^0(A_{\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}}) \arrow[u, hook] \arrow[urr, hook, "\psi"'] & & S_2(\mathbf{Z})[I] \arrow[u, hook] \end{tikzcd}$$

DEFINITION 3.2. The *Manin constant* of A is the (lattice) index

$$c_A = [S_2(\mathbf{Z})[I] : \psi(H^0(A_{\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}}))].$$

Theorem 3.3 below asserts that $c_A \in \mathbf{Z}$, so we may also consider the Manin module of A, which is the quotient $M = S_2(\mathbf{Z})[I]/\psi(H^0(A_{\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}}))$, and the Manin ideal of A, which is the annihilator of M in $\mathbf{T}$.

If A is an elliptic curve, then c_A is the usual Manin constant as in Definition 2.2. The constant c as defined above was also considered by Gross [Gro82, 2.5, p.222] and Lang [Lan91, III.5, p.95]. The constant c_A was defined for the winding quotient in [Aga99], where it was called the generalized Manin constant. A Manin constant is defined in the context of $\mathbf{Q}$-curves in [GL01].

3.2 MOTIVATION: CONNECTION WITH THE CONJECTURE OF BIRCH AND SWINNERTON-DYER

On a Néron model, the global differentials are the same as the invariant differentials, so $H^0(A_{\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}})$ is a free $\mathbf{Z}$-module of rank $d = \dim(A)$. The *real measure* Ω_A of A is the measure of $A(\mathbf{R})$ with respect to the measure given by $\bigwedge^d H^0(A_{\mathbf{Z}}, \Omega^1_{A/\mathbf{Z}})$. This quantity is of interest because it appears in the conjecture of Birch and Swinnerton-Dyer, which expresses the ratio $L^{(r)}(A,1)/\Omega_A$, in terms of arithmetic invariants of A, where $r = \mathrm{ord}_{s=1} L(A,s)$ (see, e.g., [Lan91, Chap. III, §5] and [AS05, §2.3]).

The differentials corresponding to a $\mathbf{Z}$-basis of $S_2(\mathbf{Z})[I]$ give a $\mathbf{Q}$-basis of $H^0(A, \Omega^1_{A/\mathbf{Q}})$; let Ω'_A denote the measure of $A(\mathbf{R})$ with respect to the wedge product of the elements of this basis. In doing calculations or proving formulas regarding the ratio in the Birch and Swinnerton-Dyer conjecture mentioned above, it is easier to work with the volume Ω'_A instead of working with Ω_A (see, e.g., [AS05, §4.2]). If one works with the easier-to-compute volume Ω'_A instead of Ω_A, it is necessary to obtain information about c_A in order to make conclusions regarding the conjecture of Birch and Swinnerton-Dyer, since $\Omega_A = c_A \cdot \Omega'_A$.

The method of Section 5 for verifying that $c_A = 1$ for specific elliptic curves is of little use when applied to general abelian varieties, since there is no simple analogue of the minimal Weierstrass model (but see [GL01] for $\mathbf{Q}$-curves). This emphasizes the need for general theorems regarding the Manin constant of quotients of dimension bigger than one.

3.3 Results and a conjecture

We start by giving several results regarding the Manin constant for quotients of arbitrary dimension. The proofs of the theorems are given in Section 4.

We have the following generalization of Edixhoven's Theorem 2.4; we give its proof in Section 4.1.

THEOREM 3.3. *The Manin constant c_A is an integer.*

For each prime $\ell \mid N$ with $\mathrm{ord}_\ell(N) = 1$, let W_ℓ be the ℓth Atkin-Lehner operator. In the following theorem, we view W_ℓ as an operator on $S_2(\mathbf{Q}_\ell)$.

THEOREM 3.4. *Recall that A is an optimal quotient of $J = J_0(N)$. Let ℓ be an odd prime such that $\ell^2 \nmid N$. If $\ell \mid N$, then suppose that $W_\ell \cdot \mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}}) \subseteq S_2(\mathbf{Z}_\ell)$. Under these hypotheses, $\ell \nmid c_A$.*

We will prove this theorem in Section 4.3.

Let J_{old} denote the abelian subvariety of J generated by the images of the degeneracy maps from levels that properly divide N (see, e.g., [Maz78, §2(b)]) and let J^{new} denote the quotient of J by J_{old}. A *new quotient* is a quotient $J \to A$ that factors through the map $J \to J^{\mathrm{new}}$. The following corollary generalizes Mazur's Theorem 2.5:

COROLLARY 3.5. *If A is an optimal new quotient of $J_0(N)$ and $\ell \mid c_A$ is a prime, then $\ell = 2$ or $\ell^2 \mid N$.*

Proof. We verify the hypothesis of Theorem 3.4. Since $W_\ell = -T_\ell$ on the new subvariety of $J = J_0(N)$, it suffices to show that the kernel $B = \ker(J \to A)$ is $\mathbf{T}$-stable, since then $\mathbf{T}$ would act on A hence preserve $\mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}})$ and its image in $S_2(\mathbf{Z}_\ell)$. We have $J_{\mathrm{old}} \subset B$ and J_{old} is $\mathbf{T}$-stable, so it suffices to show that B/J_{old} is $\mathbf{T}$-stable. This follows because the newform abelian subvarieties $A_f^\vee$ occur with multiplicity one in J, and B/J_{old} is isogenous to a product of simple abelian varieties. □

REMARK 3.6. Note that the hypothesis in the third sentence of Theorem 3.4 is satisfied if W_ℓ preserves $S_2(\mathbf{Z}_\ell)$ (as a subgroup of $S_2(\mathbf{Q}_\ell)$). This latter condition need not be hold when A is not new. For example, if $A = J_0(33)$, then

$$W_3 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & -\frac{4}{3} \\ \frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \end{pmatrix}$$

with respect to the basis

$$\begin{aligned} f_1 &= q - q^5 - 2q^6 + 2q^7 + \cdots, \\ f_2 &= q^2 - q^4 - q^5 - q^6 + 2q^7 + \cdots, \\ f_3 &= q^3 - 2q^6 + \cdots \end{aligned}$$

for $S_2(\mathbf{Z})$. Thus W_3 does not preserve $S_2(\mathbf{Z}_3)$. In fact, the Manin constant is not 1 in this case (see Section 3.4.2).

At the same time, the hypothesis in the third sentence of Theorem 3.4 sometimes does hold for non-new abelian varieties. For example, take $J = J_0(33)$ and $\ell = 3$. Then W_3 acts as an endomorphism of J, and a computation shows that the characteristic polynomial of W_3 on J_{new} is $(x-1)$ and on J_{old} is $(x-1)(x+1)$. Consider the optimal elliptic curve quotient $A = J/(W_3+1)J$, which is isogenous to $J_0(11)$. Then A is an optimal old quotient of J, and W_3 acts as -1 on A, so W_3 preserves the corresponding spaces of modular forms. Thus Theorem 3.4 tells us that $3 \nmid c_A$.

The following theorem generalizes Raynaud's Theorem 2.6 (see [GL01] for generalizations to $\mathbf{Q}$-curves).

THEOREM 3.7. *If $f \in S_2(\Gamma_0(N))$ is a newform and ℓ is a prime such that $\ell^2 \nmid N$, then* $\mathrm{ord}_\ell(c_{A_f}) \leq \dim A_f$.

Note that in light of Theorem 3.4, this theorem gives new information only at $\ell = 2$, when $2 \parallel N$. We prove the theorem in Section 4.5

Let π denote the natural quotient map $J \to A$. When we compose π with its dual $A^\vee \to J^\vee$, we get an isogeny $\phi : A^\vee \to A$ (for details, see [ARS]).

DEFINITION 3.8 (MODULAR EXPONENT). The *modular exponent* m_A of A is the exponent of the group $\ker(\phi)$.

When A is an elliptic curve, the modular exponent is just the modular degree of A (see, e.g., [AU96, p. 278]).

THEOREM 3.9. *If $f \in S_2(\Gamma_0(N))$ is a newform and $\ell \mid c_{A_f}$ is a prime, then $\ell^2 \mid N$ or $\ell \mid m_A$.*

Again, in view of Theorem 3.4, this theorem gives new information only at $\ell = 2$, when $2 \parallel N$. We prove the theorem in Section 4.4.

The theorems above suggest that the Manin constant is 1 for quotients associated to newforms of square-free level. In the case when the level is not square free, computations of [FpS+01] involving Jacobians of genus 2 curves that are quotients of $J_0(N)^{\mathrm{new}}$ show that $c_A = 1$ for 28 two-dimensional newform quotients. These include quotients having the following non-square-free levels:

$$3^2 \cdot 7, \quad 3^2 \cdot 13, \quad 5^3, \quad 3^3 \cdot 5, \quad 3 \cdot 7^2, \quad 5^2 \cdot 7, \quad 2^2 \cdot 47, \quad 3^3 \cdot 7.$$

The above observations suggest the following conjecture, which generalizes Conjecture 2.3:

CONJECTURE 3.10. *If f is a newform on $\Gamma_0(N)$ or $\Gamma_1(N)$, then $c_{A_f} = 1$.*

3.4 EXAMPLES OF NONTRIVIAL MANIN CONSTANTS

We present two sets of examples in which the Manin constant is not one.

3.4.1 Joyce's example

Using results of [Kil02], Adam Joyce [Joy05] proves the following:

Proposition 3.11 (Joyce). *There is a new optimal quotient of* $J_0(431)$ *with Manin constant* 2.

Joyce's methods also produce examples with Manin constant 2 at levels 503 and 2089. For the convenience of the reader, we breifly discuss his example at level 431. There are exactly two elliptic curves E_1 and E_2 of prime conductor 431, and $E_1 \cap E_2 = 0$ as subvarieties of $J_0(431)$, so $A = E_1 \times E_2$ is an optimal quotient of $J_0(431)$ attached to a saturated ideal I. If f_i is the newform corresponding to E_i, then one finds that $f_1 \equiv f_2 \pmod 2$, and so $g = (f_1 - f_2)/2 \in S_2(\mathbf{Z})[I]$. However g is not in the image of $\mathrm{H}^0(A_{\mathbf{Z}}, \Omega_{A/\mathbf{Z}})$. Thus the Manin constant of A is divisible by 2.

3.4.2 The Atkin-Lehner obstruction

Let $\Gamma = \Gamma_0(N)$ or $\Gamma_1(N)$ and $J = \mathrm{Jac}(X_\Gamma)$.

Proposition 3.12. *If the Atkin-Lehner operator* W_ℓ *does not preserve* $S_2(\mathbf{Z}_\ell)$, *then* $\ell \mid c_J$.

Proof. If $\ell \nmid c_J$, then the image of $\mathrm{H}^0(J_{\mathbf{Z}_\ell}, \Omega_{J/\mathbf{Z}_\ell})$ in $S_2(\mathbf{Z}_\ell)$ equals $S_2(\mathbf{Z}_\ell)$. By the Néron mapping property, W_ℓ preseves $\mathrm{H}^0(J_{\mathbf{Z}_\ell}, \Omega_{J/\mathbf{Z}_\ell})$, i.e., it preserves $S_2(\mathbf{Z}_\ell)$. This contradicts the hypothesis. □

For example, we find by computation for each prime $\ell \leq 100$ that W_ℓ does not leave $S_2(\Gamma_0(11\ell); \mathbf{Z})$ invariant. Proposition 3.12 then implies that the Manin constant of $J_0(11\ell)$ is divisible by ℓ for these values of ℓ.

4 Proofs of Theorems

4.1 Proof of Theorem 3.3

Suppose Γ is a subgroup of $\Gamma_0(N)$ that contains $\Gamma_1(N)$. Let $J = \mathrm{Jac}(X_\Gamma)$ and $J' = J_1(N)$. Suppose A is an optimal quotient of J. By [CES03, §6.1.2] the Manin constant of J' is an integer, so $\mathrm{H}^0(J'_{/\mathbf{Z}}, \Omega_{J'/\mathbf{Z}}) \hookrightarrow S_2(\Gamma_1(N); \mathbf{Z})$. The maps $J' \to J \to A$ induce a chain of inclusions

$$\mathrm{H}^0(A_{\mathbf{Z}}, \Omega_{A/\mathbf{Z}}) \hookrightarrow \mathrm{H}^0(J_{\mathbf{Z}}, \Omega_{J/\mathbf{Z}}) \hookrightarrow \mathrm{H}^0(J'_{/\mathbf{Z}}, \Omega_{J'/\mathbf{Z}}) \hookrightarrow S_2(\Gamma_1(N); \mathbf{Z}) \hookrightarrow \mathbf{Z}[[q]].$$

Combining this chain of inclusions with commutativity of the diagram

$$\begin{tikzcd} & S_2(\Gamma_1(N)) \arrow[dr, "q-\exp"] & \\ S_2(\Gamma) \arrow[ur, "f(q)\mapsto f(q)"] \arrow[rr, "q-\exp"] & & \mathbf{C}[[q]] \end{tikzcd}$$

implies that the image of $\mathrm{H}^0(A_{\mathbf{Z}}, \Omega_{A/\mathbf{Z}})$ lies in $S_2(\mathbf{Z})[I]$, as claimed.

4.2 Two lemmas

In this section, we state two lemmas that will be used in the proofs of Theorems 3.4, 3.7, and 3.9.

LEMMA 4.1. *Suppose $i : A \hookrightarrow B$ is an injective homomorphism of finitely generated torsion free abelian groups. Let C be the torsion subgroup of $B/i(A)$. Then a prime p divides $\#C$ if and only if the induced map $A \otimes \mathbf{F}_p \to B \otimes \mathbf{F}_p$ is not injective.*

Proof. Let D denote the quotient $B/i(A)$. Tensor the exact sequence $0 \to A \to B \to D \to 0$ with $\mathbf{F}_p$. The associated long exact sequences reveals that $\ker(A \otimes \mathbf{F}_p \to B \otimes \mathbf{F}_p) \cong D_{\mathrm{tor}}[p]$. $\square$

Suppose ℓ is a prime such that $\ell^2 \nmid N$. Let $\mathcal{X}$ be the smooth locus of a minimal proper regular model for $X_0(N)$ over $\mathbf{Z}_\ell$, and let $\Omega_{\mathcal{X}}$ denote the sheaf of regular differentials as in [MR91, §7] (in the notation of [Maz78, §2(e)], it is the sheaf Ω base-changed to $\mathbf{Z}_\ell$).

Recall that π denotes the quotient map $J_0(N) \to A$. Let G be a subgroup of $H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}})$, and consider the chain of inclusions

$$G \hookrightarrow H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}}) \xrightarrow{\pi^*} H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \cong H^0(\mathcal{X}, \Omega_{\mathcal{X}}) \xrightarrow{q\text{-exp}} \mathbf{Z}_\ell[[q]], \quad (1)$$

where the map q-exp is the q-expansion map on differentials as in [Maz78, §2(e)] (actually, Mazur works over $\mathbf{Z}$; our map is obtained by tensoring with $\mathbf{Z}_\ell$). Denote by Φ the composite of the maps above. We have the following commutative diagram

$$\begin{tikzcd} & S_2(\mathbf{Z}_\ell) \arrow[dr, "F\text{-exp}"] & \\ H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}}) \arrow[ur, "\psi"] \arrow[rr, "\Phi"] & & \mathbf{Z}_\ell[[q]], \end{tikzcd} \quad (2)$$

where ψ is as in Section 3.1 and F-exp is the Fourier expansion map.

We say that a subgroup B of group C is *saturated* (in C) if the cokernel C/B is torsion free.

LEMMA 4.2. *If $\ell \mid N$, then suppose either that A is a newform quotient, or that ℓ is odd and $W_\ell \cdot \psi(G) \subseteq S_2(\mathbf{Z}_\ell)$. Assume that the induced map $G \otimes \mathbf{F}_\ell \to H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell$ is injective. Then the image of G under the composite of the maps in (1) above is saturated in $\mathbf{Z}_\ell[[q]]$.*

Proof. By Lemma 4.1, it suffices to prove that the induced map $\Phi_\ell : G \otimes \mathbf{F}_\ell \to \mathbf{Z}_\ell[[q]] \otimes \mathbf{F}_\ell = \mathbf{F}_\ell[[q]]$ is injective.

Suppose $\ell \nmid N$. Then by the q-expansion principle, the q-expansion map $H^0(\mathcal{X}_{\mathbf{F}_\ell}, \Omega_{\mathcal{X}_{\mathbf{F}_\ell}}) \to \mathbf{F}_\ell[[q]]$ is injective. The injectivity of Φ_ℓ now follows since by hypothesis the induced map $G \otimes \mathbf{F}_\ell \to H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell = H^0(\mathcal{X}_{\mathbf{F}_\ell}, \Omega_{\mathcal{X}_{\mathbf{F}_\ell}})$ is injective.

Next suppose that $\ell \mid N$. By hypothesis, $\psi(G) + W_\ell \cdot \psi(G) \subseteq S_2(\mathbf{Z}_\ell)$ (note that if A is a newform quotient, this follows since W_ℓ acts as 1 or -1). Hence, in the commutative diagram (2), the map Φ_ℓ factors as $G \otimes \mathbf{F}_\ell \xrightarrow{\psi} (\psi(G) + W_\ell \cdot \psi(G)) \otimes \mathbf{F}_\ell \xrightarrow{F\text{-exp}} \mathbf{F}_\ell[[q]]$, and so $\ker(\Phi_\ell)$ is invariant under W_ℓ. If the characteristic ℓ is odd, then since W_ℓ is an involution, $\ker(\Phi_\ell)$ is a direct sum of $+1$ and -1 eigenspaces for W_ℓ. If A is associated to a single newform, then W_ℓ acts either as $+1$ or as -1. Thus in either case, it suffices to prove that if $\omega \in \ker(\Phi_\ell)$ is in either the $+1$ or -1 eigenspace for the action of W_ℓ on $\ker(\Phi_\ell)$, then $\omega = 0$.

Suppose $\omega \in \ker(\Phi_\ell)$ is in the -1 eigenspace. Since $\ell^2 \nmid N$, we have $\ell \,||\, N$, and so the reduction $\mathcal{X}_{\mathbf{F}_\ell}$ is a union of two copies of $X_0(N/\ell)_{\mathbf{F}_\ell}$ identified transversely at the supersingular points. These two copies are swapped under the action of the Atkin-Lehner involution W_ℓ. Since $\omega \in \ker(\Phi_\ell)$, the q-expansion principle implies that ω vanishes on the irreducible component containing the cusp ∞. Then $W_\ell(\omega)$ vanishes on the other irreducible component. Since $W_\ell(\omega) = -\omega$, we see that ω is zero on both components; hence $\omega = 0$. Here, we have been thinking of $G \otimes \mathbf{F}_\ell$ as a subgroup of $H^0(\mathcal{X}_{\mathbf{F}_\ell}, \Omega_{\mathcal{X}_{\mathbf{F}_\ell}})$, which we can do by the hypothesis that the induced map $G \otimes \mathbf{F}_\ell \to H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell = H^0(\mathcal{X}_{\mathbf{F}_\ell}, \Omega_{\mathcal{X}_{\mathbf{F}_\ell}})$ is injective. A similar argument shows that if $\omega \in \ker(\Phi_\ell)$ is in the $+1$ eigenspace for the action of W_ℓ, then $\omega = 0$. $\square$

4.3 Proof of Theorem 3.4

We continue to use the notation of Section 4.2 and assume in addition that the hypotheses of Theorem 3.4 are satisfied. To show that $\ell \nmid c_A$, we have to show that c_A is a unit in $\mathbf{Z}_\ell$. For this, it suffices to check that the image of $H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}})$ in $\mathbf{Z}_\ell[[q]]$ is saturated, since the image of $S_2(\Gamma_0(N); \mathbf{Z}_\ell)[I]$ is saturated in $\mathbf{Z}_\ell[[q]]$. In view of Lemma 4.1, taking $G = H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}})$, it suffices to show that the map $H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell \to H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell$ is injective.

Since A is optimal, $\ell \neq 2$, and J has good or semistable reduction at ℓ, [Maz78, Cor 1.1] yields an exact sequence

$$0 \to H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}}) \to H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \to H^0(B_{\mathbf{Z}_\ell}, \Omega_{B_{\mathbf{Z}_\ell}}) \to 0$$

where $B = \ker(J \to A)$. Since $H^0(B_{\mathbf{Z}_\ell}, \Omega_{B_{\mathbf{Z}_\ell}})$ is torsion free, by Lemma 4.1 the map $H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell \to H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell$ is injective, as was to be shown.

4.4 Proof of Theorem 3.9

We continue to use the notation of Section 4.2 and assume in addition that A is a newform quotient, and that $\ell \nmid m_A$. We have to show that then $\ell \nmid c_A$. As before, it suffices to check that the image of $H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}})$ in $\mathbf{Z}_\ell[[q]]$ is saturated, since the image of $S_2(\Gamma_0(N); \mathbf{Z}_\ell)[I]$ is saturated in $\mathbf{Z}_\ell[[q]]$. We use Lemma 4.2, with $G = H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}})$; thus it suffices to show that the map $H^0(A_{\mathbf{Z}_\ell}, \Omega_{A_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell \to H^0(J_{\mathbf{Z}_\ell}, \Omega_{J_{\mathbf{Z}_\ell}}) \otimes \mathbf{F}_\ell$ is injective.

The composition of pullback and pushforward in the following diagram is multiplication by the modular exponent of A:

$$\begin{tikzcd} & \mathrm{H}^0(J_{\mathbf{Z}_\ell}, \Omega_{J/\mathbf{Z}_\ell}) \arrow[dr, "\pi_*"] & \\ \mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{A/\mathbf{Z}_\ell}) \arrow[ur, "\pi^*"] \arrow[rr, "m_A"] & & \mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{A/\mathbf{Z}_\ell}) \end{tikzcd}$$

Let $\overline{\pi}_*$ and $\overline{\pi}^*$ denote the maps obtained by tensoring the diagram above with $\mathbf{F}_\ell$. Then $\overline{\pi}_* \circ \overline{\pi}^*$ is multiplication by an integer coprime to ℓ from the finite dimension $\mathbf{F}_\ell$-vector space $\mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{A/\mathbf{Z}_\ell}) \otimes \mathbf{F}_\ell$ to itself, hence an isomorphism. In particular, $\overline{\pi}^*$ is injective, which is what was left to show.

Remark 4.3. Adam Joyce observed that one can also obtain injectivity of $\overline{\pi}^*$ as a consequence of Prop. 7.5.3(a) of [BLR90].

4.5 Proof of Theorem 3.7

Theorem 3.7 asserts that if $A = A_f$ is a quotient of $J = J_0(N)$ attached to a newform f, and ℓ is a prime such that $\ell^2 \nmid N$, then $\mathrm{ord}_\ell(c_A) \leq \dim(A)$. Our proof follows [AU96], except at the end we argue using indexes instead of multiples.

Let B denote the kernel of the quotient map $J \to A$. Consider the exact sequence $0 \to B \to J \to A \to 0$, and the corresponding complex $B_{\mathbf{Z}_\ell} \to J_{\mathbf{Z}_\ell} \to A_{J_{\mathbf{Z}_\ell}}$ of Néron models. Because $J_{\mathbf{Z}_\ell}$ has semiabelian reduction (since $\ell^2 \nmid N$), Theorem A.1 of the appendix of [AU96, pg. 279–280], due to Raynaud, implies that there is an integer r and an exact sequence

$$0 \to \mathrm{Tan}(B_{\mathbf{Z}_\ell}) \to \mathrm{Tan}(J_{\mathbf{Z}_\ell}) \to \mathrm{Tan}(A_{\mathbf{Z}_\ell}) \to (\mathbf{Z}/\ell\mathbf{Z})^r \to 0.$$

Here Tan is the tangent space at the 0 section; it is a free abelian group of rank equal to the dimension. Note that Tan is $\mathbf{Z}_\ell$-dual to the cotangent space, and the cotangent space is isomorphic to the space of global differential 1-forms. The theorem of Raynaud mentioned above is the generalization to $e = \ell - 1$ of [Maz78, Cor. 1.1], which we used above in the proof of Theorem 3.4.

Let C be the cokernel of $\mathrm{Tan}(B_{\mathbf{Z}_\ell}) \to \mathrm{Tan}(J_{\mathbf{Z}_\ell})$. We have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tan}(B_{\mathbf{Z}_\ell}) \arrow[r] & \mathrm{Tan}(J_{\mathbf{Z}_\ell}) \arrow[rr] \arrow[dr, two heads] & & \mathrm{Tan}(A_{\mathbf{Z}_\ell}) \arrow[r] & (\mathbf{Z}/\ell\mathbf{Z})^r \arrow[r] & 0. \\ & & & C \arrow[ur, hook] & & & \end{tikzcd} \tag{3}$$

Since $C \subset \mathrm{Tan}(A_{\mathbf{Z}_\ell})$, so C is torsion free, and hence C is a free $\mathbf{Z}_\ell$-module of rank $d = \dim(A)$. Let $C^* = \mathrm{Hom}_{\mathbf{Z}_\ell}(C, \mathbf{Z}_\ell)$ be the $\mathbf{Z}_\ell$-linear dual of C.

Applying the $\mathrm{Hom}_{\mathbf{Z}_\ell}(-,\mathbf{Z}_\ell)$ functor to the two short exact sequences in (3), we obtain exact sequences

$$0 \to C^* \to \mathrm{H}^0(J_{\mathbf{Z}_\ell}, \Omega_{J/\mathbf{Z}_\ell}) \to \mathrm{H}^0(B_{\mathbf{Z}_\ell}, \Omega_{B/\mathbf{Z}_\ell}) \to 0,$$

and

$$0 \to \mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{A/\mathbf{Z}_\ell}) \to C^* \to (\mathbf{Z}/\ell\mathbf{Z})^r \to 0. \tag{4}$$

The $(\mathbf{Z}/\ell\mathbf{Z})^r$ on the right in (4) occurs as $\mathrm{Ext}^1_{\mathbf{Z}_\ell}((\mathbf{Z}/\ell\mathbf{Z})^r, \mathbf{Z}_\ell)$. Also, (4) implies that $r \leq d = \dim(A)$.

Using Lemma 4.2, with $G = C^*$, we see that the image of C^* in $\mathbf{Z}_\ell[[q]]$ under the compostie of the maps in (1) is saturated. The Manin constant for A at ℓ is the index of the image via q-expansion of $\mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{\mathbf{Z}_\ell})$ in $\mathbf{Z}_\ell[[q]]$ in its saturation. Since the image of C^* in $\mathbf{Z}_\ell[[q]]$ is saturated, the image of C^* is the saturation of the image of $\mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{\mathbf{Z}_\ell})$, so the Manin index at ℓ is the index of $\mathrm{H}^0(A_{\mathbf{Z}_\ell}, \Omega_{\mathbf{Z}_\ell})$ in C^*, which is ℓ^r by (4), hence is at most ℓ^d.

5 Appendix by J. Cremona: Verifying that $c = 1$

Let f be a normalised rational newform for $\Gamma_0(N)$. Let Λ_f be its period lattice; that is, the lattice of periods of $2\pi i f(z)dz$ over $H_1(X_0(N), \mathbf{Z})$.

We know that $E_f = \mathbf{C}/\Lambda_f$ is an elliptic curve E_f defined over $\mathbf{Q}$ and of conductor N. This is the optimal quotient of $J_0(N)$ associated to f. Our goal is two-fold: to identify E_f (by giving an explicit Weierstrass model for it with integer coeffients); and to show that the associated Manin constant for E_f is 1. In this section we will give an algorithm for this; our algorithm applies equally to optimal quotients of $J_1(N)$.

As input to our algorithm, we have the following data:

1. a $\mathbf{Z}$-basis for Λ_f, known to a specific precision;
2. the type of the lattice Λ_f (defined below); and
3. a complete isogeny class of elliptic curves $\{E_1, \ldots, E_m\}$ of conductor N, given by minimal models, all with $L(E_j, s) = L(f, s)$.

So E_f is isomorphic over $\mathbf{Q}$ to E_{j_0} for a unique $j_0 \in \{1, \ldots, m\}$.

The justification for this uses the full force of the modularity of elliptic curves defined over $\mathbf{Q}$: we have computed a full set of newforms f at level N, and the same number of isogeny classes of elliptic curves, and the theory tells us that there is a bijection between these sets. Checking the first few terms of the L-series (i.e., comparing the Hecke eigenforms of the newforms with the traces of Frobenius for the curves) allows us to pair up each isogeny class with a newform.

We will assume that one of the E_j, which we always label E_1, is such that Λ_f and Λ_1 (the period lattice of E_1) are approximately equal. This is true in practice, because our method of finding the curves in the isogeny class is to compute the coefficients of a curve from numerical approximations to the c_4 and c_6 invariants of $\mathbf{C}/\Lambda_f$; in all cases these are very close to integers which are the invariants of the minimal model of an elliptic curve of conductor N, which we call E_1. The other curves in the isogeny class are then computed from E_1. For the algorithm described here, however, it is irrelevant how the curves E_j were obtained, provided that Λ_1 and Λ_f are close (in a precise sense defined below).

Normalisation of lattices: every lattice Λ in $\mathbf{C}$ which defined over $\mathbf{R}$ has a unique $\mathbf{Z}$-basis ω_1, ω_2 satisfying one of the following:

- TYPE 1: ω_1 and $(2\omega_2 - \omega_1)/i$ are real and positive; or
- TYPE 2: ω_1 and ω_2/i are real and positive.

For Λ_f we know the type from modular symbol calculations, and we know ω_1, ω_2 to a certain precision by numerical integration; modular symbols provide us with cycles $\gamma_1, \gamma_2 \in H_1(X_0(N), \mathbf{Z})$ such that the integral of $2\pi i f(z)dz$ over γ_1, γ_2 give ω_1, ω_2.

For each curve E_j we compute (to a specific precision) a $\mathbf{Z}$-basis for its period lattice Λ_j using the standard AGM method. Here, Λ_j is the lattice of periods of the Néron differential on E_j. The type of Λ_j is determined by the sign of the discriminant of E_j: type 1 for negative discriminant, and type 2 for positive discriminant.

For our algorithm we will need to know that Λ_1 and Λ_f are approximately equal. To be precise, we know that they have the same type, and also we verify, for a specific postive ε, that

$$\left|\frac{\omega_{1,1}}{\omega_{1,f}} - 1\right| < \varepsilon \qquad \text{and} \qquad \left|\frac{\mathrm{im}(\omega_{2,1})}{\mathrm{im}(\omega_{2,f})} - 1\right| < \varepsilon. \tag{*}$$

Here $\omega_{1,j}$, $\omega_{2,j}$ denote the normalised generators of Λ_j, and $\omega_{1,f}$, $\omega_{2,f}$ those of Λ_f.

Pulling back the Néron differential on E_{j_0} to $X_0(N)$ gives $c \cdot 2\pi i f(z)dz$ where $c \in \mathbf{Z}$ is the Manin constant for f. Hence

$$c\Lambda_f = \Lambda_{j_0}.$$

Our task is now to

1. identify j_0, to find which of the E_j is (isomorphic to) the "optimal" curve E_f; and
2. determine the value of c.

Our main result is that $j_0 = 1$ and $c = 1$, provided that the precision bound ε in (*) is sufficiently small (in most cases, $\varepsilon < 1$ suffices). In order to state this precisely, we need some further definitions.

A result of Stevens says that in the isogeny class there is a curve, say E_{j_1}, whose period lattice Λ_{j_1} is contained in every Λ_j; this is the unique curve in the class with minimal Faltings height. (It is conjectured that E_{j_1} is the $\Gamma_1(N)$-optimal curve, but we do not need or use this fact. In many cases, the $\Gamma_0(N)$- and $\Gamma_1(N)$-optimal curves are the same, so we expect that $j_0 = j_1$ often; indeed, this holds for the vast majority of cases.)

For each j, we know therefore that $a_j = \omega_{1,j_1}/\omega_{1,j} \in \mathbf{N}$ and also $b_j = \mathrm{im}(\omega_{2,j_1})/\mathrm{im}(\omega_{2,j}) \in \mathbf{N}$. Let B be the maximum of a_1 and b_1.

PROPOSITION 5.1. *Suppose that (*) holds with $\epsilon = B^{-1}$; then $j_0 = 1$ and $c = 1$. That is, the curve E_1 is the optimal quotient and its Manin constant is* 1.

Proof. Let $\varepsilon = B^{-1}$ and $\lambda = \frac{\omega_{1,1}}{\omega_{1,f}}$, so $|\lambda - 1| < \varepsilon$. For some j we have $c\Lambda_f = \Lambda_j$. The idea is that $\mathrm{lcm}(a_1, b_1)\Lambda_1 \subseteq \Lambda_{j_1} \subseteq \Lambda_j = c\Lambda_f$; if $a_1 = b_1 = 1$, then the closeness of Λ_1 and Λ_f forces $c = 1$ and equality throughout. To cover the general case it is simpler to work with the real and imaginary periods separately.

Firstly,

$$\frac{\omega_{1,j}}{\omega_{1,f}} = c \in \mathbf{Z}.$$

Then

$$c = \frac{\omega_{1,1}}{\omega_{1,f}}\frac{\omega_{1,j}}{\omega_{1,1}} = \frac{a_1}{a_j}\lambda.$$

Hence

$$0 \leq |\lambda - 1| = \frac{|a_j c - a_1|}{a_1} < \varepsilon.$$

If $\lambda \neq 1$, then $\varepsilon > |\lambda - 1| \geq a_1^{-1} \geq B^{-1} = \varepsilon$, contradiction. Hence $\lambda = 1$, so $\omega_{1,1} = \omega_{1,f}$. Similarly, we have

$$\frac{\mathrm{im}(\omega_{2,j})}{\mathrm{im}(\omega_{2,f})} = c \in \mathbf{Z}$$

and again we can conclude that $\mathrm{im}(\omega_{2,1}) = \mathrm{im}(\omega_{2,f})$, and hence $\omega_{2,1} = \omega_{2,f}$.

Thus $\Lambda_1 = \Lambda_f$, from which the result follows. □

THEOREM 5.2. *For all $N < 60000$, every optimal elliptic quotient of $J_0(N)$ has Manin constant equal to* 1. *Moreover, the optimal curve in each class is the one whose identifying number on the tables [Cre] is* 1 *(except for class* 990h *where the optimal curve is* 990h3*).*

Proof. For all $N < 60000$ we used modular symbols to find all newforms f and their period lattices, and also the corresponding isogeny classes of curves. In all cases we verified that (*) held with the appropriate value of ε. (The case of 990h is only exceptional on account of an error in labelling the curves several years ago, and is not significant.) □

REMARK 5.3. In the vast majority of cases, the value of B is 1, so the precision needed for the computation of the periods is very low. For $N < 60000$, out of 258502 isogeny classes, only 136 have $B > 1$: we found $a_1 = 2$ in 13 cases,

$a_1 = 3$ in 29 cases, and $a_1 = 4$ and $a_1 = 5$ once each (for $N = 15$ and $N = 11$ respectively); $b_1 = 2$ in 93 cases; and no larger values. Class $17a$ is the only one for which both a_1 and b_1 are greater than 1 (both are 2).

Finally, we give a slightly weaker result for $60000 < N < 130000$; in this range we do not know Λ_f precisely, but only its projection onto the real line. (The reason for this is that we can find the newforms using modular symbols for $H_1^+(X_0(N), \mathbf{Z})$, which has half the dimension of $H_1(X_0(N), \mathbf{Z})$; but to find the exact period lattice requires working in $H_1(X_0(N), \mathbf{Z})$.) The argument is similar to the one given above, using $B = a_1$.

THEOREM 5.4. *For all N in the range $60000 < N < 130000$, every optimal elliptic quotient of $J_0(N)$ has Manin constant equal to 1, except for the following cases where the Manin constant is either 1 or 2:*

$$62416a, 67664a, 71888e, 72916a, 75092a, 85328d, 86452a, 96116a,$$
$$106292b, 112290a, 112290a, 115664a, 121168e, 125332a.$$

Proof. We continue to use the notation above. We do not know the lattice Λ_f but only (to a certain precision) a positive real number $\omega_{1,f}^+$ such that either Λ_f has type 1 and $\omega_{1,f} = 2\omega_{1,f}^+$, or Λ_f has type 2 and $\omega_{1,f} = \omega_{1,f}^+$. Curve E_1 has lattice Λ_1, and the ratio $\lambda = \omega_{1,1}^+/\omega_{1,f}^+$ satisfies $|\lambda - 1| < \varepsilon$. In all cases this holds with $\varepsilon = \frac{1}{3}$, which will suffice.

First assume that $a_1 = 1$.

If the type of Λ_f is the same as that of Λ_1 (for example, this must be the case if all the Λ_j have the same type, which will hold whenever all the isogenies between the E_j have odd degree) then from $c\Lambda_f = \Lambda_j$ we deduce as before that $\lambda = 1$ exactly, and $c = a_1/a_j = 1/a_j$, hence $c = a_j = 1$. So in this case we have that $c = 1$, though there might be some ambiguity in which curve is optimal if $a_j = 1$ for more than one value of j.

Assume next that Λ_1 has type 1 but Λ_f has type 2. Now $\lambda = \omega_{1,1}/2\omega_{1,f}$. The usual argument now gives $ca_j = 2$. Hence either $c = 1$ and $a_j = 2$, or $c = 2$ and $a_j = 1$. To see if the latter case could occur, we look for classes in which $a_1 = 1$ and Λ_1 has type 1, while for some $j > 1$ we also have $a_j = 1$ and Λ_j of type 2. This occurs 29 times for $60000 < N < 130000$, but for 15 of these the level N is odd, so we know that c must be odd. The remaining 14 cases are those listed in the statement of the Theorem. We note that in all of these 14 cases, the isogeny class consists of two curves, E_1 of type 1 and E_2 of type 2, with $[\Lambda_1 : \Lambda_2] = 2$, so that E_2 rather than E_1 has minimal Faltings height.

Next suppose that Λ_1 has type 2 but Λ_f has type 1. Now $\lambda = 2\omega_{1,1}/\omega_{1,f}$. The usual argument now gives $2ca_j = 1$, which is impossible; so this case cannot occur.

Finally we consider the cases where $a_1 > 1$. There are only three of these for $60000 < N < 130000$: namely, $91270a$, $117622a$ and $124973b$, where $a_1 = 3$. In each case the Λ_j all have the same type (they are linked via 3-isogenies) and the usual argument shows that $ca_j = 3$. But none of these levels is divisible by 3, so $c = 1$ in each case. $\square$

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