Stationary Statistical Properties of Rayleigh-Bénard Convection at Large Prandtl Number

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Abstract

This is the third in a series of our study of Rayleigh-Bénard convection at large Prandtl number. Here we investigate whether stationary statistical properties of the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number are related to those of the infinite Prandtl number model for convection that is formally derived from the Boussinesq system via setting the Prandtl number to infinity. We study asymptotic behavior of stationary statistical solutions, or invariant measures, to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. In particular, we show that the invariant measures of the Boussinesq system for Rayleigh-Bénard convection converge to those of the infinite Prandtl number model for convection as the Prandtl number approaches infinity. We also show that the Nusselt number for the Boussinesq system (a specific statistical property of the system) is asymptotically bounded by the Nusselt number of the infinite Prandtl number model for convection at large Prandtl number. We discover that the Nusselt numbers are saturated by ergodic invariant measures. Moreover, we derive a new upper bound on the Nusselt number for the Boussinesq system at large Prandtl number of the form

$$\mathrm{Ra}^{1/3}(\ln \mathrm{Ra})^{1/3} + c\frac{\mathrm{Ra}^{7/2}\ln \mathrm{Ra}}{\mathrm{Pr}^2},$$

which asymptotically agrees with the (optimal) upper bound on Nusselt number for the infinite Prandtl number model for convection.

1 Introduction

We continue our investigation into the asymptotic behavior of solutions at large Prandtl number of the following *Boussinesq system for Rayleigh-Bénard convection* (*nondimensional*):

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) + \nabla p = \Delta\mathbf{u} + \mathrm{Ra}\,\mathbf{k}\theta, \quad \nabla\cdot\mathbf{u} = 0, \tag{1.1}$$

$$\frac{\partial\theta}{\partial t} + \mathbf{u}\cdot\nabla\theta - u_3 = \Delta\theta, \tag{1.2}$$

$$\mathbf{u}|_{z=0,1} = 0, \tag{1.3}$$

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$$\theta|_{z=0,1} = 0, \tag{1.4}$$

$$\mathbf{u}|_{t=0} = \mathbf{u}_0, \quad \theta|_{t=0} = \theta_0, \tag{1.5}$$

where $\mathbf{u}$ is the fluid velocity field, p is the modified pressure, θ is the deviation of the temperature field from the pure conduction state $1 - z$, $\mathbf{k}$ is the unit upward vector, Ra is the Rayleigh number, Pr is the Prandtl number, and the fluids occupy the (nondimensionalized) region

$$\Omega = [0, L_x] \times [0, L_y] \times [0, 1] \tag{1.6}$$

with periodicity in the horizontal directions assumed for simplicity.

At very large Prandtl number, we may formally set the Prandtl number to infinity and arrive at the following *infinite Prandtl number model* (*nondimensional*):

$$\nabla p^0 = \Delta \mathbf{u}^0 + \mathrm{Ra}\,\mathbf{k}\theta^0, \quad \nabla \cdot \mathbf{u}^0 = 0, \tag{1.7}$$

$$\frac{\partial \theta^0}{\partial t} + \mathbf{u}^0 \cdot \nabla \theta^0 - u_3^0 = \Delta \theta^0, \tag{1.8}$$

$$\mathbf{u}^0|_{z=0,1} = 0, \tag{1.9}$$

$$\theta^0|_{z=0,1} = 0, \tag{1.10}$$

which is relevant for fluids such as silicone oil and the earth's mantle as well as many gases under high pressure [3, 4, 19]. One observes that the Navier-Stokes equations in the Boussinesq system have been replaced by the Stokes equations in the infinite Prandtl number model.

The fact that the velocity field is linearly *slaved* by the temperature field has been exploited in several recent, very interesting works on rigorous estimates on the rate of heat convection in this infinite Prandtl number setting (see [9, 11, 14, 16] and the references therein, as well as the work of [4, 22]).

An important natural question is whether such an approximation is valid.

In our previous works [37, 40], we have shown that the infinite Prandtl number model is a reasonable model for convection at large Prandtl number in the sense that suitable weak solutions to the Boussinesq system converge to those of the infinite Prandtl number model on any fixed finite time interval [37], and the global attractors of the Boussinesq system converge to that of the infinite Prandtl number model [40] as the Prandtl number approaches infinity.

It is well-known that for complex systems such as the Boussinesq system where turbulent/chaotic behavior abound (see, for instance, [3, 7, 19, 23, 32]), the statistical properties for such systems are much more important and physically relevant than single trajectories [18, 28, 29]. In particular, if a complex system reaches a statistical equilibrium state, it is the stationary statistical properties characterized by the invariant measures that are important [18, 28, 35]. Hence it is natural and essential to ask whether the stationary statistical properties (in terms of invariant measures) remain close.

Recall that the invariant measures are supported on the global attractors. Therefore, the upper semicontinuity of the global attractors that we derived earlier [38, 40] indicates that the statistical properties of the Boussinesq system may be close to that of the infinite Prandtl number model even for this singular perturbation problem.

The main purpose of this manuscript is to show that general stationary statistical properties (in terms of invariant measures) of the Boussinesq system are close to general stationary statistical properties (invariant measures) of the infinite Prandtl number model at large Prandtl number. Specific statistical properties such as time-averaged heat transport in the vertical direction characterized by the long-time-averaged Nusselt number are also related in the sense that the Nusselt number for the Boussinesq system is asymptotically bounded by the Nusselt number of the infinite Prandtl number model at large Prandtl number. Moreover, we show that the Nusselt numbers are saturated by ergodic invariant measures. We also derive an upper bound on the Nusselt number for the Boussinesq system of the form

$$\mathrm{Ra}^{1/3}(\ln \mathrm{Ra})^{1/3} + c\,\frac{\mathrm{Ra}^{7/2}\ln \mathrm{Ra}}{\mathrm{Pr}^2}$$

at large Prandtl number. This bound asymptotically agrees with the best known (and physically relevant) bound for the infinite Prandtl number model ($c\,\mathrm{Ra}^{1/3}$ modulo a logarithmic term [9, 16, 30]); see also [20, 21]. These results further justify the infinite Prandtl number model for convection as an approximate model for convection at large Prandtl number.

Throughout this manuscript, we assume the physically important case of high Rayleigh number

$$\mathrm{Ra} \geq 1 \tag{1.11}$$

so that we may have nontrivial dynamics.

We also follow the mathematical tradition of denoting our small parameter as ε, i.e.,

$$\varepsilon = \frac{1}{\mathrm{Pr}}. \tag{1.12}$$

For convenience, we recall from [40] the following a priori estimates for solutions on the union of all global attractors ($\bigcup_{\varepsilon\leq\varepsilon_0}\mathcal{A}_\varepsilon$ where $\mathcal{A}_\varepsilon$ is the global attractor at $\mathrm{Pr} = 1/\varepsilon$) for the Boussinesq system at large Prandtl number:

$$\left|\frac{\partial}{\partial t}\mathbf{u}\right|_{L^2} \leq c\,\mathrm{Ra}^{9/4}, \tag{1.13}$$

$$\frac{1}{t}\int_{t_0}^{t_0+t}\left|\nabla\frac{\partial}{\partial t}\mathbf{u}(s)\right|_{L^2}^2 ds \leq c\,\mathrm{Ra}^{\frac{9}{2}}, \tag{1.14}$$

$$|\mathbf{u}(t)|_{H^1} \leq c\,\mathrm{Ra}, \tag{1.15}$$

$$|\mathbf{u}(t)|_{H^2} \le c\mathrm{Ra}^{5/2}, \tag{1.16}$$

$$\frac{1}{t}\int_{t_0}^{t_0+t} |\mathbf{u}(s)|_{H^2}^2\, ds \le cRa^2, \tag{1.17}$$

$$|\theta(t)|_{H^2} \le c\mathrm{Ra}^8, \tag{1.18}$$

$$\left|\frac{\partial\theta}{\partial t}(t)\right|_{L^2} \le cRa^8, \tag{1.19}$$

as long as

$$\varepsilon\,\mathrm{Ra} = \frac{\mathrm{Ra}}{\mathrm{Pr}} \le c_0, \tag{1.20}$$

where c_0 is an absolute constant.

The rest of the manuscript is organized as follows: In Section 2 we introduce the definition of stationary statistical solutions (invariant measures) to the Boussinesq and infinite Prandtl number system. We prove that invariant measures for the Boussinesq system must contain subsequences that converge to invariant measures of the infinite Prandtl number model as the Prandtl number approaches infinity. In Section 3 we show that the Nusselt numbers for the Boussinesq system are asymptotically bounded by that of the infinite Prandtl number model. In Section 4 we derive a new upper bound on the Nusselt number for the Boussinesq system at large Prandtl number which agrees with the (optimal) upper bound on Nusselt number for the infinite Prandtl number model for convection. In Section 5 we offer concluding remarks.

2 Upper Semicontinuity of Invariant Measures

For convenience, we recall the following function spaces that are standard for the mathematical treatment of Boussinesq equations:

We denote the phase space of the Boussinesq system as

$$X = H \times L^2 \tag{2.1}$$

where H is the divergence-free subspace of $(L^2)^3$ with zero normal component in the vertical direction and periodic in the horizontal directions [40]. The phase space for the infinite Prandtl number model is simply L^2.

We also denote

$$Y = V \times H^1_{0,\mathrm{per}} \tag{2.2}$$

where $H^1_{0,\mathrm{per}}$ is the subspace of H^1 that is zero at $z = 0, 1$ and periodic in the horizontal directions, and V is the divergence-free subspace of $(H^1_{0,\mathrm{per}})^3$.

Denoting

$$\mathbf{F}_\varepsilon((\mathbf{u},\theta)) = \left(-B(\mathbf{u},\mathbf{u}) + \frac{1}{\varepsilon}(-A\mathbf{u} + \mathrm{Ra}\,P(\mathbf{k}\theta)), -\mathbf{u}\cdot\nabla\theta + u_3 - \Delta\theta\right) \tag{2.3}$$

where $B(\mathbf{u},\mathbf{u}) = P((\mathbf{u}\cdot\nabla)\mathbf{u})$ is the standard bilinear term in the analysis of incompressible fluids [10, 15, 18, 27], P represents the Leray-Hopf projection, and A denotes the Stokes operator with the associated boundary condition; we then rewrite the Boussinesq system as an abstract (generalized) dynamical system

$$\frac{d}{dt}(\mathbf{u},\theta) = \mathbf{F}_\varepsilon((\mathbf{u},\theta)). \tag{2.4}$$

Similarly, denoting

$$F_0(\theta) = -\operatorname{Ra} A^{-1}(\mathbf{k}\theta)\cdot\nabla\theta + \operatorname{Ra}(A^{-1}(\mathbf{k}\theta))_3 - \Delta\theta, \tag{2.5}$$

we can rewrite the infinite Prandtl number model as

$$\frac{d}{dt}\theta = F_0(\theta). \tag{2.6}$$

We now introduce the concept of stationary invariant measures for the Boussinesq system and the infinite Prandtl number model, which are similar to the case of the Navier-Stokes system [18, 35].

DEFINITION 2.1 A *stationary statistical solution for the Boussinesq system* (*with Prandtl number* $\operatorname{Pr} = 1/\varepsilon$) is a probability measure μ_ε on the phase space X such that

$$\int_X \|(\mathbf{u},\theta)\|_Y^2 \, d\mu_\varepsilon((\mathbf{u},\theta)) < \infty, \tag{2.7}$$

$$\int_X \big(\mathbf{F}_\varepsilon((\mathbf{u},\theta)), \Phi'((\mathbf{u},\theta))\big) d\mu_\varepsilon((\mathbf{u},\theta)) = 0, \tag{2.8}$$

for any test functional Φ that is bounded on bounded sets of X and Fréchet-differentiable for $(\mathbf{u},\theta)\in Y$ with $\Phi'((\mathbf{u},\theta))\in Y$, and the derivative is continuous and bounded as a function from Y to Y.

In addition,

$$\begin{aligned}\int_X \{|\nabla\mathbf{u}|_{L^2}^2 - \operatorname{Ra}\theta u_3\} d\mu_\varepsilon((\mathbf{u},\theta)) &\le 0,\\ \int_X \{|\nabla\theta|_{L^2}^2 - \theta u_3\} d\mu_\varepsilon((\mathbf{u},\theta)) &\le 0.\end{aligned} \tag{2.9}$$

The set of all stationary statistical solutions for the Boussinesq system at Prandtl number $\operatorname{Pr} = 1/\varepsilon$ is denoted $\mathcal{IM}_\varepsilon$.

DEFINITION 2.2 Likewise, a *stationary statistical solution for the infinite Prandtl number model* is a probability measure μ_0 on the phase space L^2 such that

$$\int_{L^2} \|\theta\|^2_{H^1_{0,\mathrm{per}}} \, d\mu_0(\theta) < \infty, \tag{2.10}$$

$$\int_{L^2} (F_0(\theta), \Phi_0'(\theta)) d\mu_0(\theta) = 0, \tag{2.11}$$

for any test functional Φ_0 that is bounded on bounded sets of L^2 and Fréchet-differentiable for $\theta \in H^1_{0,\mathrm{per}}$ with $\Phi'(\theta) \in H^1_{0,\mathrm{per}}$ and whose derivative is continuous and bounded as a function from $H^1_{0,\mathrm{per}}$ to $H^1_{0,\mathrm{per}}$. Also,

$$\int_{L^2} \{|\nabla\theta|^2_{L^2} - \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta\} d\mu_0(\theta) \le 0. \tag{2.12}$$

The set of all stationary statistical solutions for the infinite Prandtl number model is denoted $\mathcal{IM}_0$.

Roughly speaking, conditions (2.7) and (2.10) express the fact that the stationary statistical solutions are supported on a smaller and smoother space; conditions (2.8) and (2.11) are functional weak formulations of the time invariance of stationary statistical solutions; conditions (2.9) and (2.12) are versions of energy estimates.

It is easy to see that both $\mathcal{IM}_\varepsilon$ and $\mathcal{IM}_0$ contain more than one element since a Dirac delta measure concentrated at any steady state solution is an invariant measure of the underlying (generalized) dynamical system, and we know that both the Boussinesq system and the infinite Prandtl number model contain multiple steady states [26, 31].

Recall that the well-posedness of the Boussinesq system is a major unsettled open problem. Hence stationary statistical solutions may not be invariant measures just as in the case of the three-dimensional Navier-Stokes system [18]. Nevertheless, we have eventual regularity for the Boussinesq system and there exists a global attractor at large Prandtl number, and the system is well-posed on the global attractor [39, 40]. Therefore, we may modify the proof for the two-dimensional Navier-Stokes system [18] and show that stationary statistical solutions to the Boussinesq system at large Prandtl number are equivalent to invariant measures, i.e., measures that are invariant under the (generalized) flow. This justifies our notation of $\mathcal{IM}$. As usual, the support of these invariant measures/stationary statistical solutions is included in the global attractor just as in the three-dimensional Navier-Stokes system case [18]. We will provide details elsewhere.

Since the infinite Prandtl number model can be viewed as the (singular) limit of the Boussinesq system as the Prandtl number approaches infinity, and since the dynamics is believed to be chaotic [3, 19, 30, 32], we naturally inquire whether the statistical properties of the Boussinesq system are related to the statistical properties of the infinite Prandtl number model. The objects that capture the statistically stationary properties of the systems are stationary statistical solutions (or invariant measures) [18, 28]. Therefore we are interested in whether the (stationary) statistical solutions for the Boussinesq system and the infinite Prandtl number model are related. Similar issues for the inviscid limit of the Navier-Stokes systems in terms of time-dependent statistical solutions can be found in [5, 6, 12, 35].

Since the Boussinesq system and the infinite Prandtl number model possess different natural phase spaces, the convergence of invariant measures has to be studied after we either take the marginal distribution of the invariant measures for the Boussinesq system onto the perturbative temperature field only or lift the invariant measures for the infinite Prandtl number model to the product space of velocity and perturbative temperature.

Our goal here is to show the following result:

THEOREM 2.3 *Let $\mu_\varepsilon \in \mathcal{I}M_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, be stationary statistical solutions (invariant measures) of the Boussinesq system at Prandtl number* $\mathrm{Pr} = 1/\varepsilon$. *Then there exists a subsequence that weakly converges to a limit μ_* as $\varepsilon \to 0$. Moreover, there exists an invariant measure μ_0 of the infinite Prandtl number model such that $\mu_* = \mathcal{L}\mu_0$ where $\mathcal{L}\mu_0$ is the natural lift of μ_0 and is defined through the relation*

$$\int_X \Phi(\mathbf{u},\theta)\,d(\mathcal{L}\mu_0)((\mathbf{u},\theta)) = \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta)\,d\mu_0(\theta) \tag{2.13}$$

for every suitable test functional Φ.

PROOF: We observe that a direct proof of the theorem utilizing the weak formulation of the invariance property ((2.8) and (2.11) in the definitions) is not convenient due to the singular nature of the limit. Instead, we first prove the upper semicontinuity in the projected sense.

There are two main ingredients in this part of the proof: a compactness result that ensures the existence of a convergent subsequence, and an argument indicating that the limit must be an invariant measure of the limit system.

Thanks to (1.16) and (1.17), there exists a constant $R_2 > 0$ such that $\operatorname{supp}\{\mu_\varepsilon\} \subset B_{R_2}((H^2)^3 \times H^2)$, $0 < \varepsilon \le \varepsilon_0$; i.e., the support of all these invariant measures is included in a ball of radius R_2 in $(H^2)^3 \times H^2$ (independent of ε) since invariant measures are supported on the global attractor [18]. Since this bounded ball in $(H^2)^3 \times H^2$ is compactly imbedded in the phase space X by Sobolev imbedding, we have, thanks to Prokhorov's compactness theorem [2, 24], that the set $\{\mu_\varepsilon, 0 < \varepsilon \le \varepsilon_0\}$ is weakly precompact in the space $\mathcal{P}M(X)$ of all probability measures on the phase space X.

Next, we plan to show that appropriate marginal distributions of $\{\mu_\varepsilon, 0 < \varepsilon \leq \varepsilon_0\}$ converge weakly to an invariant measure μ_0 of the infinite Prandtl number model.

For this purpose, we make a change of variable for the Boussinesq system and introduce the new variable

$$\mathbf{v} = \mathbf{u} - \mathrm{Ra}\, A^{-1}(\mathbf{k}\theta), \tag{2.14}$$

which measures the deviation of the velocity component of the Boussinesq system from that of the infinite Prandtl number model.

We then have a new set of measures $\tilde{\mu}_\varepsilon$ on the $(\mathbf{v}, \theta)$ space defined as

$$\int \Phi(\mathbf{u}, \theta) d\mu_\varepsilon((\mathbf{u}, \theta)) = \int \Phi(\mathbf{v} + \mathrm{Ra}\, A^{-1}(\mathbf{k}\theta), \theta) d\tilde{\mu}_\varepsilon((\mathbf{v}, \theta)) \tag{2.15}$$

for all appropriate test functionals Φ.

The precompactness of the set $\{\mu_\varepsilon, 0 < \varepsilon \leq \varepsilon_0\}$ implies the precompactness of the set $\{\tilde{\mu}_\varepsilon, 0 < \varepsilon \leq \varepsilon_0\}$. Hence, the marginal distribution of $\tilde{\mu}_\varepsilon$ in θ, denoted $M\tilde{\mu}_\varepsilon$ and defined by

$$\int \Phi_0(\theta) dM\tilde{\mu}_\varepsilon(\theta) = \int \Phi_0(\theta) d\tilde{\mu}_\varepsilon((\mathbf{v}, \theta)) = \int \Phi_0(\theta) d\mu_\varepsilon((\mathbf{u}, \theta)), \tag{2.16}$$

is also precompact in the space $\mathcal{P}M(L^2)$ of all probability measures on L^2.

Therefore, without loss of generality we may assume

$$M\tilde{\mu}_\varepsilon \rightharpoonup \mu_0 \tag{2.17}$$

in $\mathcal{P}M(L^2)$.

Our goal now is to show that μ_0 must be a member of $\mathcal{I}M_0$, the set of stationary statistical solutions (invariant measures) for the infinite Prandtl number model.

We need to verify the three conditions in the definition of $\mathcal{I}M_0$, (2.10), (2.11), and (2.12).

It is easy to see that the first condition is satisfied thanks to the uniform a priori estimates (1.16) and (1.17).

It is also easy to see that for any point $(\mathbf{u}, \theta) \in \mathcal{A}_\varepsilon$ and the associated trajectory passing through this given point, we have

$$A\mathbf{v} = -\varepsilon\left(\frac{\partial \mathbf{u}}{\partial t} + B(\mathbf{u}, \mathbf{u})\right). \tag{2.18}$$

Hence, thanks to the uniform a priori estimates (1.13) and (1.16), the regularity of the Stokes operator, and Sobolev inequalities

$$\begin{aligned} |\mathbf{v}|_{H^2} &\leq c_2\varepsilon\left(\left|\frac{\partial \mathbf{u}}{\partial t}\right|_{L^2} + |\mathbf{u}|_{H^2}|\nabla\mathbf{u}|_{L^2}\right) \\ &\leq c_3\varepsilon. \end{aligned} \tag{2.19}$$

Therefore,

$$
\begin{aligned}
&\int_{L^2} \{|\nabla\theta|_{L^2}^2 - \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta\} d\mu_0(\theta) \\
&\quad = \lim_{\varepsilon\to 0} \int_{L^2} \{|\nabla\theta|_{L^2}^2 - \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta\} dM\tilde{\mu}_\varepsilon(\theta) \\
&\quad = \lim_{\varepsilon\to 0} \int_{X} \{|\nabla\theta|_{L^2}^2 - \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta\} d\mu_\varepsilon((\mathbf{u},\theta)) \\
&\quad \le \limsup_{\varepsilon\to 0} \int_{X} \{|\nabla\theta|_{L^2}^2 - u_3\theta\} d\mu_\varepsilon((\mathbf{u},\theta)) \\
&\qquad + \limsup_{\varepsilon\to 0} \int_{X} \{(u_3 - \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3)\theta\} d\mu_\varepsilon((\mathbf{u},\theta)) \\
&\quad \le \limsup_{\varepsilon\to 0} \int_{X} |\mathbf{v}|_{L^2}|\theta|_{L^2}\, d\mu_\varepsilon((\mathbf{u},\theta)) \\
&\quad = 0, \qquad (2.20)
\end{aligned}
$$

where we have utilized the weak convergence of $M\tilde{\mu}_\varepsilon$, the definition of marginal distribution with $\Phi_0(\theta) = |\nabla\theta|_{L^2}^2 - \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta$, the assumption that μ_ε is an invariant measure for the Boussinesq system, and the uniform estimates on $\mathbf{v}$.

Hence we have verified (2.9) and (2.12).

As for the second condition, we take special test functionals

$$
\Phi((\mathbf{u},\theta)) = \Phi_0(\theta) \qquad (2.21)
$$

for some test functional Φ_0 for the infinite Prandtl number model.

We then have

$$
\begin{aligned}
&\left| \int_{L^2} (F_0(\theta), \Phi_0'(\theta)) d\mu_0(\theta) \right| \\
&\quad = \left| \lim_{\varepsilon\to 0} \int_{L^2} (F_0(\theta), \Phi_0'(\theta)) d(M\tilde{\mu}_\varepsilon)(\theta) \right| \\
&\quad = \left| \lim_{\varepsilon\to 0} \int_{X} (F_0(\theta), \Phi_0'(\theta)) d\mu_\varepsilon((\mathbf{u},\theta)) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \left| \limsup_{\varepsilon\to 0} \int_X (-\mathbf{u}\cdot\nabla\theta + u_3 - \Delta\theta, \Phi_0'(\theta)) d\mu_\varepsilon((\mathbf{u},\theta)) \right| \\
&\quad + \left| \limsup_{\varepsilon\to 0} \int_X (\mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3 - u_3, \Phi_0'(\theta)) d\mu_\varepsilon((\mathbf{u},\theta)) \right| \\
&\quad + \left| \limsup_{\varepsilon\to 0} \int_X \big(-(\mathbf{u} - \mathrm{Ra}\, A^{-1}(\mathbf{k}\theta))\cdot\nabla\theta, \Phi_0'(\theta)\big) d\mu_\varepsilon((\mathbf{u},\theta)) \right| \\
&\le \lim_{\varepsilon\to 0} \int_X |\mathbf{v}|_{L^2} \|\Phi_0'(\theta)\|_{L^2}\, d\mu_\varepsilon((\mathbf{u},\theta)) \\
&\quad + \lim_{\varepsilon\to 0} \int_X |\mathbf{v}|_{L^\infty} |\nabla\theta|_{L^2} \|\Phi_0'(\theta)\|_{L^2}\, d\mu_\varepsilon((\mathbf{u},\theta)) \\
&= 0. \qquad (2.22)
\end{aligned}
$$

This verifies condition (2.11) and thus completes the proof of the upper semicontinuity in the projected sense.

Next, we show the upper semicontinuity in the lifted sense, i.e, $\mu_\varepsilon \rightharpoonup \mathcal{L}\mu_0 = \mu_*$.

It is easy to check that $\mathcal{L}\mu_0 \in \mathcal{P}M(X)$ and that the marginal distribution of $\mathcal{L}\mu_0 \in \mathcal{P}M(X)$ defined above (through $\mathbf{v}$) is μ_0.

Now fix a test functional Φ that is bounded on bounded set of X, Fréchet-differentiable for $(\mathbf{u},\theta) \in Y$ with $\Phi'((\mathbf{u},\theta)) \in Y$, and whose derivative is continuous and bounded as a function from Y to Y; we have

$$
\begin{aligned}
&\left| \int_X \Phi(\mathbf{u},\theta) d\mu_\varepsilon((\mathbf{u},\theta)) - \int_X \Phi(\mathbf{u},\theta) d(\mathcal{L}\mu_0)((\mathbf{u},\theta)) \right| \\
&\quad = \left| \int_X \Phi(\mathbf{u},\theta) d\mu_\varepsilon((\mathbf{u},\theta)) - \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta) d\mu_0(\theta) \right| \\
&\quad \le \left| \int_X \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta) d\mu_\varepsilon((\mathbf{u},\theta)) - \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta) d\mu_0(\theta) \right| \\
&\qquad + \left| \int_X (\Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta) - \Phi(\mathbf{u},\theta)) d\mu_\varepsilon((\mathbf{u},\theta)) \right| \\
&\quad = \left| \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta) d(M\tilde{\mu}_\varepsilon)(\theta) - \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta) d\mu_0(\theta) \right| \\
&\qquad + \left| \int_X (\Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta),\theta) - \Phi(\mathbf{u},\theta)) d\mu_\varepsilon((\mathbf{u},\theta)) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \left| \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta), \theta) d(M\tilde{\mu}_\varepsilon)(\theta) - \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta), \theta) d\mu_0(\theta) \right| \\
&\quad + \left| \int_X \sup_{(\mathbf{u},\theta)\in \mathrm{supp}(\mu_\varepsilon)} \|\Phi'(\mathbf{u},\theta)\|_Y \, |\mathbf{v}|_{H^1} \, d\mu_\varepsilon((\mathbf{u},\theta)) \right| \\
&\leq \left| \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta), \theta) d(M\tilde{\mu}_\varepsilon)(\theta) - \int_{L^2} \Phi(\mathrm{Ra}\, A^{-1}(\mathbf{k}\theta), \theta) d\mu_0(\theta) \right| \\
&\quad + c\varepsilon \sup_{(\mathbf{u},\theta)\in \mathrm{supp}(\mu_\varepsilon)} \|\Phi'(\mathbf{u},\theta)\|_Y \\
&\to 0 \quad \text{as } \varepsilon \to 0, \qquad (2.23)
\end{aligned}
$$

where we have used the weak convergence of the marginal distribution proved in the first half, the mean value property, the boundedness of the Fréchet derivative of Φ, and the a priori estimates on $\mathbf{v}$.

This completes the proof of the theorem. □

What we have shown here is the upper semicontinuity of the set of invariant measures in this singular limit setting. This is reminiscent of the upper semicontinuity of the global attractors [40] for the same problem as well as well-known results on upper semicontinuity of global attractors for regular perturbations of dissipative dynamical systems [33]. We do not expect continuity in general since, for instance, the set of equilibrium states for an ODE can be a discontinuous function of a parameter, and each delta function on the phase space that is supported on a steady state is an invariant measures [40].

We could also formulate a general theorem on upper semicontinuity of statistical solutions with respect to a certain parameter for two time-scale problems of relaxation type just as we did for the upper semicontinuity of the global attractors [40]. However, we refrain from this since it is necessary to write the statistical analogue of the energy inequality (condition (2.9) and (2.12) in the definitions of stationary statistical solutions), which depends on the specific structure of the equation.

3 Upper Semicontinuity of the Nusselt Number

Among all statistical properties of the Boussinesq system for Rayleigh-Bénard convection, one of the most prominent is the long-time averaged Nusselt number measuring the heat transport in the vertical direction,

$$
\begin{aligned}
(\mathrm{Nu})_\varepsilon &= \sup_{(\mathbf{u}_0,\theta_0)\in X} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega |\nabla T(\mathbf{x},s)|^2\, d\mathbf{x}\, ds,\\
&= 1 + \sup_{(\mathbf{u}_0,\theta_0)\in X} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega u_3(\mathbf{x},s)T(\mathbf{x},s) d\mathbf{x}\, ds,\\
&= 1 + \sup_{(\mathbf{u}_0,\theta_0)\in X} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s) d\mathbf{x}\, ds, \qquad (3.1)
\end{aligned}
$$

where $T = 1 - z + \theta$ is the temperature field and $(\mathbf{u}, \theta)$ are suitable weak solutions to the Boussinesq system with initial data $(\mathbf{u}_0, \theta_0)$.

The Nusselt number for the infinite Prandtl number is defined similarly as

$$
(\mathrm{Nu})_0 = 1 + \sup_{\theta_0\in L^2} \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3(\mathbf{x},s)\theta(\mathbf{x},s) d\mathbf{x}\, ds. \qquad (3.2)
$$

A natural question to ask is whether the Nusselt number of the Boussinesq system is related to the Nusselt number of the infinite Prandtl number model.

The first observation is the following lemma, which states that the Nusselt number defined above is a statistical property of the system with respect to a certain ergodic invariant measure:

LEMMA 3.1 *There exists at least one ergodic invariant measure $\nu_\varepsilon \in \mathcal{I}M_\varepsilon$ for each $\varepsilon \in [0, \varepsilon_0]$ such that*

$$
\begin{aligned}
(\mathrm{Nu})_\varepsilon &= 1 + \frac{1}{L_xL_y}\int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu_\varepsilon((\mathbf{u},\theta))\\
&= 1 + \sup_{\mu\in\mathcal{I}M_\varepsilon} \frac{1}{L_xL_y}\int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu((\mathbf{u},\theta))\\
&= 1 + \lim_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)\, d\mathbf{x}\, ds \qquad (3.3)\\
&\qquad \forall (\mathbf{u}_0,\theta_0) \in \mathrm{supp}(\nu_\varepsilon),
\end{aligned}
$$

$$
\begin{aligned}
(\mathrm{Nu})_0 &= 1 + \frac{1}{L_x L_y} \int\limits_{L^2} \int\limits_{\Omega} \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3 \theta \, d\mathbf{x} \, d\nu_0(\theta) \\
&= 1 + \sup_{\mu \in \mathcal{IM}_0} \frac{1}{L_x L_y} \int\limits_{X} \int\limits_{\Omega} \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3 \theta \, d\mathbf{x} \, d\mu(\theta) \\
&= 1 + \lim_{t \to \infty} \frac{1}{t L_x L_y} \int_0^t \int\limits_{\Omega} \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3(\mathbf{x}, s) \theta(\mathbf{x}, s) d\mathbf{x} \, ds \qquad (3.4) \\
&\qquad \forall \theta_0 \in \mathrm{supp}(\nu_0).
\end{aligned}
$$

PROOF: We show only the case of $0 < \varepsilon \leq \varepsilon_0$.

For each fixed $\varepsilon \in (0, \varepsilon_0]$, there exists $(\mathbf{u}_{0j}, \theta_{0j}) \in \mathcal{A}_\varepsilon \subset X$ such that

$$
\begin{gathered}
(\mathrm{Nu})_\varepsilon = 1 + \lim_{j \to \infty} \limsup_{t \to \infty} \frac{1}{t L_x L_y} \int_0^t \int\limits_{\Omega} u_3(\mathbf{x}, s) \theta(\mathbf{x}, s) d\mathbf{x} \, ds \\
(\mathbf{u}(\mathbf{x}, 0), \theta(\mathbf{x}, 0)) = (\mathbf{u}_{0j}, \theta_{0j}).
\end{gathered} \qquad (3.5)
$$

For each fixed orbit (corresponding to suitable weak solutions [39, 40]), the long-time average is a statistical property in the sense that any fixed choice of generalized limit of the time average is equivalent to the spatial average with respect to a suitable stationary statistical solution by an application of the Krylov-Bogoliubov theory (see [18, 35] for the case of the Navier-Stokes system). In particular, for the chosen orbit (suitable weak solution), after an application of the Hahn-Banach theorem, there is a special generalized limit of the time averaging that is consistent with lim sup on the particular functional ($u_3\theta$) and particular orbit. Therefore, there exists $\mu_{\varepsilon, (\mathbf{u}_{0j}, \theta_{0j})} \in \mathcal{IM}_\varepsilon$ (this is an abuse of notation since there might be many orbits corresponding to the same initial data) such that

$$
\begin{aligned}
(3.6) \quad \limsup_{t \to \infty} \frac{1}{t L_x L_y} \int_0^t \int\limits_{\Omega} u_3(\mathbf{x}, s) \theta(\mathbf{x}, s) d\mathbf{x} \, ds = \\
\frac{1}{L_x L_y} \int\limits_{X} \int\limits_{\Omega} u_3 \theta \, d\mathbf{x} \, d\mu_{\varepsilon, (\mathbf{u}_{0j}, \theta_{0j})}(\mathbf{u}, \theta).
\end{aligned}
$$

We leave the details to the interested reader.

These stationary statistical solutions are weakly compact in $\mathcal{PM}(X)$ for each fixed $\varepsilon \in (0, \varepsilon_0]$ due to the uniform a priori estimates (1.16) and (1.17) and Prokhorov's theorem [2, 24]. Therefore, without loss of generality we may assume

$$
\mu_{\varepsilon, (\mathbf{u}_{0j}, \theta_{0j})} \rightharpoonup \mu_\varepsilon \quad \text{as } j \to \infty \qquad (3.7)
$$

for some $\mu_\varepsilon \in \mathcal{IM}_\varepsilon$.

This implies

$$
\begin{aligned}
(\mathrm{Nu})_\varepsilon &= 1 + \lim_{j\to\infty} \limsup_{t\to\infty} \frac{1}{tL_xL_y} \int_0^t \int_\Omega u_3(\mathbf{x},s)\theta(\mathbf{x},s)d\mathbf{x}\,ds \\
&= 1 + \lim_{j\to\infty} \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu_{\varepsilon,(\mathbf{u}_{0j},\theta_{0j})}(\mathbf{u},\theta) \\
&= 1 + \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu_\varepsilon((\mathbf{u},\theta)). \qquad (3.8)
\end{aligned}
$$

Now consider the following extremal subset of $\mathcal{IM}_\varepsilon$:

$$
\mathcal{SIM}_\varepsilon = \left\{ \mu \in \mathcal{IM}_\varepsilon \,\middle|\, \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu = \sup_{\nu\in\mathcal{IM}_\varepsilon} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu \right\}. \qquad (3.9)
$$

The subset $\mathcal{SIM}_\varepsilon$ is nonempty by the uniform a priori estimates (1.16) and (1.17) and Prokhorov's theorem. Indeed, suppose we have $\nu_{\varepsilon,j} \in \mathcal{IM}_\varepsilon$ such that

$$
\lim_{j\to\infty} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu_{\varepsilon,j} = \sup_{\nu\in\mathcal{IM}_\varepsilon} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu, \qquad (3.10)
$$

(the supremum is finite due to the a priori estimates (1.16) and (1.17)). The set $\{\nu_{\varepsilon,j},\ j \geq 1\}$ must be weakly precompact in $\mathcal{PM}(X)$ thanks to the uniform a priori estimates (1.16) and (1.17) and Prokhorov's theorem. Hence it must contain a subsequence that converges to some $\nu_\varepsilon \in \mathcal{PM}(X)$. It is then easy to verify that $\nu_\varepsilon \in \mathcal{SIM}_\varepsilon$.

Notice that $\mathcal{M}(X)$, the space of all finite Borel measures on X, form a locally convex topological space with the topology generated by weak convergence, and notice that $\mathcal{SIM}_\varepsilon$ is a compact subset of $\mathcal{M}(X)$. Therefore, the extremal set[1] of $\mathcal{SIM}_\varepsilon$ is nonempty thanks to the Krein-Milman theorem [18, 24]. Let ν_ε be an extremal point of $\mathcal{SIM}_\varepsilon$. Then ν_ε is necessarily an extremal point of $\mathcal{IM}_\varepsilon$, which further implies, after repeating the proof of a well-known result for dynamical systems [36] and utilizing the eventual regularity of the Boussinesq system [39, 40], that ν_ε must be ergodic. Therefore,

$$
\begin{aligned}
(\mathrm{Nu})_\varepsilon &= 1 + \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu_\varepsilon(\mathbf{u},\theta) \\
&\leq 1 + \sup_{\mu\in\mathcal{IM}_\varepsilon} \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\mu((\mathbf{u},\theta)) \\
&= 1 + \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u},\theta)
\end{aligned}
$$

[1] An *extremal point* of a set is a point that cannot be expressed as a proper convex combination of two other (distinct) points in the set.

$$= 1 + \lim_{t\to\infty} \frac{1}{tL_xL_y} \int_0^t \int_\Omega u_3(\mathbf{x}, s)\theta(\mathbf{x}, s) d\mathbf{x}\, ds \quad \forall(\mathbf{u}_0, \theta_0) \in \operatorname{supp}(\nu_\varepsilon)$$

$$\leq (\mathrm{Nu})_\varepsilon. \tag{3.11}$$

This completes the proof of the lemma. □

Although the lemma establishes that the Nusselt number is a statistical property with respect to appropriate ergodic invariant measures, the limit of the Nusselt number may not be directly related to the Nusselt number of the limit system (infinite Prandtl number model) since those invariant measures corresponding to the Nusselt number are special and their limit may not be invariant measures corresponding to the Nusselt number for the limit system. In other words, the weak limits of the set $\mathcal{SIM}_\varepsilon$ may not be associated with $\mathcal{SIM}_0$. Nevertheless, we are still able to establish the following relationship, which can be interpreted as upper semicontinuity of the Nusselt number in this singular limit.

THEOREM 3.2

$$\limsup_{\varepsilon\to 0} (\mathrm{Nu})_\varepsilon \leq (\mathrm{Nu})_0. \tag{3.12}$$

PROOF: Let $\nu_\varepsilon \in \mathcal{IM}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$, be ergodic invariant measures corresponding to the Nusselt number that we discussed in Lemma 3.1. Thanks to Theorem 2.3, we know that the set of $\{M\tilde{\nu}_\varepsilon, 0 < \varepsilon \leq \varepsilon_0\}$ is weakly precompact in $\mathcal{PM}(L^2)$. Without loss of generality, we assume it weakly converges to some $\mu_0 \in \mathcal{IM}_0$, i.e., $M\tilde{\nu}_\varepsilon \rightharpoonup \mu_0$. This implies

$$\begin{aligned}
\limsup_{\varepsilon\to 0} (\mathrm{Nu})_\varepsilon &= 1 + \limsup_{\varepsilon\to 0} \frac{1}{L_xL_y} \int_X \int_\Omega u_3\theta\, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u}, \theta) \\
&\leq 1 + \limsup_{\varepsilon\to 0} \frac{1}{L_xL_y} \int_X \int_\Omega \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta\, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u}, \theta) \\
&\quad + \limsup_{\varepsilon\to 0} \frac{1}{L_xL_y} \int_X \int_\Omega (u_3 - \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3)\theta\, d\mathbf{x}\, d\nu_\varepsilon(\mathbf{u}, \theta) \\
&\leq 1 + \limsup_{\varepsilon\to 0} \frac{1}{L_xL_y} \int_{L^2} \int_\Omega \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta\, d\mathbf{x}\, dM\tilde{\nu}_\varepsilon(\theta) \\
&\quad + \limsup_{\varepsilon\to 0} \frac{1}{L_xL_y} \int_X |v_3|_{L^2}\, |\theta|_{L^2}\, d\nu_\varepsilon(\mathbf{u}, \theta) \\
&= 1 + \frac{1}{L_xL_y} \int_{L^2} \int_\Omega \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3\theta\, d\mathbf{x}\, d\mu_0(\theta)
\end{aligned}$$

$$\leq 1 + \frac{1}{L_x L_y} \int_{L^2} \int_{\Omega} \mathrm{Ra}(A^{-1}(\mathbf{k}\theta))_3 \theta \, d\mathbf{x} \, d\nu_0(\theta)$$
$$= (\mathrm{Nu})_0. \tag{3.13}$$

This ends the proof of the theorem. □

It is worthwhile to recall that we do not expect continuity of statistical properties on parameters for general dynamical systems. In fact, it is easy to find ODE counterexamples where we have bifurcation. However, there is hope that at large Rayleigh number, both the Boussinesq system and the infinite Prandtl number model possess enough mixing so that the invariant measures that saturate the Nusselt number, i.e., ν_ε, are unique for each $\varepsilon \in [0, \varepsilon_0]$ since we believe the Nusselt number will be saturated by turbulent trajectories (the set of invariant measures itself always contains more than one element for large Rayleigh number due to the existence of multiple steady states). If this assumption is valid, we then may have continuity of the Nusselt number. Still we do not have the rate of convergence. The rate of convergence can be derived if we look at upper bounds on the Nusselt number instead. This is the goal of the next section.

4 Estimates of the Nusselt Number

The semicontinuity result in the previous section indicates that the Nusselt number for the limit system (infinite Prandtl number model) is an asymptotic bound for the Nusselt number associated with the Boussinesq system at large Prandtl number. However, no explicit convergence rate is given. The goal of this section is to derive a convergence rate result for the upper bound of the Nusselt numbers associated with the Boussinesq and infinite Prandtl number models. More specifically, we intend to derive an upper bound on the Nusselt number $(\mathrm{Nu})_\varepsilon$ that is consistent with the best-known upper bound [1, 9, 16, 30] on the Nusselt number $(\mathrm{Nu})_0$ for the infinite Prandtl number model.

The approach we take here is to view the Boussinesq system at large Prandtl number as a perturbation of the infinite Prandtl number model for convection. More specifically, we write the Boussinesq system as

$$\nabla p = \Delta \mathbf{u} + \mathrm{Ra}\, \mathbf{k} T + \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \tag{4.1}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \Delta T, \tag{4.2}$$

$$\mathbf{u}|_{z=0,1} = 0, \tag{4.3}$$

$$T|_{z=0} = 1, \quad T|_{z=1} = 0, \tag{4.4}$$

where

$$\mathbf{f} = -\varepsilon\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right). \tag{4.5}$$

We follow the background temperature profile method for the Boussinesq system proposed by Constantin and Doering [8, 9, 16], which is a generalization of E. Hopf's original idea [34]. One difference here is we *do not enforce the spectral constraint* in choosing our background temperature profile. In fact, we will choose a background profile that "almost" satisfies the spectral constraint.

Let $\tau = \tau(z)$ be a background temperature profile that satisfies the boundary condition for the temperature field T, and let

$$\theta = T - \tau. \tag{4.6}$$

Then the new perturbative temperature field[2] θ satisfies the equation

$$\frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = -u_3\tau' + \Delta\theta + \tau''. \tag{4.7}$$

Therefore, the Nusselt number can be written as

$$\begin{aligned}
(\mathrm{Nu})_\varepsilon &= \sup_{(\mathbf{u}_0,\theta_0)\in X} \langle|\nabla T|^2\rangle \\
&= \int_0^1 (\tau')^2\,dz + \sup_{(\mathbf{u}_0,\theta_0)\in X} [\langle|\nabla\theta|^2\rangle - 2\langle\theta\tau''\rangle] \\
&= \int_0^1 (\tau')^2\,dz - \inf_{(\mathbf{u}_0,\theta_0)\in X} \langle|\nabla\theta|^2 + 2\tau' u_3\theta\rangle \\
&= \int_0^1 (\tau')^2\,dz - \inf_{(\mathbf{u}_0,\theta_0)\in X} \langle Q^{(\tau)}(\theta)\rangle, \qquad (4.8)
\end{aligned}$$

where

$$Q^{(\tau)}(\theta) = |\nabla\theta|^2 + 2\tau' u_3\theta \tag{4.9}$$

and $\langle\cdot\rangle$ denotes the space-time average defined as

$$\langle g\rangle = \limsup_{t\to\infty} \frac{1}{tL_xL_y}\int_0^t \int_\Omega g(\mathbf{x}, s)d\mathbf{x}\,ds. \tag{4.10}$$

It is a straightforward calculation, based on the alternative form of the Boussinesq system introduced in this section, that the vertical velocity field u_3 and the

[2] This perturbative temperature field away from the background temperature field τ is different from the perturbative temperature field utilized in the previous sections, which is the perturbation away from the pure conduction state.

perturbative temperature field θ satisfy the following equation,

$$\Delta^2 u_3 = -\operatorname{Ra}\Delta_H\theta - \Delta_H f_3 + \frac{\partial^2 f_1}{\partial x\partial z} + \frac{\partial^2 f_2}{\partial y\partial z}, \tag{4.11}$$

$$u_3\big|_{z=0,1} = 0, \quad \frac{\partial u_3}{\partial z}\bigg|_{z=0,1} = 0, \quad \theta\big|_{z=0,1} = 0, \tag{4.12}$$

where $\Delta_H = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is the horizontal Laplace operator. This relationship implies that the velocity field is almost slaved to the perturbative temperature field.

In terms of the horizontal Fourier coefficients $\hat{\theta}_{\mathbf{m}}$, $\hat{u}_{3\mathbf{m}}$, and $\hat{f}_{j\mathbf{m}}$, $j = 1,2,3$, where $\mathbf{m} = (m_1, m_2)$ is the horizontal Fourier wave number, the relationship between the vertical velocity and the perturbative temperature can be written as

$$\left(\mathbf{m}^2 - \frac{d^2}{dz^2}\right)^2 \hat{u}_{3\mathbf{m}} = \operatorname{Ra}\mathbf{m}^2\hat{\theta}_{\mathbf{m}} + \mathbf{m}^2\hat{f}_{3\mathbf{m}} + im_1\frac{d}{dz}\hat{f}_{1\mathbf{m}} + im_2\frac{d}{dz}\hat{f}_{2\mathbf{m}}, \tag{4.13}$$

$$\hat{u}_{3\mathbf{m}}\big|_{z=0,1} = 0, \quad \frac{d}{dz}\hat{u}_{3\mathbf{m}}\bigg|_{z=0,1} = 0, \quad \hat{\theta}_{\mathbf{m}}\big|_{z=0,1} = 0, \tag{4.14}$$

where $\mathbf{m}^2 = m_1^2 + m_2^2$ as usual.

Therefore, we have the following lemma, which is a modification/generalization of the proposition in [16]:

LEMMA 4.1 *The following inequality holds for all* $\mathbf{m}$*:*

$$\begin{aligned}\operatorname{Re}\int_0^1 \frac{\hat{\theta}_{\mathbf{m}}\hat{u}_{3\mathbf{m}}^*}{z}\,dz &\ge \frac{1}{\operatorname{Ra}}\int_0^1 \frac{|\hat{u}_{3\mathbf{m}}|^2}{z^3}\,dz \\ &\quad - \frac{3}{2\operatorname{Ra}}\left(|\hat{f}_{3\mathbf{m}}|_{L^2}^2 + \frac{|\hat{f}_{1\mathbf{m}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{m}}|_{L^2}}{\mathbf{m}^2}\right).\end{aligned} \tag{4.15}$$

PROOF OF LEMMA 4.1: We multiply (4.13) by $\zeta = \hat{u}_{3\mathbf{m}}^*/z$ (here $*$ denotes complex conjugation), integrate over $[0,1]$, and take the real part; we deduce

$$\begin{aligned}&\operatorname{Re}\int_0^1 \frac{\hat{\theta}_{\mathbf{m}}\hat{u}_{3\mathbf{m}}^*}{z}\,dz \\ &= \operatorname{Re}\int_0^1 \left\{\frac{(\mathbf{m}^4\hat{u}_{3\mathbf{m}} - 2\mathbf{m}^2\hat{u}_{3\mathbf{m}}'' + \hat{u}_{3\mathbf{m}}'''')\hat{u}_{3\mathbf{m}}^*}{\operatorname{Ra}\mathbf{m}^2 z} - \frac{\hat{f}_{3\mathbf{m}}\hat{u}_{3\mathbf{m}}^*}{\operatorname{Ra} z}\right. \\ &\qquad\left. - \frac{im_1\hat{f}_{1\mathbf{m}}'\hat{u}_{3\mathbf{m}}^*}{\mathbf{m}^2\operatorname{Ra} z} - \frac{im_2\hat{f}_{2\mathbf{m}}'\hat{u}_{3\mathbf{m}}^*}{\mathbf{m}^2\operatorname{Ra} z}\right\}dz \\ &\qquad\text{(integration by parts and Cauchy-Schwarz inequality)} \\ &\ge \frac{\mathbf{m}^2}{\operatorname{Ra}}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2}^2 - \frac{2}{\operatorname{Ra}}\operatorname{Re}\int_0^1 \frac{\hat{u}_{3\mathbf{m}}''\hat{u}_{3\mathbf{m}}^*}{z}\,dz + \frac{1}{\operatorname{Ra}\mathbf{m}^2}\operatorname{Re}\int_0^1 \frac{\hat{u}_{3\mathbf{m}}''''\hat{u}_{3\mathbf{m}}^*}{z}\,dz \\ &\quad - \frac{1}{\operatorname{Ra}}|\hat{f}_{3\mathbf{m}}|_{L^2}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} - \frac{1}{\operatorname{Ra}|\mathbf{m}|}\left(\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} + \left|\frac{\hat{u}_{3\mathbf{m}}'}{z^{1/2}}\right|_{L^2}\right)\left(\left|\frac{\hat{f}_{1\mathbf{m}}}{z^{1/2}}\right|_{L^2} + \left|\frac{\hat{f}_{2\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right) \\ &\qquad\text{(lemma 1 of [16])}\end{aligned} \tag{4.16}$$

$$
\begin{aligned}
&= \frac{\mathbf{m}^2}{\mathrm{Ra}}|z^{1/2}\zeta|^2_{L^2} + \frac{2}{\mathrm{Ra}}|z^{1/2}\zeta'|^2_{L^2} + \frac{1}{\mathrm{Ra}\,\mathbf{m}^2}|z^{1/2}\zeta''|^2_{L^2} \\
&\quad - \frac{1}{\mathrm{Ra}}|\hat{f}_{3\mathbf{m}}|_{L^2}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} - \frac{1}{\mathrm{Ra}\,|\mathbf{m}|}\left(\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}|_{L^2} + |\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right)\left(\left|\frac{\hat{f}_{1\mathbf{m}}}{z^{1/2}}\right|_{L^2} + \left|\frac{\hat{f}_{2\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right) \\
&\quad \text{(Cauchy-Schwarz inequality)} \\
&\ge \frac{2}{\mathrm{Ra}}|z^{1/2}\zeta'|^2_{L^2} + \frac{2}{\mathrm{Ra}}|z^{1/2}\zeta|_{L^2}\,|z^{1/2}\zeta''|_{L^2} \\
&\quad - \frac{1}{\mathrm{Ra}}|\hat{f}_{3\mathbf{m}}|_{L^2}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} - \frac{1}{\mathrm{Ra}\,|\mathbf{m}|}\left(\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} + \left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right)\left(\left|\frac{\hat{f}_{1\mathbf{m}}}{z^{1/2}}\right|_{L^2} + \left|\frac{\hat{f}_{2\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right) \\
&\quad \text{(lemma 2 of [16])} \\
&\ge \frac{2}{\mathrm{Ra}}|z^{1/2}\zeta'|^2_{L^2} + \frac{2}{\mathrm{Ra}}\left|\frac{\zeta}{z^{1/2}}\right|^2_{L^2} \\
&\quad - \frac{1}{\mathrm{Ra}}|\hat{f}_{3\mathbf{m}}|_{L^2}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} - \frac{1}{\mathrm{Ra}\,|\mathbf{m}|}\left(\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} + \left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right)\left(\left|\frac{\hat{f}_{1\mathbf{m}}}{z^{1/2}}\right|_{L^2} + \left|\frac{\hat{f}_{2\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right) \\
&\quad \text{(Cauchy-Schwarz inequality)} \\
&\ge \frac{2}{\mathrm{Ra}}\left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|^2_{L^2} - \frac{1}{\mathrm{Ra}}|\hat{f}_{3\mathbf{m}}|_{L^2}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} \\
&\quad - \frac{1}{\mathrm{Ra}\,|\mathbf{m}|}\left(\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} + \left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right)\left(\left|\frac{\hat{f}_{1\mathbf{m}}}{z^{1/2}}\right|_{L^2} + \left|\frac{\hat{f}_{2\mathbf{m}}}{z^{1/2}}\right|_{L^2}\right) \\
&\quad \text{(Hardy-type inequality and Cauchy-Schwarz inequality)} \\
&\ge \frac{2}{\mathrm{Ra}}\left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|^2_{L^2} - \frac{1}{\mathrm{Ra}}|\hat{f}_{3\mathbf{m}}|_{L^2}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|_{L^2} \\
&\quad - \frac{2}{\mathrm{Ra}\,|\mathbf{m}|}\left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2}\left(|\hat{f}_{1\mathbf{m}}|^{1/2}_{L^2}\left|\frac{\hat{f}_{1\mathbf{m}}}{z}\right|^{1/2}_{L^2} + |\hat{f}_{2\mathbf{m}}|^{1/2}_{L^2}\left|\frac{\hat{f}_{2\mathbf{m}}}{z}\right|^{1/2}_{L^2}\right) \\
&\quad \text{(Hardy inequality)} \\
&\ge \frac{2}{\mathrm{Ra}}\left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|^2_{L^2} - \frac{1}{\mathrm{Ra}}|\hat{f}_{3\mathbf{m}}|_{L^2}\left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2} \\
&\quad - \frac{2}{\mathrm{Ra}\,|\mathbf{m}|}\left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|_{L^2}(|\hat{f}_{1\mathbf{m}}|^{1/2}_{L^2}|\hat{f}'_{1\mathbf{m}}|^{1/2}_{L^2} + |\hat{f}_{2\mathbf{m}}|^{1/2}_{L^2}|\hat{f}'_{2\mathbf{m}}|^{1/2}_{L^2}) \\
&\quad \text{(Cauchy-Schwarz inequality)} \\
&\ge \frac{1}{\mathrm{Ra}}\left|\frac{\hat{u}'_{3\mathbf{m}}}{z^{1/2}}\right|^2_{L^2} - \frac{3}{2\,\mathrm{Ra}}\left(|\hat{f}_{3\mathbf{m}}|^2_{L^2} + \frac{1}{|\mathbf{m}|^2}(|\hat{f}_{1\mathbf{m}}|_{L^2}|\hat{f}'_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2}|\hat{f}'_{2\mathbf{m}}|_{L^2})\right) \\
&\quad \text{(Hardy-type inequality)} \\
&\ge \frac{1}{\mathrm{Ra}}\left|\frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}}\right|^2_{L^2} - \frac{3}{2\,\mathrm{Ra}}\left(|\hat{f}_{3\mathbf{m}}|^2_{L^2} + \frac{1}{|\mathbf{m}|^2}(|\hat{f}_{1\mathbf{m}}|_{L^2}|\hat{f}'_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2}|\hat{f}'_{2\mathbf{m}}|_{L^2})\right) \qquad (4.17)
\end{aligned}
$$

where we have performed integration by parts (there is no singularity in our integration by parts thanks to the boundary conditions listed in (4.14)), applied following lemma 1 of [16]: *Assume* $w(0) = w(1) = w'(0) = w'(1) = 0$; *we have*

$$-\operatorname{Re}\int_0^1 \frac{w''w^*}{z}\,dz = \int_0^1 z\left|\left(\frac{w}{z}\right)'\right|^2 dz,$$
$$\operatorname{Re}\int_0^1 \frac{w''''w*}{z}\,dz = \int_0^1 z\left|\left(\frac{w}{z}\right)''\right|^2 dz,$$

and lemma 2 from [16]: *Assume* $\zeta(0) = \zeta(1) = 0$; *we have*

$$\left|\frac{\zeta}{z^{1/2}}\right|_{L^2}^2 \le |z^{1/2}\zeta|_{L^2}\,|z^{1/2}\zeta''|_{L^2},$$

as well as a Hardy inequality: *Assume* $g(0) = g(1) = 0$; *we have*

$$\int_0^1 \frac{g^2}{z^2} \le \int_0^1 g'^2,$$

and a Hardy-type inequality: *Assume* $g(0) = g'(0) = 0 = g(1)$; *we have*

$$\int_0^1 \frac{g^2}{z^3} \le \int_0^1 \frac{g'^2}{z},$$

which can be shown by using the fundamental theorem of calculus and Cauchy-Schwarz on g^2/z^2.

This completes the proof of Lemma 4.1. □

Next we borrow an idea from [16] and introduce the following (nonmonotonic) background flow τ for $\delta \in (0, \frac{1}{2})$,

$$\tau(z) = \begin{cases} 1 - z/\delta, & 0 \le z \le \delta, \\ 1/2 + \lambda(\delta)\ln z/(1-z), & \delta \le z \le 1-\delta, \\ (1-z)/\delta & 1-\delta \le z \le 1, \end{cases} \tag{4.18}$$

with

$$\lambda(\delta) = \frac{1}{2\ln(1-\delta)/\delta}. \tag{4.19}$$

Now in terms of the Fourier coefficients, we have

$$\begin{aligned} Q^{(\tau)}(\theta) &= \sum_{\mathbf{m}} \left(|\hat\theta'_{\mathbf{m}}|^2 + |\mathbf{m}|^2|\hat\theta_{\mathbf{m}}|^2 + \tau'(\hat u_{3\mathbf{m}}\hat\theta^*_{\mathbf{m}} + \hat u^*_{3\mathbf{m}}\hat\theta_{\mathbf{m}})\right) \\ &= \sum_{\mathbf{m}} \left(Q^{(\tau)}_{\mathbf{m},\text{lower}} + Q^{(\tau)}_{\mathbf{m},\text{upper}}\right) \end{aligned} \tag{4.20}$$

where

$$\begin{aligned} Q^{(\tau)}_{\mathbf{m},\text{lower}} = {}& \int_0^{1/2} (|\hat\theta'_{\mathbf{m}}|^2 + \mathbf{m}^2|\hat\theta_{\mathbf{m}}|^2)dz + 2\lambda\int_0^1 \frac{\operatorname{Re}[\hat\theta_{\mathbf{m}}\hat u^*_{3\mathbf{m}}]}{z}\,dz \\ & - 2\int_0^\delta \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)\operatorname{Re}[\hat\theta_{\mathbf{m}}\hat u^*_{3\mathbf{m}}]dz \end{aligned} \tag{4.21}$$

$$Q^{(\tau)}_{\mathbf{m},\mathrm{upper}} = \int_{1/2}^{1} (|\hat{\theta}'_{\mathbf{m}}|^2 + \mathbf{m}^2 |\hat{\theta}_{\mathbf{m}}|^2) dz + 2\lambda \int_0^1 \frac{\mathrm{Re}[\hat{\theta}_{\mathbf{m}} \hat{u}^*_{3\mathbf{m}}]}{1-z} dz \\ - 2\int_{1-\delta}^{1} \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) \mathrm{Re}[\hat{\theta}_{\mathbf{m}} \hat{u}^*_{3\mathbf{m}}] dz \tag{4.22}$$

where we have rewritten the terms so that the stable stratification of $\tau(z)$ in the bulk may help to asymptotically dominate the negative contributions to $Q^{(\tau)}$ from the boundary layer.

It is easy to check that

$$Q^{(\tau)}_{\mathbf{m},\mathrm{lower}} = \int_0^{1/2} (|\hat{\theta}'_{\mathbf{m}}|^2 + \mathbf{m}^2 |\hat{\theta}_{\mathbf{m}}|^2) dz + 2\lambda \int_0^1 \frac{\mathrm{Re}[\hat{\theta}_{\mathbf{m}} \hat{u}^*_{3\mathbf{m}}]}{z} dz \\ - 2\int_0^{\delta} \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) \mathrm{Re}[\hat{\theta}_{\mathbf{m}} \hat{u}^*_{3\mathbf{m}}] dz$$

(by lemma 2)

$$\geq \int_0^{1/2} |\hat{\theta}'_{\mathbf{m}}|^2 dz + \frac{2\lambda}{\mathrm{Ra}} \int_0^1 \frac{|\hat{u}_{3\mathbf{m}}|^2}{z^3} dz \\ - 2\int_0^{\delta} \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right) |\hat{\theta}_{\mathbf{m}}|\, |\hat{u}_{3\mathbf{m}}| dz \\ - \frac{3\lambda}{\mathrm{Ra}} \left(|\hat{f}_{3\mathbf{m}}|^2_{L^2} + \frac{|\hat{f}_{1\mathbf{m}}|_{L^2} |\frac{d}{dz}\hat{f}_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2} |\frac{d}{dz}\hat{f}_{2\mathbf{m}}|_{L^2}}{\mathbf{m}^2} \right)$$

(by Hölder's inequality)

$$\geq \int_0^{1/2} |\hat{\theta}'_{\mathbf{m}}|^2 dz + \frac{2\lambda}{\mathrm{Ra}} \int_0^1 \frac{|\hat{u}_{3\mathbf{m}}|^2}{z^3} dz \\ - 2\left(\int_0^{\delta} z^4 \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)^2 dz \right)^{1/2} \left| \frac{\hat{\theta}_{\mathbf{m}}}{z^{1/2}} \right|_{L^\infty(0,1/2)} \left| \frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}} \right|_{L^2} \\ - \frac{3\lambda}{\mathrm{Ra}} \left(|\hat{f}_{3\mathbf{m}}|^2_{L^2} + \frac{|\hat{f}_{1\mathbf{m}}|_{L^2} |\frac{d}{dz}\hat{f}_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2} |\frac{d}{dz}\hat{f}_{2\mathbf{m}}|_{L^2}}{\mathbf{m}^2} \right)$$

(by a one-dimensional Sobolev inequality)

$$\geq \int_0^{1/2} |\hat{\theta}'_{\mathbf{m}}|^2 dz + \frac{2\lambda}{\mathrm{Ra}} \int_0^1 \frac{|\hat{u}_{3\mathbf{m}}|^2}{z^3} dz \\ - 2\left(\int_0^{\delta} z^4 \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)^2 dz \right)^{1/2} |\hat{\theta}'_{\mathbf{m}}|_{L^2(0,1/2)} \left| \frac{\hat{u}_{3\mathbf{m}}}{z^{3/2}} \right|_{L^2} \\ - \frac{3\lambda}{\mathrm{Ra}} \left(|\hat{f}_{3\mathbf{m}}|^2_{L^2} + \frac{|\hat{f}_{1\mathbf{m}}|_{L^2} |\frac{d}{dz}\hat{f}_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2} |\frac{d}{dz}\hat{f}_{2\mathbf{m}}|_{L^2}}{\mathbf{m}^2} \right)$$

(by Cauchy-Schwarz)

$$\geq \left[\frac{2\lambda}{\mathrm{Ra}} - \int_0^\delta z^4 \left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)^2 dz\right] \times \int_0^1 \frac{|\hat{u}_{3\mathbf{m}}|^2}{z^3}\,dz$$
$$- \frac{3\lambda}{\mathrm{Ra}}\left(|\hat{f}_{3\mathbf{m}}|_{L^2}^2 + \frac{|\hat{f}_{1\mathbf{m}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{m}}|_{L^2}}{\mathbf{m}^2}\right) \tag{4.23}$$

where we have used Lemma 4.1 and the Cauchy-Schwarz inequality as well as a one-dimensional Sobolev inequality (calculus inequality): *Suppose* $g(0) = 0$; *we then have*

$$\left|\frac{g(z)}{z^{1/2}}\right|_{L^\infty(0,\frac{1}{2})} \leq |g'|_{L^2(0,\frac{1}{2})}.$$

This can be proved via applying the fundamental theorem of calculus on g followed by Cauchy-Schwarz.

Noting that

$$\int_0^\delta z^4\left(\frac{1}{\delta} + \frac{\lambda}{z} + \frac{\lambda}{1-z}\right)^2 dz = \frac{\delta^3}{5} \times \left\{1 + \mathcal{O}\left(\frac{1}{|\ln\delta|}\right)\right\} \quad \text{as } \delta \to 0, \tag{4.24}$$

a sufficient asymptotic condition for the nonnegativity of the principal part (the part not involving $\mathbf{f}$) of $Q^{(\tau)}_{\text{lower}}$ (and also $Q^{(\tau)}_{\text{upper}}$ and hence $Q^{(\tau)}$)[3] is

$$\mathrm{Ra}\,\delta^3 = 10\lambda = \frac{5}{\ln(1-\delta)/\delta}. \tag{4.25}$$

This is satisfied asymptotically by

$$\delta \sim \left(\frac{15}{\mathrm{Ra}\ln\mathrm{Ra}}\right)^{1/3}, \quad \lambda \sim 15\ln\mathrm{Ra}, \quad \text{as } \mathrm{Ra} \to \infty. \tag{4.26}$$

Combining this with (4.8), (4.20), and (4.23), we arrive at

$$(\mathrm{Nu})_\varepsilon \leq \int_0^1 (\tau')^2\,dz$$
$$+ \frac{6\lambda}{\mathrm{Ra}}\sum_{\mathbf{m}}\left\langle |\hat{f}_{3\mathbf{m}}|_{L^2}^2 + \frac{|\hat{f}_{1\mathbf{m}}|_{L^2}|\frac{d}{dz}\hat{f}_{1\mathbf{m}}|_{L^2} + |\hat{f}_{2\mathbf{m}}|_{L^2}|\frac{d}{dz}\hat{f}_{2\mathbf{m}}|_{L^2}}{\mathbf{m}^2}\right\rangle$$
$$\sim \frac{2}{\delta} + \frac{6\lambda}{\mathrm{Ra}}\langle|\mathbf{f}|_{L^2}|\mathbf{f}|_{H^1}\rangle$$
$$\sim 2 \times \left(\frac{\mathrm{Ra}\ln\mathrm{Ra}}{15}\right)^{1/3} + \frac{6\lambda}{\mathrm{Ra}}\langle|\mathbf{f}|_{L^2}|\mathbf{f}|_{H^1}\rangle. \tag{4.27}$$

It is then an easy exercise to check, thanks to the a priori estimates on $\mathbf{u}$ (equations (1.13)–(1.17)),

$$\langle|\mathbf{f}|_{L^2}|\mathbf{f}|_{H^1}\rangle \leq c\varepsilon^2\,\mathrm{Ra}^{9/2}. \tag{4.28}$$

[3] Note here that we deviate from the Constantin-Doering approach in the sense that the spectral constraint is not enforced exactly, but only asymptotically (modulus the part involving $\mathbf{f}$).

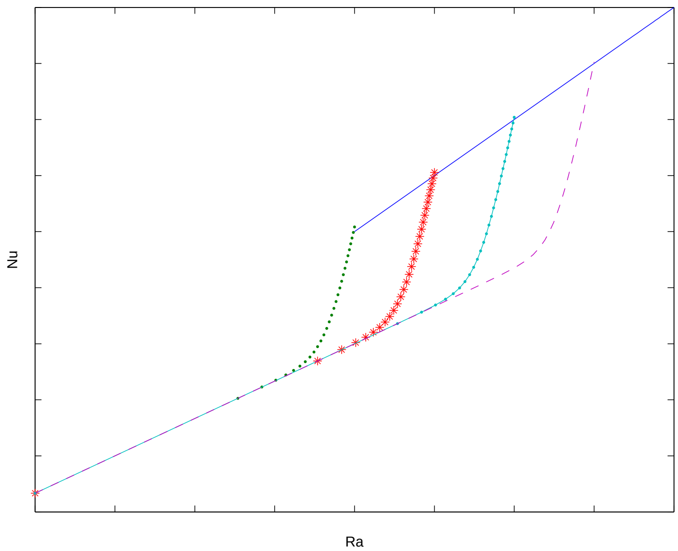

FIGURE 4.1. Schematic log-log plot of the new upper bound on the Nusselt number (4.30) versus Rayleigh number for different values of Prandtl number.

Combining the estimates above and the uniform bound of $c' \,\mathrm{Ra}^{1/2}$ [8], we have the following result:

THEOREM 4.2 *There exists a constant c independent of* Ra *and* Pr *such that*

$$(\mathrm{Nu})_\varepsilon \le \mathrm{Ra}^{1/3}(\ln \mathrm{Ra})^{1/3} + c\frac{\mathrm{Ra}^{7/2}\ln \mathrm{Ra}}{\mathrm{Pr}^2}, \tag{4.29}$$

as long as the large Prandtl number assumption (1.20) $\mathrm{Ra}\,/\,\mathrm{Pr} \le c_0$ *is satisfied. Moreover,*

$$(\mathrm{Nu})_\varepsilon \le \min\left\{\mathrm{Ra}^{1/3}(\ln \mathrm{Ra})^{1/3} + c\,\frac{\mathrm{Ra}^{7/2}\ln \mathrm{Ra}}{\mathrm{Pr}^2},\, c' \,\mathrm{Ra}^{1/2}\right\}. \tag{4.30}$$

Notice that the second bound in the theorem does not explicitly require the large Prandtl number assumption (1.20) since it is implicitly satisfied for large Rayleigh number if the uniform Prandtl number bound $\mathrm{Ra}^{1/2}$ dominates the new upper bound.

The upper bound above fits the common belief that the Nusselt number at large Rayleigh number should be eventually independent of the Prandtl number at large Prandtl number, and the Nusselt number should scale like $\mathrm{Ra}^{1/3}$ for large Prandtl number modulo logarithmic terms. In fact, there is even evidence of uniform-in-Prandtl-number scaling of $\mathrm{Ra}^{1/3}$ for the Nusselt number [1]. However, the correction term here is not very satisfactory (of the order of $\mathrm{Ra}^{7/2}\,/\,\mathrm{Pr}^2$), which grows faster than the known uniform-in-Prandtl-number bound of $\mathrm{Ra}^{1/2}$ [8] at large

Rayleigh number unless the Prandtl number grows much faster than the Rayleigh number. Although the correction term can be improved by refining the estimates from [40], we are not yet able to derive a bound that is consistent with the uniform $\mathrm{Ra}^{1/2}$ bound for the Boussinesq system and the uniform $\mathrm{Ra}^{1/3}$ bound for the infinite Prandtl number model in the sense that we are not yet able to derive a bound on the Nusselt number for the Boussinesq system that consists of the bound for the infinite Prandtl number model ($\mathrm{Ra}^{1/2}$) plus a correction term of the form $\mathrm{Ra}^{\alpha} / \mathrm{Pr}^{\beta}$, with $\alpha \leq \frac{1}{2}$ and $\beta > 0$. However, see [41] for a uniform $Ra^{1/3}$ (modulo logarithmic correction) upper bound on the Nusselt number at large Prandtl number.

5 Concluding Remarks

We have demonstrated that the infinite Prandtl number model is a good effective model for the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number in terms of stationary statistical properties.

More specifically, we have established the upper semicontinuity of the set of invariant measures for the Boussinesq system as the Prandtl number approaches infinity (with the limit being the infinite Prandtl number model). Therefore, equilibrium statistics of the Boussinesq system can be asymptotically dominated by equilibrium statistics of the infinite Prandtl number model. This complements our result on the upper semicontinuity of the global attractors [40]. We are not able to show the continuity at this point since the set of invariant measures may contain multiple elements, and we may experience hysteresis-type phenomena. One way to obtain continuity is by adding appropriate noises as these noises will connect different branches of the attractor and render the uniqueness of the invariant measure [13, 17, 25]. The uniqueness of the invariant measure (at any fixed Prandtl number) leads to the continuity in the Prandtl number (including the singular limit of $\mathrm{Pr} \to \infty$, $\varepsilon \to 0$) of the invariant measure. This would be an example of noise-induced statistical stability. The noise may be justified as accounting for neglected small effects of various physical mechanisms not represented in the system.

We have also established the upper semicontinuity of the Nusselt number as the Prandtl number approaches infinity. This implies that the Nusselt number for the infinite Prandtl number model asymptotically bounds the Nusselt number for the Boussinesq system at large Prandtl number. This is not a direct consequence of the upper semicontinuity of the set of invariant measures, since the limit of the sequence of invariant measures corresponding to the Nusselt numbers for the Boussinesq system as the Prandtl number approaches infinity may not be an invariant measure of the infinite Prandtl number model corresponding to the Nusselt number of the limit system. Again, we do not have continuity of the Nusselt number. Yet we strongly believe that continuity is true at large Rayleigh number since we expect a unique, strongly mixing trajectory/invariant measure that saturates the Nusselt number at large Rayleigh number. Of course, adding appropriate noise leads to the uniqueness of the invariant measure, which further leads to the continuity of the Nusselt numbers with respect to the Prandtl number. A by-product that we derived here is that the Nusselt numbers are saturated by ergodic invariant measures.

A more concrete result that we obtained here is an upper bound on the Nusselt number for the Boussinesq system of the form

$$\mathrm{Ra}^{1/3}(\ln \mathrm{Ra})^{1/3} + c\,\frac{\mathrm{Ra}^{7/2}\ln \mathrm{Ra}}{\mathrm{Pr}^2}.$$

This bound asymptotically agrees with the optimal bound for the Nusselt number of the infinite Prandtl number model ($\mathrm{Ra}^{1/3}$ modulo a logarithmic term) to the leading order at large Prandtl number. This is the first result of this kind.

Finally, we remark that the results and techniques derived here may be applied to many other systems with two disparate time scales of relaxation type.

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Bibliography

[1] Amati, G.; Koal, K.; Massaioli, F.; Sreenivasan, K. R.; Verzicco, R. Turbulent thermal convection at high Rayleigh numbers for a Boussinesq fluid of constant Prandtl number. *Phys. Fluids* **17** (2005), no. 12, 121701-1–121701-4.

[2] Billingsley, P. *Weak convergence of measures: Applications in probability*. Conference Board of the Mathematical Sciences Regional Conference Series in Applied Mathematics, 5. Society for Industrial and Applied Mathematics, Philadelphia, 1971.

[3] Bodenschatz, E.; Pesch, W.; Ahlers, G. Recent developments in Rayleigh-Bénard convection. *Annual review of fluid mechanics*, vol. 32, 709–778. Annual Reviews, Palo Alto, Calif., 2000.

[4] Busse, F. H. Fundamentals of thermal convection. *Mantle convection: Plate tectonics and global dynamics*, 23–95. Gordon and Breach, New York, 1989.

[5] Chae, D. The vanishing viscosity limit of statistical solutions of the Navier-Stokes equations. I. 2-D periodic case. *J. Math. Anal. Appl*. **155** (1991), no. 2, 437–459.

[6] Chae, D. The vanishing viscosity limit of statistical solutions of the Navier-Stokes equations. II. The general case. *J. Math. Anal. Appl*. **155** (1991), no. 2, 460–484.

[7] Chandrasekhar, S. *Hydrodynamic and hydro-magnetic stability*. International Series of Monographs on Physics. Clarendon, Oxford, 1961.

[8] Constantin, P.; Doering, C. R. Heat transfer in convective turbulence. *Nonlinearity* **9** (1996), no. 4, 1049–1060.

[9] Constantin, P.; Doering, C. R. Infinite Prandtl number convection. *J. Statist. Phys*. **94** (1999), no. 1-2, 159–172.

[10] Constantin, P.; Foias, C. *Navier-Stokes equations*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, 1988.

[11] Constantin, P.; Hallstrom, C.; Poutkaradze, V. Logarithmic bounds for infinite Prandtl number rotating convection. *J. Math. Phys*. **42** (2001), no. 2, 773–783.

[12] Constantin, P.; Wu, J. Statistical solutions of the Navier-Stokes equations on the phase space of vorticity and the inviscid limits. *J. Math. Phys*. **38** (1997), no. 6, 3031–3045.

[13] Da Prato, G.; Zabczyk, J. *Ergodicity for infinite dimensional systems*. Cambridge University Press, Cambridge–New York, 1996.

[14] Doering, C. R.; Constantin, P. On upper bounds for infinite Prandtl number convection with or without rotation. *J. Math. Phys*. **42** (2001), no. 2, 784–795.

[15] Doering, C. R.; Gibbon, J. D. *Applied analysis of the Navier-Stokes equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 1995.

[16] Doering, C. R.; Otto, F.; Reznikoff, M. G. Bounds on vertical heat transport for infinite Prandtl number Rayleigh-Bénard convection. *J. Fluid Mech*. **560** (2006), 229–241.

[17] E, W. Stochastic hydrodynamics. *Current developments in mathematics, 2000*, 109–147. International, Somerville, Mass., 2001.

[18] Foias, C.; Manley, O.; Rosa, R.; Temam, R. *Navier-Stokes equations and turbulence*. Encyclopedia of Mathematics and Its Applications, 83. Cambridge University Press, Cambridge, 2001.

[19] Getling, A. V. *Rayleigh-Bénard convection. Structures and dynamics*. Advanced Series in Nonlinear Dynamics, 11. World Scientific, River Edge, N.J., 1998.

[20] Grossmann, S.; Lohse, D. Scaling in thermal convection: A unifying theory. *J. Fluid Mech*. **407** (2000), 27–56.

[21] Howard, L. Heat transport by turbulent convection. *J. Fluid Mech*. **17** (1963), 405–432.

[22] Ierley, G. R.; Kerswell, R. R.; Plasting, S. C. Infinite-Prandtl-number convection. Part 2. A singular limit of upper bound theory. *J. Fluid Mech*. **560** (2006), 159–227.

[23] Kadanoff, L. P. Turbulent heat flow: structures and scaling. *Phys. Today* **54** (2001), no. 8, 34–39.

[24] Lax, P. D. *Functional analysis*. Pure and Applied Mathematics (New York). Wiley-Interscience, New York, 2002.

[25] Lee, J.; Wu, M-Y. Ergodicity for dissipative Boussinesq equations with random forcing. *J. Statist. Phys*. **117** (2004), no. 5-6, 929–973.

[26] Ma, T.; Wang, S. Dynamic bifurcation and stability in the Rayleigh-Bénard convection. *Commun. Math. Sci*. **2** (2004), no. 2, 159–183.

[27] Majda, A. J.; Bertozzi, A. L. *Vorticity and incompressible flow*. Cambridge Texts in Applied Mathematics, 27. Cambridge University Press, Cambridge, 2002.

[28] Majda, A. J.; Wang, X. *Non-linear dynamics and statistical theories for basic geophysical flows*. Cambridge University Press, Cambridge, 2006.

[29] Monin, A. S.; Yaglom, A. M. *Statistical fluid mechanics*: *Mechanics of turbulence*. MIT Press, Cambridge, Mass., 1975.

[30] Nikolaenko, A.; Brown, E.; Funfschilling, D.; Ahlers, G. Heat transport by turbulent Rayleigh-Bénard convection in cylindrical cells with aspect ratio one and less. *J. Fluid Mech*. **523** (2005), 251–260.

[31] Rabinowitz, P. H. Existence and nonuniqueness of rectangular solutions of the Bénard problem. *Arch. Rational Mech. Anal*. **29** (1968), 32–57.

[32] Siggia, E. D. High Rayleigh number convection. *Annual review of fluid mechanics*, vol. 26, 137–168. Annual Reviews, Palo Alto, Calif., 1994.

[33] Temam, R. M. *Infinite-dimensional dynamical systems in mechanics and physics*. 2nd ed. Applied Mathematical Sciences, 68. Springer, New York, 1997.

[34] Temam, R. M. *Navier-Stokes equations. Theory and numerical analysis*. Reprint of the 1984 edition. AMS Chelsea, Providence, R.I., 2001.

[35] Vishik, M. I.; Fursikov, A. V. *Mathematical problems of statistical hydromechanics*. Mathematics and Its Applications. Kluwer, Dordrecht, The Netherlands, 1988.

[36] Walters, P. *An introduction to ergodic theory*. Graduate Texts in Mathematics, 79. Springer, New York–Berlin, 2000.

[37] Wang, X. Infinite Prandtl number limit of Rayleigh-Bénard convection. *Comm. Pure Appl. Math*. **57** (2004), no. 10, 1265–1282.

[38] Wang, X. Large Prandtl number behavior of the Boussinesq system of Rayleigh-Bénard convection. *Appl. Math. Lett*. **17** (2004), no. 7, 821–825.

[39] Wang, X. A note on long time behavior of solutions to the Boussinesq system at large Prandtl number. *Nonlinear partial differential equations and related analysis*, 315–323. Contemporary Mathematics, 371. American Mathematical Society, Providence, R.I., 2005.

[40] Wang, X. Asymptotic behavior of global attractors to the Boussinesq system for Rayleigh-Bénard convection at large Prandtl number. *Comm. Pure Appl. Math*. **60** (2007), no. 9, 1293–1318.

[41] Wang, X. Bound on vertical heat transport at large Prandtl number. Submitted to *Phys. D*.

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