Conjugate Harmonic Functions and Clifford Algebras

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Abstract

We generalize a Hardy-Littlewood inequality and a Privalov inequality for conjugate harmonic functions in the plane to components of Clifford-valued monogenic functions.

1 Introduction

Throughout this paper a domain $\Omega \subset \mathbb{R}^n$ is a connected open set. Given $u : \Omega \to \mathbb{R}$ we write

$$\|u\|_{p,\Omega} = \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}, \qquad p > 0.$$

We denote the Lipschitz norm of u over Ω by

$$\|u\|^L_{k,\Omega} = \sup_{\substack{x_1, x_2 \in \Omega \\ x_1 \neq x_2}} \frac{|u(x_1) - u(x_2)|}{|x_1 - x_2|^k}$$

for $0 < k \leq 1$. In [3], Hardy and Littlewood proved the following result.

Theorem 1.1 *If $u+iv$ is analytic in a disk D centered at z_0, then there exists a constant C, depending only on p, such that*

$$\|u - u(z_0)\|_{p,D} \leq C \|v\|_{p,D} . \tag{1.1}$$

Similarly, Theorem 2.1 is given in [9] by Privalov.

Theorem 1.2 *If $u + iv$ is analytic in a disk D, then there exists a constant C, depending only on k, such that*

$$\|u\|_{k,D}^{L} \leq C \|v\|_{k,D}^{L}. \tag{1.2}$$

In fact Theorem 1.2 follows from Theorem 1.1 [7].

We prove versions of these theorems for components of monogenic functions valued in the universal Clifford algebras over $\mathbb{R}^n$. See Theorem 3.1.

In [12], Stein and Weiss studied systems of conjugate harmonic functions in $\mathbb{R}^n$. These are vectors of harmonic functions $(u_1, u_2, \ldots, u_n)$, which satisfy

$$\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \quad \text{and} \tag{1.3}$$

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \quad \text{for all} \quad i \text{ and } j. \tag{1.4}$$

Notice for $x_1 = x$, $x_2 = y$, $u_2 = u$ and $u_1 = v$ these are the usual Cauchy-Riemann equations.

The results of this paper hold in the special case of such Stein-Weiss systems. In this special case the results appear in [6] as well as versions for quasiregular mappings. The quasiregular theory was published in [4] and subsequently developed in [7] and [8]. The quasiregular case is a "one- dimensional" analytic theory in the sense that the properties of one component often determine those of the rest. The theory we present here is a "one- codimensional" analytic theory. The results for Stein-Weiss systems that appear in [6] have never been published.

We fix an orthonormal basis of $\mathbb{R}^n$, $(e_1, e_2, \ldots, e_n)$, and denote by $\mathcal{U}_n$ the (real) Clifford algebra spanned by the reduced multi-indexed products

$$e_\alpha = e_{\alpha_1} \cdots e_{\alpha_k},$$

$1 \leq \alpha_1 < \ldots < \alpha_k \leq n$, with the rule

$$e_j e_k + e_k e_j = -2\delta_{jk}.$$

Here $\delta_{jk} = 0$ if $j \neq k$, 1 if $j = k$. We have an increasing chain

$$\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathcal{U}_3 \subset \ldots \subset \mathcal{U}_n \subset \ldots.$$

Here $\mathbb{H}$ is the quaternions. As such a function $F : \Omega \to \mathcal{U}_n$, can be represented as $F = \sum_\alpha F_\alpha e_\alpha$ where each $F_\alpha : \Omega \to \mathbb{R}$. We define a norm

$$|F| = \left(\sum_\alpha |F_\alpha|^2 \right)^{\frac{1}{2}}.$$

We consider here a Dirac operator D, defined as follows. If $F = \sum_\alpha F_\alpha e_\alpha$, then

$$DF = \sum_\alpha \left(\sum_{i=1}^{n} \frac{\partial F_\alpha}{\partial x_i} e_i e_\alpha \right).$$

Definition 1.3 *A Clifford-valued function $F : \Omega \to \mathcal{U}_n$ is monogenic if $DF = 0$.*

We also define the Clifford Laplacian $\Delta F = \sum_\alpha \Delta F_\alpha e_\alpha$. Since $D^2 = -\Delta$, it follows that the coefficients of any monogenic function are harmonic in the usual sense.

We denote the length of a multi-index α by $|\alpha|$ and decompose any Clifford-valued function into its even and odd parts. We write $F = F_{\text{even}} + F_{\text{odd}} = \sum_{|\alpha| \text{ even}} F_\alpha e_\alpha + \sum_{|\alpha| \text{ odd}} F_\alpha e_\alpha$. Notice that the Dirac operator D maps even parts to odd parts and odd parts to even parts:

$$D(F_{\text{even}}) = (DF)_{\text{odd}} \qquad \text{and}$$

$$D(F_{\text{odd}}) = (DF)_{\text{even}}.$$

As such, F is monogenic if and only if $D(F_{\text{even}}) = 0$ and $D(F_{\text{odd}}) = 0$.

Definition 1.4 *A system of conjugate harmonic functions in a Clifford algebra consists of the coefficients of F_{even} or F_{odd} for some monogenic function F.*

A way to realize the Stein-Weiss systems in this context is through the natural embedding of $\mathbb{R}^n$ into $\mathcal{U}_n$, namely

$$(x_1, x_2, \dots, x_n) \mapsto x_1 e_1 + \dots + x_n e_n.$$

Here

$$D\left(\sum_{1}^{n} u_i e_i \right) = -\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} + \sum_{i=1}^{n} \sum_{i \neq j}^{n} \frac{\partial u_i}{\partial x_j} e_j e_i.$$

As such $D(F_{\text{odd}}) = 0$ is equivalent to (1.3) and (1.4) where $F = u_1 e_1 + \dots + u_n e_n$. We will use the notation

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n} \right) = \frac{\partial u}{\partial x_1} e_1 + \dots + \frac{\partial u}{\partial x_n} e_n$$

for $u:\Omega\to\mathbb{R}$. Notice that $DF=0$ is equivalent to 2^n linear equations involving the components of the vectors ∇F_α. Because the operator D intertwines parity, 2^{n-1} of these equations is a system for the even coefficients and the rest for the odd.

Example 1.5 *Let $F=u_0+u_1e_1+u_2e_2+u_3e_3+u_4e_1e_2+u_5e_1e_3+u_6e_2e_3+u_7e_1e_2e_3$. If $DF=0$, then*

$$\nabla u_0=\left(-\frac{\partial u_4}{\partial x_2}-\frac{\partial u_5}{\partial x_2}-\frac{\partial u_5}{\partial x_3}\right)e_1+\left(\frac{\partial u_4}{\partial x_1}+\frac{\partial u_5}{\partial x_1}-\frac{\partial u_6}{\partial x_3}\right)e_2+\left(\frac{\partial u_6}{\partial x_2}\right)e_3$$

and

$$\nabla u_1=\left(-\frac{\partial u_2}{\partial x_2}-\frac{\partial u_3}{\partial x_3},\frac{\partial u_2}{\partial x_1}-\frac{\partial u_7}{\partial x_3},\frac{\partial u_3}{\partial x_1}+\frac{\partial u_7}{\partial x_2}\right).$$

Notice that the second equation is a part of the Stein-Weiss system (1.3) and (1.4) when $u_7=0$.

Clearly such representations hold generally and we have the following simple estimate.

Lemma 1.6 *If $\{u_\alpha\}$ is a system of conjugate harmonic functions in the Clifford algebra $\mathcal{U}_n$, defined in $\Omega\subset\mathbb{R}^n$, then for each α,*

$$|\nabla u_\alpha|^2\le C(n)\sum_{\beta\ne\alpha}|\nabla u_\beta|^2 \tag{1.5}$$

in Ω. Here $C(n)$ is a constant that depends only on n.

We mention [1], [10] and [11] as references for Clifford analysis.

2 Notations and Domains

We assume throughout that w is a Muckenhoupt weight and write $w\in A^q_M(\Omega)$, $1<q<\infty$, $1\le M<\infty$, when $w\ge 0$ a.e. and

$$\frac{1}{|Q|}\int_Q w\le M\left(\frac{1}{|Q|}\int_Q\frac{1}{w^{(1-q)}}\right)^{1-q}$$

for all cubes $Q\subset\Omega$. Here $|Q|$ is the volume of Q. For $u:\Omega\to\mathbb{R}$ we write for $0<p<\infty$,

$$\|u\|^\sharp_{p,\Omega\mu}=\inf_{a\in\mathbb{R}}\left(\int_\Omega|u-a|^p d\mu\right)^{\frac{1}{p}}$$

where $d\mu = wdx$ is weighted Lebesgue measure. We define the Hardy-Littlewood sharp maximal function for $0 < p < \infty$,

$$M_p^\sharp(u,\mu)(x) = \sup_{\substack{Q\subset\Omega \\ x\in Q}} \mu(Q)^{-\frac{1}{p}} \|u\|_{p,Q,\mu}^\sharp.$$

Also the sharp BMO norm is

$$\|u\|_{\Omega,\mu}^{\text{BMO}} = \sup_{x\in\Omega} M_1^\sharp(u,\mu)(x).$$

Definition 2.1 *A domain Ω is a δ-John domain, $0 < \delta$, if there exists a point $x_0 \in \Omega$ which can be joined with any other point $x \in \Omega$ by a continuous curve $\gamma \subset \Omega$ which satisfies*

$$\delta\,|\xi - x| \leq \ d(\xi, \partial\Omega)$$

for all $\xi \in \gamma$.

John domains do not have external cusps. Using the geometry of John domains, weak local weighted L^p-estimates patch together to form global estimates. It is in this way that Theorem 3.2 is obtained. See [4] and [7].

Definition 2.2 *For $0 < k \leq 1$, Ω is a Lip_k-extension domain if there is a constant N such that every pair of points $x_1, x_2 \in \Omega$ can be joined by a continuous curve $\gamma \subset \Omega$ for which*

$$\int_\gamma d(\gamma(s), \partial\Omega)^{k-1} ds \leq N\,|x_1 - x_2|^k\,.$$

Theorem 2.3 appears in [2].

Theorem 2.3 *Suppose that Ω is a Lip_k-extension domain. If there are constants C_1 and C_2, $C_2 < 1$, so that*

$$|f(x_1) - f(x_2)| \leq C_1|x_1 - x_2|^k,$$

for all $x_1, x_2 \in \Omega$ with $|x_1 - x_2| \leq C_2 d(x_1, \partial\Omega)$, then there is a constant C_3, depending only on C_1, C_2, N and k, so that

$$|f(x_1) - f(x_2)| \leq C_3\,|x_1 - x_2|^k\,,$$

for all $x_1, x_2 \in \Omega$.

3 Main Results

Theorem 3.1 *Assume that* $\{u_\alpha\}$ *is a system of conjugate harmonic functions in* $\mathcal{U}_n$ *defined in* Ω *and* $w \in A_M^q(\Omega)$. *In each case* C *is a constant that is independent of* $\{u_\alpha\}$.

a) For $0 < p < \infty$,

$$M_p^\sharp(u_\alpha, x) \le C \sum_{\beta \ne \alpha} M_p^\sharp(u_\beta, x) \tag{3.1}$$

where $C = C(p, q, M, n)$.

b)

$$\|u_\alpha\|_{\Omega,\mu}^{\mathrm{BMO}} \le C \sum_{\beta \ne \alpha} \|u_\beta\|_{\Omega,\mu}^{\mathrm{BMO}} \tag{3.2}$$

where $C = C(q, M, n)$.

c) If Ω *is a* δ*-John domain,* $0 < p < \infty$, *then*

$$\|u_\alpha\|_{p,\Omega,\mu}^\sharp \le C \sum_{\beta \ne \alpha} \|u_\beta\|_{p,\Omega,\mu}^\sharp \tag{3.3}$$

where $C = C(p, q, M, \delta, n)$.

d) If Ω *is a* Lip_k*-extension domain with constant* N, $0 < k \le 1$, *then*

$$\|u_\alpha\|_{k,\Omega}^L \le C \sum_{\beta \ne \alpha} \|u_\beta\|_{k,\Omega}^L \tag{3.4}$$

where $C = C(N, n, k)$.

Proof of Theorem 3.1. We first prove c). Assertions a), b) and d) then follow. We use the following theorem which is a special case of Theorem 3.1 in [7] to supply a brief proof. The basic local results can be derived from the mean value property of harmonic functions and improvement of reverse Holder inequalities. The global result in John domains follows by patching together the weak local results and requires the special geometry of these domains (see [6] and [4]).

Theorem 3.2 *Suppose that* $0 < p < \infty$, Ω *is a* δ*-John domain,* $w \in A_M^q(\Omega)$ *and* u *and* v *are harmonic in* Ω. *If there is a constant* A *such that*

$$\|\nabla u\|_{2,Q} \le A \|\nabla v\|_{2,2Q}, \tag{3.5}$$

for all cubes Q *with* $2Q \subset \Omega$, *then there is a constant* B, *depending only on* A,p,n,q,δ *and* M, *so that*

$$\inf_{c \in \mathbb{R}} \|u - c\|_{p,\Omega,w} \le B \inf_{c \in \mathbb{R}} \|v - c\|_{p,\Omega,w}. \tag{3.6}$$

Indeed if $\{u_\alpha\}$ is a system of conjugate harmonic functions in Ω, then $u = u_\alpha$ and $v = \sum_{\beta \neq \alpha} u_\beta$ are harmonic for each α. Moreover (1.5) shows that (3.5) holds and so (3.3) follows from (3.6). Since cubes are John domains, a) and b) of Theorem 3.1 follow from c). Locally the Lipschitz norm $\|u\|^L_{k,\Omega}$, $0 < k \leq 1$, is equivalent to the norm $\sup_{Q \subset \Omega} |Q|^{-1-(k/n)} \|u - u_Q\|_{1,Q}$ where the supremum is over all local cubes Q. See [5] for this result. Hence if Ω is a Lip_k-extension domain then d) follows using Theorem 2.3.

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