SOME CONVOLUTION INEQUALITIES AND THEIR APPLICATIONS

Daniel M. Oberlin

Department of Mathematics, Florida State University

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Abstract. We introduce a class of convolution inequalities and study the implications of these inequalities for certain problems in harmonic analysis.

§1 Introduction

Let λ be a nonnegative Borel measure on $\mathbb{R}^n$. Two familiar problems in harmonic analysis are

(C) *determine the indices p and q such that the convolution estimate*

$$\|\lambda * f\|_q \leq C(\lambda, p, q)\ \|f\|_p$$

holds for $f \in L^p(\mathbb{R}^n)$,

and

(R*) *determine the indices p and q such that the adjoint Fourier restriction estimate*

$$\|\widehat{fd\lambda}\|_q \leq C(\lambda, p, q)\ \|f\|_{L^p(\lambda)}$$

holds for $f \in L^p(\mathbb{R}^n)$.

Results on problems (C) and (R*) generally deal either with pushing the range of indices for which the estimates hold in cases where the measures are "nondegenerate" or with extending such previously-known results to cover degenerate cases. The results in this paper are of the latter type. In particular, our interest here is to study the implications for problems (C) and (R*) of certain inequalities which λ might satisfy. Our study of these inequalities was originally motivated by the hope that, with the proper choice of measure λ on the possibly degenerate manifold M, there would exist convolution and restriction estimates which are uniform over large classes of M. And this hope was, in turn, motivated by the paper [8] of Drury, where was first pointed out the suitability to problems in harmonic analysis of affinely invariant measures.

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To begin, we will say that λ satisfies (1) if, letting $|E|$ denote the Lebesgue measure of $E \subseteq \mathbb{R}^n$ (in this paper E will always denote a Borel subset of $\mathbb{R}^n$), we have

$$\|\lambda * \chi_E\|_{L^2(\lambda)} \leq c\ |E|^{\frac{1}{2}} \tag{1}$$

for some positive constant c and all $E \subseteq \mathbb{R}^n$. We will also be interested in a weakened version of (1): if there exists some real-valued Borel function ω on $\mathrm{supp}(\lambda)$ such that the inequality

$$\int \Big(\int_{\{\omega(y_1) \geq \omega(y_2)\}} \chi_E(y_2 - y_1) d\lambda(y_1) \Big)^2 d\lambda(y_2) \leq c\ |E| \tag{2}$$

holds for some positive constant c and all $E \subseteq \mathbb{R}^n$, then we will say that λ satisfies (2).

We will prove the following theorems.

Theorem 1. *If λ satisfies (1), then the inequality $\|\lambda * f\|_3 \leq C(c)\ \|f\|_{\frac{3}{2}}$ holds for $f \in L^{\frac{3}{2}}(\mathbb{R}^n)$.*

Theorem 2. *If λ satisfies (2), then the inequality $\|\lambda * \chi_E\|_3 \leq C(c)\ |E|^{\frac{2}{3}}$ holds for $E \subseteq \mathbb{R}^n$.*

Theorem 3. *If λ satisfies (1), then the inequality $\|(\chi_A d\lambda) * (\chi_B d\lambda)^{\sim}\|_{L^{2,\infty}(\mathbb{R}^n)} \leq c\ \lambda(A)^{\frac{1}{4}} \lambda(B)^{\frac{1}{4}}$ holds for Borel subsets A and B of the support of λ.*

Theorem 4. *If λ satisfies (2), then the inequality $\|\widehat{fd\lambda}\|_q \leq C(c,p)\ \|f\|_{L^p(\lambda)}$ holds for $f \in L^p(\lambda)$ whenever $1 \leq p < 4$ and $\frac{1}{p} + \frac{3}{q} = 1$.*

Notes: (a) The estimate in Theorem 2 is equivalent to a weak-type version of the estimate in Theorem 1.
(b) We will prove Theorem 4 by observing that (2) implies the conclusion of Theorem 3 when $A = B$ (which inequality we regard as a weak endpoint estimate for the adjoint Fourier restriction problem) and then showing how that inequality implies the conclusion of Theorem 4.
(c) Concerning the organization of this paper, the proofs for the theorems will be found in §2. Proofs for the examples are in §3.

Here is an easy example of a measure λ satisfying (1): let $d\lambda$ be dt on the curve (t, t^2) in $\mathbb{R}^2$. Then, for $E \subseteq \mathbb{R}^2$,

$$\int \frac{1}{2} \Big(\int \chi_E(t-s, t^2 - s^2)\ ds \Big)^2 dt \leq \int \int \chi_E(t-s, t^2-s^2)\ 2|t-s|\ dsdt = |E|,$$

giving (1) with $c = 2^{1/2}$. Of course the convolution and restriction results which follow for this λ from Theorems 1-4 are, except perhaps for Theorem 3, not new. But our study of inequalities like (1) and (2) began with a desire to obtain similar results with constants uniform over a large class of measures on curves generalizing dt on (t, t^2). Indeed, the proofs for Theorems 2 and 4 are simple abstractions of parts of the proofs in [16] and [19], respectively (see the discussion after Example 1 below), while the proof of Theorem 1 bears the same relationship to the proof of Theorem 1 in [15].

Here are five examples of measures satisfying (1) or (2):

Example 1. *Suppose ϕ is a real-valued $C^{(2)}$ function on (a,b) with ϕ'' positive and nondecreasing. Let $d\lambda$ be the measure $\phi''(t)^{1/3}dt$ on the curve $\big(t,\phi(t)\big)$ in $\mathbb{R}^2$. Then, for $E \subseteq \mathbb{R}^2$, we have*

$$\int_a^b \Big(\int_t^b \chi_E\big(t-s,\phi(t)-\phi(s)\big)\phi''(s)^{1/3}ds\Big)^2 \phi''(t)^{1/3}dt \leq 4|E|. \tag{3}$$

Here the measure λ is called the affine arclength measure on the curve $\big(t,\phi(t)\big)$. As previously mentioned, its study in harmonic analysis was initiated by Drury [8]. Thus this example shows that (2) holds uniformly for affine arclength measure on a large class of curves in $\mathbb{R}^2$ (with $\omega(t)=t$). The uniform convolution and restriction results furnished by this example and Theorems 2 and 4 are the main results in [16] and [19], respectively, while the proof of (3) can be found in either of those papers. It can be shown that there are examples of this type for which the stronger inequality (1) fails. It is an open question whether a uniform version of Theorem 1 holds anyway.

Example 2. *If p is a real-valued polynomial of degree N and p'' is of constant sign on an interval I, then (1) holds uniformly for affine arclength measure on the graph of p over I:*

$$\int_I \Big(\int_I \chi_E\big(t-s,p(t)-p(s)\big)|p''(s)|^{\frac13}ds\Big)^2 |p''(t)|^{\frac13}dt \leq C(N)\ |E|.$$

Generalizing the (t,t^2) example, this is a consequence of Lemma 3.1 in [2] and an argument originally appearing in [15]. The uniform convolution estimate which then follows from Theorem 1 was originally proved in [18].

The following definition is required for Examples 3, 6, and 8:

Definition. *Suppose $\Omega \subseteq \mathbb{R}^p$. Say that a mapping $F:\Omega \mapsto \mathbb{R}^p$ has generic multiplicity ℓ if card $F^{-1}(y) \leq \ell$ for almost all $y \in \mathbb{R}^p$.*

Example 3. *Suppose D is an open subset of $\mathbb{C}$ and that $\phi: D \to \mathbb{C}$ is analytic. Suppose that ϕ' and the map $(z,w)\mapsto \big(z-w,\phi(z)-\phi(w)\big)$ from D^2 into $\mathbb{C}^2$ both have generic multiplicities ℓ. Let dw be two-dimensional Lebesgue measure on $\mathbb{C}$. Then the following inequality holds for $E \subseteq \mathbb{C}^2$ with constant c depending only on ℓ:*

$$\int_D \Big(\int_{\{w\in D:|\phi''(w)|\geq|\phi''(z)|\}} \chi_E\big(z-w,\phi(z)-\phi(w)\big)\,|\phi''(w)|^{2/3}dw\Big)^2 |\phi''(z)|^{2/3}dz \leq c\ |E|.$$

Thus (2) holds with $d\lambda = |\phi''(z)|^{2/3}dz$ on the graph

$$\big\{\big(z,\phi(z)\big) : z \in D\big\}$$

and with $\omega(z) = |\phi''(z)|$. This is an analogue of Example 1. Earlier convolution and restriction estimates for measures on 2-dimensional manifolds in $\mathbb{R}^4$ can be found in [9] and [5] respectively. There are also examples where (1) holds in this setting:

Example 4. *Suppose that D is an open and convex subset of $\mathbb{C}$. Suppose additionally that ϕ is a polynomial of degree N and that there is a constant A such that the inequality*

$$|w-z|\int_0^1 |\phi''\big(z+t(w-z)\big)|dt \le A\ |\phi'(z)-\phi'(w)| \tag{4}$$

holds for $z,w\in D$. Then for $E\subseteq\mathbb{C}^2$ there is the inequality

$$\int\limits_D \Big(\int\limits_D \chi_E\big(z-w,\phi(z)-\phi(w)\big)|\phi''(w)|^{2/3}dw\Big)^2|\phi''(z)|^{2/3}dz \le c\ |E|.$$

with $c=c(A,N)$.

For example, if $\phi(z)=z^N$ and D is a sector with vertex 0 and angle $\le\frac{\pi}{4N}$, then (4) holds.

Convolution and restriction results such as Theorems 1–4 are most familiar when, as in Examples 1–4, n is even and λ is a measure on an $\frac{n}{2}$-dimensional manifold in $\mathbb{R}^n$. Here is an example on a (degenerate) 2-manifold in $\mathbb{R}^3$, the light cone.

Example 5. *Consider the cone $C=\{(y,|y|):y\in\mathbb{R}^2\}\subseteq\mathbb{R}^{2+1}=\mathbb{R}^3$. Then the inequality*

$$\int\Big(\int\limits_{\{|y_1|\le|y_2|\}} \chi_E(y_2-y_1,|y_2|-|y_1|)|y_1|^{-1}dy_1\Big)^2|y_2|^{-1}dy_2 \le c\ |E| \tag{5}$$

holds for $E\subseteq\mathbb{R}^3$.

This is (2) with $d\lambda=|y|^{-1}dy$ on C and with $\omega(y,|y|)=|y|^{-1}$. The convolution result implied by Example 5 and Theorem 2 is weaker than the appropriate special case of the main theorem in [12] (see also Theorem 1 in [14]). The restriction result which follows from Theorem 4 is originally due to Barcelo [3], though the current proof is simpler than his or the proof in [10]. An analogue of (5) also holds for the measure $dsdt$ on the surface $(s-t,s^2-t^2,s^3-t^3)$ in $\mathbb{R}^3$.

Theorems 1–4 admit certain generalizations. For example, the two following results are extensions of Theorems 1 and 2, respectively.

Theorem 5. *Suppose that λ is a nonnegative Borel measure on $\mathbb{R}^n$ satisfying the inequality*

$$\|\lambda*\chi_E\|_{L^m(\lambda)} \le c\ |E|^{\frac{m-1}{m}} \tag{6}$$

for $E\subseteq\mathbb{R}^n$. Then the convolution estimate

$$\|\lambda*f\|_{m+1} \le C(c)\ \|f\|_{\frac{m+1}{m}}$$

holds for $f\in L^{\frac{m+1}{m}}(\mathbb{R}^n)$.

Theorem 6. *Suppose that λ is a nonnegative Borel measure on $\mathbb{R}^n$. Suppose that ω is a real-valued Borel function on supp(λ) and suppose the inequality*

$$\int\Big(\int\limits_{\{\omega(y_1)\ge\omega(y_2)\}} \chi_E(y_2-y_1)d\lambda(y_1)\Big)^m d\lambda(y_2) \le c\ |E|^{m-1} \tag{7}$$

holds for $E \subseteq \mathbb{R}^n$. Then the convolution estimate

$$\|\lambda * \chi_E\|_{m+1} \leq C(c)\ |E|^{\frac{m}{m+1}}$$

holds whenever $E \subseteq \mathbb{R}^n$.

When $m = n$, Theorems 5 and 6 describe convolution estimates which one might expect when λ is a measure on an $(n-1)$-dimensional manifold in $\mathbb{R}^n$. Here are two examples:

Example 6. *Suppose ϕ is a real-valued $C^{(2)}$ function defined on a subset Ω of $\mathbb{R}^{n-1}$. Define $\gamma : \Omega \mapsto \mathbb{R}^n$ by $\gamma(y) = \big(y, \phi(y)\big)$. Then the inequality*

$$\int_\Omega \Big(\int_{\{|\det H_\phi(y_1)| \geq |\det H_\phi(y_2)|\}} \chi_E(y_2 - y_1) |\det H_\phi(y_1)|^{\frac{1}{n+1}}\, dy_1 \Big)^n |\det H_\phi(y_2)|^{\frac{1}{n+1}}\, dy_2 \leq c\, |E|^{n-1}$$

holds for $E \subseteq \mathbb{R}^n$ with constant c depending only on n and the generic multiplicities of the gradient of ϕ and the map

$$\Psi : \Omega^n \to (\mathbb{R}^n)^{n-1}, \quad \Psi(y_1, \dots, y_n) = \big(\gamma(y_1) - \gamma(y_2), \dots, \gamma(y_1) - \gamma(y_n)\big).$$

Thus (7) holds with $m = n$ and $\omega(x) = |\det H_\phi(x)|$. This is (2) in [17]. (The conclusion that follows from Example 6 and Theorem 6 is the principal result of [17]). The stronger hypothesis (6) of Theorem 5 seems substantially more difficult to verify in any generality. The next example may be compared with Example 5:

Example 7. *Consider the paraboloid $P = \{(y, |y|^2) : y \in \mathbb{R}^2\} \subseteq \mathbb{R}^{2+1} = \mathbb{R}^3$. Then the inequality*

$$\int \Big(\int \chi_E(y_2 - y_1, |y_2|^2 - |y_1|^2) dy_1 \Big)^3 dy_2 \leq c\, |E|^2$$

holds for $E \subseteq \mathbb{R}^3$.

If $d\lambda = dx$ on P, this is (6) with $m = 3$. The convolution result which follows from Example 7 and Theorem 5 is well-known. But, as will be pointed out after the proof of Theorem 5, the hypothesis of Theorem 5 actually yields a slightly stronger conclusion.

Not surprisingly, the inequalities of Examples 6 and 7 seem irrelevant to Fourier restriction. But, following a suggestion of A. Seeger and S. Wainger, we will prove the following result:

Theorem 7. *Suppose that λ is a nonnegative Borel measure on $\mathbb{R}^n$ satisfying the inequality*

$$(8) \quad \int \Big(\int_{\{\omega(y_j) \geq \omega(y_1)\}} \chi_E(y_1 + y_2 + \cdots + y_m) d\lambda(y_2) \cdots d\lambda(y_m) \Big)^{\frac{m+2}{2}} d\lambda(y_1) \leq c\, |E|$$

for some nonnegative integer $m \geq 3$, some real-valued Borel function ω on supp(λ), and all $E \subseteq \mathbb{R}^n$. Then the adjoint restriction estimate

$$(9) \quad \|\widehat{f d\lambda}\|_q \leq C(c, p)\ \|f\|_{L^p(\lambda)}$$

holds whenever $\frac{1}{p}+\frac{m(m+1)}{2}\frac{1}{q}=1$ *and* $1\leq p<m+2$.

Example 8. *Let* $\gamma(t)=(\gamma_1(t),\dots,\gamma_n(t)):[a,b]\to\mathbb{R}^n$ *parametrize a* $C^{(n)}$ *curve in* $\mathbb{R}^n$. *Write* $D(t)=|\det[\gamma^{'}(t),\dots,\gamma^{(n)}(t)]|$. *Let*

$$\Delta(t_1,\dots,t_n)=|\det[\gamma^{'}(t_1),\dots,\gamma^{'}(t_n)]|$$

and let $V(t_1,\dots,t_n)$ *be the (absolute value of) the Vandermonde determinant given by* $\Delta(t_1,\dots,t_n)$ *when* $\gamma_j(t)=t^j$. *Suppose there is a positive constant* A *such that*

$$A^{-1}\Delta(t_1,\dots,t_n)\leq V(t_1,\dots,t_n)\cdot\sup_{1\leq j\leq n}D(t_j)\leq A^{-1}\Delta(t_1,\dots,t_n)$$

for $t_j\in[a,b]$. *Then there is a constant* c, *depending only on* A, n, *and the generic multiplicity of the map* $(t_1,\dots,t_n)\mapsto\gamma(t_1)+\cdots+\gamma(t_n)$ *such that the inequality*

$$\int\Big(\int_{\{D(t_j)\leq D(t_1)\}}\chi_E\big(\gamma(t_1)+\cdots+\gamma(t_n)\big)\prod_2^n D(t_j)^{\frac{2}{n(n+1)}}dt_2\cdots dt_n\Big)^{\frac{n+2}{2}}D(t_1)^{\frac{2}{n(n+1)}}dt_1$$
$$\leq c\,|E|$$

holds for $E\subseteq\mathbb{R}^n$.

That is, if λ is the measure on the curve γ given by $d\lambda=D(t)^{\frac{2}{n(n+1)}}dt$, then λ satisfies (8) with $m=n$ and $\omega\big(\gamma(t)\big)=D(t)^{-1}$. As in Example 1, the measure λ is affine arclength on γ. The range of p in the adjoint restriction estimate furnished by Theorem 8 is short of the conjectured optimal range (see Drury [6]) and also short of the range in Drury's three-dimensional result for degenerate curves (Theorem 3 in [8]) but the same as the range in Christ's local result ((A) of Theorem 1.1 in [5]) for nondegenerate curves. The hypothesis (8) is particularly easy to verify when $[a,b]\subseteq[0,\infty)$, $\gamma_j(t)=t^j$ for $1\leq j<n$, and $\gamma_n(t)$ is a polynomial with nonnegative coefficients.

§2 Proofs of the Theorems

Proof of Theorems 5 and 1: The proof is an adaptation of an idea from the proof of Theorem 1 in [15]. See [1] and [2] for other applications of the same idea. Assuming the hypothesis (6), we will actually establish the multilinear estimate

$$\int\prod_{i=1}^{m+1}\lambda*f_i(x)dx\leq C(c)\prod_{i=1}^{m+1}\|f_i\|_{\frac{m+1}{m},m+1}$$

where the norms on the right hand side of the inequality are Lorentz-space norms, the functions f_i are nonnegative and Borel measurable, and dx denotes Lebesgue measure on $\mathbb{R}^n$. By symmetry and a multilinear argument of Christ [4] (see [7] for a more detailed exposition of Christ's idea), it is sufficient to establish the estimate

$$\int\prod_{i=1}^{m+1}\lambda*f_i(x)dx\leq C(c)\,\|f_1\|_1\prod_{i=2}^{m+1}\|f_i\|_{\frac{m}{m-1},1}. \tag{10}$$

We may assume that, for $2 \le i \le m+1$, each f_i is of the form χ_{E_i} for Borel $E_i \subseteq \mathbb{R}^n$. An application of Fubini's theorem to the left hand side of the inequality above leads to

$$\int f_1(x) \int \prod_{i=2}^{m+1} \big(\lambda * \chi_{(E_i - x)}(y)\big) d\lambda(y) dx \le \|f_1\|_1 \prod_{i=2}^{m+1} \|\lambda * \chi_{(E_i-x)}\|_{L^m(d\lambda)}.$$

According to (6) this is bounded by

$$C(c)\ \|f_1\|_1 \prod_{i=2}^{m+1} |E_i|^{\frac{m-1}{m}},$$

which is the right hand side of (10). This proves Theorem 5.

The proof shows that the Lebesgue space norm $\|\cdot\|_{\frac{m+1}{m}}$ in the conclusion of Theorem 5 may be replaced by the weaker Lorentz space norm $\|\cdot\|_{\frac{m+1}{m}, m+1}$. That is the "slightly stronger conclusion" mentioned after Example 7. Of course Theorem 1 is the special case $m = 2$ of Theorem 5.

Proof of Theorems 6 and 2: We estimate

$$\|\lambda * \chi_E\|_{m+1}^{m+1} = \int \Big(\int \chi_E(x-y) d\lambda(y)\Big)^{m+1} dx$$

$$\le m \int\int \chi_E(x-y_2)\Big(\int_{\{\omega(y_1)\ge\omega(y_2)\}} \chi_E(x-y_1) d\lambda(y_1)\Big)^m d\lambda(y_2)\ dx$$

$$= m \int\int \chi_E(x)\Big(\int_{\{\omega(y_1)\ge\omega(y_2)\}} \chi_E(x+y_2-y_1) d\lambda(y_1)\Big)^m d\lambda(y_2)\ dx \le m\cdot c\ |E|^m$$

by the hypothesis (7) and the fact that $|E - x| = |E|$. This argument is just an abstraction of the first part of the proof of Theorem 1 in [17]. Again, Theorem 2 is the special case $m = 2$ of Theorem 6.

Proof of Theorem 4: By interpolation with the case $p = 1$ it is enough to establish the estimate

$$\|\widehat{\chi_A d\lambda}\|_{q,\infty} \le C(c,p)\ \lambda(A)^{\frac{1}{p}} \tag{11}$$

for all Borel subsets A of $\operatorname{supp}(\lambda)$ whenever $1 \le p < 4$ and $\frac{1}{p} + \frac{3}{q} = 1$. By Hunt's generalization of the Hausdorff-Young theorem, this will follow from the inequality

$$\|(\chi_A d\lambda) * (\chi_A d\lambda)^{\sim}\|_{r,\infty} \le C(c,p)\ \lambda(A)^{\frac{2}{p}} \tag{12}$$

if $\frac{1}{p} = \frac{3}{2}\frac{1}{r} - \frac{1}{2}$ (so that $\frac{1}{r} + \frac{2}{q} = 1$). Now (12) is true for all (p,r) of interest if it is true for the two extreme cases $(p,r) = (1,1)$ and $(p,r) = (4,2)$. The first of these is easy and so it is enough to establish the estimate

$$\|(\chi_A d\lambda) * (\chi_A d\lambda)^{\sim}\|_{2,\infty} \le C(c)\ \lambda(A)^{\frac{1}{2}}, \tag{13}$$

which we regard as a weak endpoint version of (11). This inequality will follow from

$$\int_A \int_A \chi_E(y_2 - y_1)d\lambda(y_1)d\lambda(y_2) \leq C(c)\ \lambda(A)^{\frac{1}{2}}|E|^{\frac{1}{2}}, \tag{14}$$

for $E \subseteq \mathbb{R}^n$. But the the left hand side of (14) is bounded by

$$\int_A \int_{\{\omega(y_1)\geq\omega(y_2)\}} \chi_E(y_2 - y_1)d\lambda(y_1)d\lambda(y_2) + \int_A \int_{\{\omega(y_1)\geq\omega(y_2)\}} \chi_{(-E)}(y_2 - y_1)d\lambda(y_1)d\lambda(y_2).$$

Applying Hölder's inequality and then (2) to each of these in turn yields (14).

Again, this argument is just an abstraction of part of the argument in [19]. Now, as noted in (b) after the statement of Theorem 4, (13) and therefore the conclusion of Theorem 4 clearly follow from the conclusion of Theorem 3.

Proof of Theorem 3: It is enough to check that

$$\int_A \int_B \chi_E(y_2 - y_1)d\lambda(y_1)d\lambda(y_2) \leq C(c)\ \lambda(A)^{\frac{1}{4}}\lambda(B)^{\frac{1}{4}}|E|^{\frac{1}{2}}$$

for $E \subseteq \mathbb{R}^n$. But, by Hölder's inequality and (1), the left hand side is bounded by

$$\lambda(B)^{\frac{1}{2}}\|\lambda * \chi_{(-E)}\|_{L^2(\lambda)} \leq c\ \lambda(B)^{\frac{1}{2}}|E|^{\frac{1}{2}}$$

and, similarly, by $c\ \lambda(A)^{\frac{1}{2}}|E|^{\frac{1}{2}}$.

Proof of Theorem 7: This is a generalization of the proof of Theorem 4. By interpolation with the case $p = 1$ it is enough to establish an endpoint analogue of (9):

$$\|\widehat{\chi_A d\lambda}\|_{\frac{m(m+2)}{2},\infty} \leq C(c)\ \lambda(A)^{\frac{1}{m+2}}$$

for Borel subsets A of $\mathrm{supp}(\lambda)$. By Hunt's generalization of the Hausdorff-Young theorem, this will follow from the inequality

$$\|(\chi_A d\lambda) * \cdots * (\chi_A d\lambda)\|^{\frac{1}{m}}_{\frac{m+2}{m},\infty} \leq C(c)\ \lambda(A)^{\frac{1}{m+2}},$$

where the convolution on the left hand side is $m-$fold. This estimate follows from

$$\int_A \cdots \int_A \chi_E(y_1 + \cdots + y_m)d\lambda(y_1)\cdots d\lambda(y_m)$$

$$\leq m \int_A \Big(\int_{\{\omega(y_j)\geq\omega(y_1)\}} \chi_E(y_1 + \cdots + y_m)d\lambda(y_2)\cdots d\lambda(y_m)\Big)d\lambda(y_1)$$

$$\leq C(c)\lambda(A)^{\frac{m}{m+2}}|E|^{\frac{2}{m+2}},$$

where we have used Hölder's inequality and (8).

§3 Proofs for the Examples

Proof for Example 2: Let T be the mapping from the Lorentz space $L^{2,1}(\mathbb{R}^2)$ into $L^2(I, |p''(t)|^{\frac{1}{3}}dt)$ defined by

$$Tf(t) = \int_I f\big(t-s, p(t)-p(s)\big)|p''(s)|^{\frac{1}{3}}ds.$$

We regard the desired inequality

$$\int_I \Big(\int_I \chi_E\big(t-s, p(t)-p(s)\big)|p''(s)|^{\frac{1}{3}}ds\Big)^2 |p''(t)|^{\frac{1}{3}}dt \le C(N)\ |E|$$

as an estimate for T and we will prove it by establishing the dual estimate

$$\|T^*g\|_{L^{2,\infty}(\mathbb{R}^2)} \le C(N)\ \|g\|_{L^2(I,|p''(t)|^{\frac{1}{3}}dt)}. \tag{15}$$

Because of the convexity of p on I, the mapping $(t,s) \mapsto \big(t-s, p(t)-p(s)\big)$ from I^2 into $\mathbb{R}^2$ is one-to-one. The absolute value of its Jacobian is $|p'(t)-p'(s)|$. Thus the equation

$$\langle f, T^*g\rangle = \langle Tf, g\rangle = \int_I\int_I f\big(t-s, p(t)-p(s)\big)\frac{|p''(t)p''(s)|^{1/3}}{|p'(t)-p'(s)|}g(t)|p'(t)-p'(s)|\ ds\ dt$$

shows that, for $y>0$, the measure of the set $\{|T^*g|>y\}$ is

$$\int\int_{E_y} |p'(t)-p'(s)|\ ds\ dt$$

where

$$E_y = \{(t,s)\in I^2 : \frac{|g(t)|}{y}\ |p''(t)p''(s)|^{1/3} \ge |p'(t)-p'(s)|\}.$$

Lemma 1 below (Lemma 3.1 from [2]) bounds the inner integral by

$$\frac{C(N)\ |g(t)|^2|p''(t)|^{1/3}}{y^2}.$$

Thus an integration in t completes the proof of (15).

Lemma 1 *There is a constant $C(N)$ such that the following is true: If r is a real-valued polynomial of degree $\le N-1$ and I is an interval with r' of constant sign on I, then for all $t\in I$ and all $B>0$ the inequality*

$$\int_{\substack{\{s\in I: B|r'(t)r'(s)|^{1/3}\\ \ge |r(t)-r(s)|\}}} |r(t)-r(s)|\ ds \le C(N)\ B^2|r'(t)|^{1/3}$$

holds.

Proof for Example 3: With notation as in Example 3, we need to establish the inequality

$$\int_D \Big(\int_{\{w \in D: |\phi''(w)| \geq |\phi''(z)|\}} \chi_E\big(z-w, \phi(z)-\phi(w)\big) |\phi''(w)|^{2/3} dw \Big)^2 |\phi''(z)|^{2/3} dz \leq c\, |E|$$

for $E \subseteq \mathbb{C}^2$ and with $c = c(\ell)$. The mapping $(z,w) \mapsto \big(z-w, \phi(z)-\phi(w)\big)$ from $D \times D$ into $\mathbb{C}^2$ has Jacobian $|\phi'(z)-\phi'(w)|^2$ so

$$\int_D \int_D \chi_E\big(z-w, \phi(z)-\phi(w)\big) |\phi'(z)-\phi'(w)|^2 dw\ dz \leq c(\ell)|E|.$$

Thus it is enough to show that

$$|\phi''(z)|^{\frac{2}{3}} \Big(\int_{\{w \in D: |\phi''(w)| \geq |\phi''(z)\}} \chi_E\big(z-w, \phi(z)-\phi(w)\big) |\phi''(w)|^{\frac{2}{3}} dw \Big)^2$$

$$\leq c(\ell) \int_D \chi_E\big(z-w, \phi(z)-\phi(w)\big) |\phi'(z)-\phi'(w)|^2 dw$$

or that

$$|\phi''(z)|^{\frac{2}{3}} \Big(\int_{\{w \in D: |\phi''(w)| \geq |\phi''(z)\}} \chi_E\big(z-w, \phi(z)-\phi(w)\big) |\phi''(w)|^{-\frac{4}{3}} |\phi''(w)|^2 dw \Big)^2$$

$$\leq c(\ell) \int_D \chi_E\big(z-w, \phi(z)-\phi(w)\big) |\phi'(z)-\phi'(w)|^2 |\phi''(w)|^{-2} |\phi''(w)|^2 dw.$$

If we write $Z = \phi'(z)$ and $W = \phi'(w)$, so that $dW = |\phi''(w)|^2 dw$, and then let

$$g(W) = \sum_{w: \phi'(w) = W} \chi_E\big(z-w, \phi(z)-\phi(w)\big) \chi_{\{w \in D: |\phi''(w)| \geq |\phi''(z)\}}(w) |\phi''(w)|^{-\frac{4}{3}},$$

so that $\|g\|_\infty^{1/2} \leq \ell^{1/2} |\phi''(z)|^{-2/3}$, this will follow from the inequality

$$\Big(\int_{\mathbb{C}} g(W) dW \Big)^2 \leq 2\pi\ \|g\|_\infty^{1/2} \int_{\mathbb{C}} g(W)^{\frac{3}{2}} |W-Z|^2 dW$$

for fixed $Z \in \mathbb{C}$ and for nonnegative Borel functions g on $\mathbb{C}$. To see the latter inequality, take $\|g\|_\infty = 1$ and choose R such that

$$\int_{\mathbb{C}} g(W) dW = \pi R^2.$$

Then, by Hölder's inequality and the assumption $\|g\|_\infty = 1$,

$$\int_{\mathbb{C}} g^{\frac{3}{2}}(W) dW \geq \pi R^2.$$

Since also $0 \leq g^{3/2} \leq 1$, the integral $\int_{\mathbb{C}} g(W)^{\frac{3}{2}}|W - Z|^2 dW$ is minimized when $g^{3/2} = \chi_{B(Z,R)}$.

Proof for Example 4: The proof is analogous to the proof for Example 2. There are two points of difference: first, the mapping $(z,w) \mapsto \big(z-w, \phi(z)-\phi(w)\big)$ may not be one-to-one. But, by an argument involving Bezout's theorem, it will have generic multiplicity $C(N)$. Second, reflecting the fact that the mapping above has Jacobian $|\phi'(z)-\phi'(w)|^2$, Lemma 1 must be replaced by

Lemma 2 *Under the hypothesis (4) there is a constant $c = c(A,N)$ such that for all $z \in D$ and all $B > 0$, the inequality*

$$\int_{\substack{\{w\in D: B|\phi''(z)\phi''(w)|^{2/3} \\ \geq |\phi'(z)-\phi'(w)|^2\}}} |\phi'(z)-\phi'(w)|^2 \; dw \leq C(N) \; B^2 |\phi''(z)|^{2/3} \tag{16}$$

holds.

Proof of Lemma 2: Without loss of generality assume $z = 0$. Using the convexity of D, there is a function $r(\theta)$ with the property

$$|\phi'(0) - \phi'(r(\theta)e^{i\theta})|^2 \leq B \; |\phi''(0) \; \phi''(r(\theta)e^{i\theta})|^{\frac{2}{3}} \tag{17}$$

for $0 \leq \theta < 2\pi$ such that the left hand side of (16) is bounded by

$$\int_0^{2\pi} \int_0^{r(\theta)} |\phi'(0) - \phi'(re^{i\theta})|^2 r \; dr \; d\theta. \tag{18}$$

Now

$$\int_0^{r(\theta)} |\phi'(0)-\phi'(re^{i\theta})|^2 r \; dr \leq \int_0^{r(\theta)} \Big(\int_0^r |\phi''(se^{i\theta})|ds\Big)^2 r \; dr \leq \frac{r(\theta)^2}{2}\Big(\int_0^{r(\theta)} |\phi''(se^{i\theta})|ds\Big)^2. \tag{19}$$

It follows from the equivalence of norms on the finite-dimensional space of polynomials of degree $\leq N-2$ that there is $C(N)$ such that

$$r(\theta) \; |\phi''(0)|^{\alpha} |\phi''(r(\theta)e^{i\theta})|^{1-\alpha} \leq C(N) \; \int_0^{r(\theta)} |\phi''(se^{i\theta})|ds$$

for $0 \leq \alpha \leq 1$. Thus

$$\begin{aligned} r(\theta)^2 \Big(\int_0^{r(\theta)} |\phi''(se^{i\theta})|ds\Big)^2 &\leq C(N)^2 \frac{\Big(\int_0^{r(\theta)} |\phi''(se^{i\theta})|ds\Big)^4}{|\phi''(0)|^{2\alpha}|\phi''(r(\theta)e^{i\theta})|^{2-2\alpha}} \\ &\leq C(N)^4 \frac{A^4 B^2 |\phi''(0) \; \phi''(r(\theta)e^{i\theta})|^{\frac{4}{3}}}{|\phi''(0)|^{2\alpha}|\phi''(r(\theta)e^{i\theta})|^{2-2\alpha}}, \end{aligned}$$

where we have used (4) and (17) to obtain the last inequality. Now take $\alpha = \frac{1}{3}$ and integrate in θ. With (19) this yields the desired bound for (18).

Proof for Example 5: It is notationally convenient to regard $\mathbb{R}^{2+1}$ as $\mathbb{C}\times\mathbb{R}$ and use coordinates $(re^{i\theta},t)$. It will be enough to establish the inequality

(20)
$$\int_0^\infty\int_0^{2\pi}\int_0^r\int_0^{2\pi}\chi_E(re^{i\theta}-se^{i\phi},r-s)\,d\phi\,ds\,g(r,\theta)\,d\theta\,dr\le c\,|E|^{\frac12}\Big(\int_0^\infty\int_0^{2\pi}g(r,\theta)^2d\theta\,dr\Big)^{\frac12}$$

for nonnegative Borel functions g. The proof is an easy consequence of two lemmas.

Lemma 3. *The following inequality holds for some positive C and all nonnegative Borel functions f on $(0,\infty)$:*

$$\int_0^\infty\Big(\frac{1}{r^{1/2}}\int_0^r f(r-s)\frac{ds}{s^{1/2}}\Big)^2dr\le C\int_0^\infty f(r)^2dr.$$

This fractional integration result is (9.9.5) in [13].

Lemma 4. *There is a positive constant C such that the following inequality holds for all Borel $S\subseteq\mathbb{R}^2$ and all $r,s\in(0,\infty)$:*

$$\int_0^{2\pi}\Big(\int_0^{2\pi}\chi_S(re^{i\theta}-se^{i\phi})\,d\phi\Big)^2d\theta\le\frac{C}{rs}|S|.$$

Proof of Lemma 4: Using the idea from the (t,t^2) example at the beginning of the paper, we have for fixed θ

$$\Big(\int_0^{2\pi}\chi_S(re^{i\theta}-se^{i\phi})\,d\phi\Big)^2\le C\int_0^{2\pi}\chi_S(re^{i\theta}-se^{i\phi})\,|\sin(\theta-\phi)|\,d\phi. \tag{21}$$

Since a Jacobian computation shows that

$$rs\int_0^{2\pi}\int_0^{2\pi}\chi_S(re^{i\theta}-se^{i\phi})\,|\sin(\theta-\phi)|\,d\phi\,d\theta\le C\,|S|,$$

integrating (21) with respect to θ completes the proof of the lemma.

If $E\subseteq\mathbb{R}^{2+1}$ and $t\in\mathbb{R}$, then E_t will stand for $\{x\in\mathbb{R}^2:(x,t)\in E\}$. Since

$$\int_0^{2\pi}\int_0^{2\pi}\chi_E(re^{i\theta}-se^{i\phi},r-s)\,d\phi\,g(r,\theta)\,d\theta\le\Big(\int_0^{2\pi}\big(\int_0^{2\pi}\chi_E(re^{i\theta}-se^{i\phi},r-s)\,d\phi\big)^2d\theta\Big)^{\frac12}\Big(\int_0^{2\pi}g(r,\theta)^2d\theta\Big)^{\frac12},$$

Lemma 4 yields the following bound for the left hand side of (20):

$$C\int_0^\infty\frac{1}{r^{\frac12}}\int_0^r|E_{r-s}|^{\frac12}\frac{ds}{s^{\frac12}}\big(\int_0^{2\pi}g(r,\theta)^2d\theta\big)^{\frac12}dr$$

$$\le C\,\Big(\int_0^\infty\big(\frac{1}{r^{\frac12}}\int_0^r|E_{r-s}|^{\frac12}\frac{ds}{s^{\frac12}}\big)^2dr\Big)^{\frac12}\Big(\int_0^\infty\int_0^{2\pi}g(r,\theta)^2d\theta\,dr\Big)^{\frac12}.$$

An application of Lemma 3 now yields (20).

Proof for Example 7: With $\int$ standing for $\int_{-\infty}^{\infty}$, the inequality to be proved is

$$\int \cdots \int \prod_1^3 \chi_E(s-u_j, t-v_j, s^2+t^2-u_j^2-v_j^2)\, ds \cdots dv_3 \le c\, |E|^2 \tag{22}$$

for $E \subseteq \mathbb{R}^3$. The proof requires two lemmas.

Lemma 5. *The following inequality holds for $a_1, a_2, b_1, b_2 \in \mathbb{R}$ and for Borel $S \subseteq \mathbb{R}^2$:*

$$\int \Big(\int \chi_S(u_1-u_2, u_1^2+a_1u_1+b_1-u_2^2-a_2u_2-b_3) du_2 \Big)^2 du_1 \le 2^{\frac{1}{2}} |S|.$$

This is proved by changing variables in the (t, t^2) example at the beginning of the paper.

Lemma 6. *The following inequality holds for all nonnegative Borel functions f on $(0,\infty)$:*

$$\int \int_{-v_1}^{v_1} \int_{-v_2}^{v_2} f(v_1) f(v_2)^{\frac{1}{2}} f(v_3)^{\frac{1}{2}} dv_3 dv_2 \frac{dv_1}{|v_1|} \le 2^{\frac{1}{2}} \Big(\int f(v)\, dv \Big)^2.$$

Proof of Lemma 6: We have

$$\int_{-v_2}^{v_2} f(v_3)^{\frac{1}{2}} dv_3 \le 2^{\frac{1}{2}} |v_2|^{\frac{1}{2}} \Big(\int_{-v_2}^{v_2} f(v_3)\, dv_3 \Big)^{\frac{1}{2}}$$

and so

$$\int \int_{-v_1}^{v_1} \int_{-v_2}^{v_2} f(v_3)^{\frac{1}{2}} dv_3 f(v_2)^{\frac{1}{2}} dv_2 f(v_1) \frac{dv_1}{|v_1|} \le 2^{\frac{1}{2}} \int \int_{-v_1}^{v_1} \Big(\int_{-v_2}^{v_2} f(v_3)\, dv_3 \Big)^{\frac{1}{2}} |v_2|^{\frac{1}{2}} f(v_2)^{\frac{1}{2}} dv_2 f(v_1) \frac{dv_1}{|v_1|}$$

$$\le 2^{\frac{1}{2}} \int \int_{-v_1}^{v_1} |v_2|^{\frac{1}{2}} f(v_2)^{\frac{1}{2}} dv_2 \Big(\int_{-v_1}^{v_1} f(v_3)\, dv_3 \Big)^{\frac{1}{2}} f(v_1) \frac{dv_1}{|v_1|} \le 2^{\frac{1}{2}} \int \Big(\int_{-v_1}^{v_1} |v_2|\, dv_2 \Big)^{\frac{1}{2}} \Big(\int_{-v_1}^{v_1} f(v_3) dv_3 \Big) f(v_1) \frac{dv_1}{|v_1|}$$

$$\le 2^{\frac{1}{2}} \Big(\int f(v)\, dv \Big)^2$$

as desired.

Let I represent the integral on the left hand side of (22) and let I' represent that integral restricted to the set

$$\{ |v_3 - t| \le |v_2 - t| \le |v_1 - t| \}.$$

Then $I \le 3I'$. In I' replace s by $s + u_1$, t by $t + v_1$, and then, for $j = 1, 2, 3$, replace v_j by $v_j - \frac{s}{t} u_1$. The result is

$$\int'' \chi_E\big(s, t, s^2+t^2+2tv_1\big) \chi_E\big(s+u_1-u_2, t+v_1-v_2, (s+u_1)^2-u_2^2+(t+v_1-\frac{s}{t}u_1)^2-(v_2-\frac{s}{t}u_1)^2\big) \cdot$$

$$\chi_E\big(s+u_1-u_3, t+v_1-v_3, (s+u_1)^2-u_3^2+(t+v_1-\frac{s}{t}u_1)^2-(v_3-\frac{s}{t}u_1)^2\big) ds \cdots dv_3,$$

where the integral $\int''$ is restricted to the set

$$\{|v_3 - v_1 - t| \le |v_2 - v_1 - t| \le |t|\}.$$

If $E \subseteq \mathbb{R}^3$ and $t \in \mathbb{R}$, then we will write E_t for

$$\{(x_1, x_3) \in \mathbb{R}^2 : (x_1, t, x_3) \in E\}.$$

Hölder's inequality and two applications of Lemma 5 bound

$$\int \chi_E\big(s+u_1-u_2, t+v_1-v_2, (s+u_1)^2 - u_2^2 + (t+v_1-\frac{s}{t}u_1)^2 - (v_2-\frac{s}{t}u_1)^2\big)\cdot$$

$$\chi_E\big(s+u_1-u_3, t+v_1-v_3, (s+u_1)^2 - u_3^2 + (t+v_1-\frac{s}{t}u_1)^2 - (v_3-\frac{s}{t}u_1)^2\big) du_1 du_2 du_3$$

by

$$2^{\frac{1}{2}} |E_{t+v_1-v_2}|^{\frac{1}{2}} |E_{t+v_1-v_3}|^{\frac{1}{2}}.$$

Thus I' is bounded by

$$2^{\frac{1}{2}} \int'' \chi_E(s, t, s^2+t^2+2tv_1)\ |E_{t+v_1-v_2}|^{\frac{1}{2}} |E_{t+v_1-v_3}|^{\frac{1}{2}} ds\ dt\ dv_3 dv_2 dv_1.$$

Replacing t by $t - v_1$ and then v_j by $v_j + t$ leads to

$$2^{\frac{1}{2}} \int''' \chi_E\big(s, -v_1, s^2+t^2-(v_1+t)^2\big)\ |E_{-v_2}|^{\frac{1}{2}} |E_{-v_3}|^{\frac{1}{2}} ds\ dt\ dv_3 dv_2 dv_1,$$

where $\int'''$ indicates that the variables v_j are restricted to

$$\{|v_3| \le |v_2| \le |v_1|\}.$$

The change of variables $(s,t) \mapsto \big(s, s^2+t^2-(v_1+t)^2\big)$ has Jacobian $-2v_1$, so this last integral is bounded by

$$2^{-\frac{1}{2}} \int \int_{-v_1}^{v_1} \int_{-v_2}^{v_2} |E_{-v_1}|\ |E_{-v_2}|^{\frac{1}{2}} |E_{-v_3}|^{\frac{1}{2}} dv_3 dv_2 \frac{dv_1}{|v_1|}.$$

Now Lemma 6 completes the proof.

Proof for Example 8: The proof is analogous to the one for Example 2. It depends on the following result.

Lemma 7. *There is $C = C(n)$ such that the following inequality holds for $y > 0$:*

$$\int_{\{V(t_1,\dots,t_n)<y\}} V(t_1,\dots,t_n)\ dt_2 \cdots dt_n \le C\ y^{\frac{n+2}{n}}.$$

This follows from a homogeneity argument and the observation (see the proof of Lemma 1 in [11]) that

$$\int_{\{V(t_1,\dots,t_n)<1\}} V(t_1,\dots,t_n)\, dt_2\cdots dt_n < \infty.$$

Let T be the mapping from $L^{\frac{n+2}{2},1}(\mathbb{R}^n)$ into $L^{\frac{n+2}{2}}([a,b], D(t)^{2/(n(n+1))}dt)$ defined by

$$Tf(t_1) = \int_{\{D(t_j)\le D(t_1)\}} f\big(\gamma(t_1)+\cdots+\gamma(t_n)\big)\prod_2^n D(t_j)^{\frac{2}{n(n+1)}}dt_2\cdots dt_n$$

where the integral is over

$$\{(t_2,\dots,t_n)\in[a,b]^{n-1} : D(t_2)\le D(t),\dots,D(t_n)\le D(t_1)\}.$$

As in the proof for Example 2, we regard the conclusion of Example 8 as an estimate for T and establish the dual estimate

$$\|T^*g\|_{L^{\frac{n+2}{n},\infty}(\mathbb{R}^n)} \le c\,\|g\|_{L^{\frac{n+2}{n}}([a,b],D(t)^{2/(n(n+1))}dt)}. \tag{23}$$

The equation

$$\langle f, T^*g\rangle = \langle Tf, g\rangle = \int_{\{D(t_j)\le D(t_1)\}} f\big(\gamma(t_1)+\cdots+\gamma(t_n)\big)\, g(t_1)\frac{\prod_1^n D(t_j)^{\frac{2}{n(n+1)}}}{\Delta(t_1,\dots,t_n)}\Delta(t_1,\dots,t_n)\, dt_1\cdots dt_n$$

shows that, for $y>0$, the measure of $\{|T^*g|>y\}$ is bounded by

$$c\int_{E_y}\Delta(t_1,\dots,t_n)\, dt_1\cdots dt_n,$$

where c depends on n and the generic multiplicity of

$$(t_1,\dots,t_n)\mapsto\big(\gamma(t_1)+\cdots+\gamma(t_n)\big)$$

and where

$$E_y = \{(t_1,\dots,t_n)\in[a,b]^n : \frac{|g(t_1)|}{y}\prod_1^n D(t_j)^{\frac{2}{n(n+1)}}\ge\Delta(t_1,\dots,t_n),\ D(t_j)\le D(t_1)\}.$$

Thus (23) will follow from

$$\int_{E_y}\Delta(t_1,\dots,t_n)\, dt_1\cdots dt_n \le c\int_a^b\Big(\frac{|g(t_1)|}{y}\Big)^{\frac{n+2}{n}} D(t_1)^{\frac{2}{n(n+1)}}dt_1, \tag{24}$$

where c depends only on n and A. For $t_1 \in [a,b]$ and $B > 0$ let $F_{t_1,B}$ be the set

$$\{(t_2,\dots,t_n) \in [a,b]^{n-1} : B\prod_1^n D(t_j)^{\frac{2}{n(n+1)}} \geq \Delta(t_1,\dots,t_n),\ D(t_j) \leq D(t_1)\}.$$

Then (24) will be a consequence of the inequality

$$\int_{F_{t_1,B}} \Delta(t_1,\dots,t_n)\, dt_2 \cdots dt_n \leq c\, B^{\frac{n+2}{n}} D(t_1)^{\frac{2}{n(n+1)}}, \tag{25}$$

where c depends only on n and A. Recalling the hypothesis

$$A^{-1}\Delta(t_1,\dots,t_n) \leq V(t_1,\dots,t_n) \cdot \sup_{1\leq j\leq n} D(t_j) \leq A^{-1}\Delta(t_1,\dots,t_n)$$

of Example 8, it follows from $\prod_1^n D(t_j)^{2/(n(n+1))} \leq D(t_1)^{1/(n+1)}$ that

$$\int_{F_{t_1,B}} \Delta(t_1,\dots,t_n)\, dt_2\cdots dt_n$$
$$\leq A\cdot D(t_1) \int_{\{V(t_1,\dots,t_n)\leq BA^{-1}D(t_1)^{\frac{2}{n+1}-1}\}} V(t_,\dots,t_n)\, dt_2\cdots dt_n.$$

Now (25) follows from Lemma 7.

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