Long-time behavior of the Hele-Shaw-Cahn-Hilliard system

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Abstract

We study the long time behavior of the Hele-Shaw-Cahn-Hilliard system (HSCH) which models two phase incompressible Darcian flow in porous media with matched density but arbitrary viscosity contrast. We demonstrate that the ω-limit set of each trajectory is a single stationary solution of the system via Lojasiewicz-Simon type technique. Moreover, a rate of convergence has been established. Eventual regularity of weak solution, as well as existence of global classical solutions if the initial data is close to an energy minimizer or the Péclet number is sufficiently small are also proved in 3D .

Keywords:
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1 Introduction

Multiphase fluid flow in porous media is of great importance in many areas of science and engineering applications. Well-known examples include groundwater study (water table, interface between air and water in soil), oil recovery in petroleum engineering (oil and water) [5]. There are also emerging applications in material science (Hele-Shaw cell), fuel cell technology (water management in PEM fuel cells), as well as biomedical science (tumor growth modeled as flow in porous media).

There are two types of approaches to multi-phase flow. The first treat the interface as a sharp one with zero width. The second one recognizes the micro-scale mixing and hence treat the interface as a transition layer with finite (small) width (the so-called diffuse interface model or phase field model) [4]. In this manuscript, we consider the long time dynamics of the following Hele-Shaw-Cahn-Hilliard system (HSCH) which is a diffuse interface model for two phase incompressible flow in porous media [7, 14]

$$u = -\frac{1}{12\eta(c)}\left(\nabla p - \frac{1}{\mathbf{M}}\mu\nabla c\right), \tag{1.1}$$

$$\nabla \cdot u = 0, \tag{1.2}$$

$$c_t + u \cdot \nabla c = \frac{1}{\mathbf{Pe}}\Delta\mu. \tag{1.3}$$

We will assume that the fluid occupies the two or three dimensional torus $\mathbb{T}^d; d = 2, 3$ for simplicity. System (1.1)–(1.3) is subject to the initial condition

$$c(t, x)|_{t=0} = c_0(x). \tag{1.4}$$ 4

Here u is the velocity of the fluid mixture, c is the order parameter which is related to the concentration of the fluid (the volume fraction of the first fluid is given by $\frac{1+c}{2}$). The chemical potential μ depends on the order parameter c and is given by

$$\mu(c) = -\mathbf{C}\Delta c + f'(c). \tag{1.5}$$

The Helmholtz free energy $f(c)$ is given by the classical double well potential

$$f(c) = (c^2 - 1)^2.$$

p is not the physical pressure but the combination of certain generalized Gibbs free energy and the gravitational potential (see [14] for more details). $\mathbf{Pe}$ is the diffusion Péclet number, $\mathbf{C}$ is the Cahn number, and $\mathbf{M}$ is the

Mach number. Furthermore, $\eta(c)$ is the kinematic viscosity coefficient of the mixture of the two fluids satisfying

(A1) $\eta \in C^\infty(\mathbb{R}^1), \quad 0 < \underline{\eta} \le \eta \le \bar{\eta} < +\infty.$

The well-posedness of this HSCH system has been established recently (global in 2D and local in 3D) [22]. The main purpose of this manuscript is to investigate the long time behavior of the system (1.1)-(1.4) although some regularity issue in 3D will be also studied. The Hele-Shaw-Cahn-Hilliard system can be viewed as appropriate limit of the Navier-Stokes-Cahn-Hilliard system (NSCH) [14]. Mathematically speaking, the difficulty is about the same since we dropped the (bad) nonlinear advection term and the (good) viscous term (replaced by the Darcian term) simultaneously in the velocity equations for the NSCH system in order to derive the HSCH system. Similar results for a phase field model for the mixture of two incompressible fluids (Navier-Stokes-Cahn-Hilliard) were obtained recently [24] (see also [1,6,9] for some related results). Derivation of various versions of the Navier-Stokes-Cahn-Hilliard system can be found [10, 16, 17] among others. Convergence of solutions of the Cahn-Hilliard equation to stationary solutions in various settings is well-known (see for instance [3, 18, 23] among many others).

Now we state the main results of this paper:

main2d **Theorem 1.1.** *In the 2D case ($d = 2$), for any $c_0(x) \in H^s(\mathbb{T}^2)$ for $s > 2$, the system* (1.1)–(1.4) *admits a unique global solution (c, u) such that*

$$c \in C([0,+\infty); H^s) \cap L^2(0,+\infty; H^{s+2}), \quad u \in C([0,+\infty); H^{s-2}) \cap L^2(0,+\infty; H^s).$$

The global solution converges to a certain equilibrium $(0, c_\infty)$ as time goes to infinity with the following convergence rate

$$\|u(\cdot,t)\|_{H^{s-2}} + \|c(\cdot,t) - c_\infty\|_{H^s} \le C(1+t)^{-\theta/(1-2\theta)}, \quad \forall\, t \ge 1. \tag{1.6}$$ rate

Here $C \ge 0$ is a constant depending on $\|c_0\|_{H^2}, \|c_\infty\|_{H^{s+2}}$ and parameters **Pe**, $\mathbf{C}, \mathbf{M}, \bar{\eta}, \underline{\eta}$. *$\theta \in (0, \frac{1}{2})$ is a constant depending only on c_∞, which is a solution to the stationary Cahn-Hilliard equation:*

$$\begin{cases} -\mathbf{C}\Delta c_\infty + f'(c_\infty) = \mu_\infty, \quad x \in \mathbb{T}^2, \\ \int_{\mathbb{T}^2} c_\infty dx = \int_{\mathbb{T}^2} c_0 dx, \end{cases} \tag{1.7}$$ sta

where μ_∞ is a constant such that

$$\mu_\infty = \int_{\mathbb{T}^2} f'(c_\infty)dx.$$

main3d **Theorem 1.2.** *The following results hold in the 3D case.*

1. *Let $c_0(x) \in H^s(\mathbb{T}^2)$ for $s > \frac{5}{2}$. If the diffusion Péclet number* **Pe** *is sufficiently small (cf.* (4.11) *below), the system* (1.1)–(1.4) *admits a unique global solution (c, u) such that $c \in C([0, +\infty); H^s) \cap L^2(0, +\infty; H^{s+2})$, $u \in C([0, +\infty); H^{s-2}) \cap L^2(0, +\infty; H^s)$.*

2. *Let $c^* \in H^1(\mathbb{T}^3)$ be a local minimizer of $E(c)$ in the sense that there exists a $\delta > 0$ such that $E(c^*) \le E(c)$ for all $c \in H^1(\mathbb{T}^3)$ satisfying $\int_{\mathbb{T}^3} cdx = \int_{\mathbb{T}^3} c^*dx$ and $\|c - c^*\|_{H^1} < \delta$. Then there exists a constant $\sigma \in (0, 1]$ which may depend on c^*, δ and coefficients of the system such that for any $c_0 \in H^3$ satisfying $\int_{\mathbb{T}^3} c_0dx = \int_{\mathbb{T}^3} c^*dx$ and $\|c_0 - c^*\|_{H^3} \le 1$, and $\|c_0 - c^*\|_{H^2} \le \sigma$, the problem* (1.1)–(1.4) *must admit a unique global classical solution.*

3. *Let (c, u) be a weak solution to problem* (1.1)-(1.4) *on $[0, +\infty)$. Then there is some a $T^* > 0$ such that (c, u) is a classical solution for $t \ge T^*$.*

Moreover, the (global) classical solution in 3D enjoys the same long-time behavior as in 2D case.

The rest of the paper is organised as follows: We recall and prove a few preliminary results related to the well-posedness of the HSCH system in section 2. Section 3 is devoted to the long time behavior of classical solutions in 2D while section 4 is dedicated to the global well-posedness and long time behavior in 3D.

2 Preliminaries

We first recall the local well-posedness of the Hele-Shaw-Cahn-Hilliard system (1.1)-(1.4) (cf. [22, Theorem 3.1]).

loc **Proposition 2.1** (Local Wellposedness)**.** *Let* $c_0(x) \in H^s(\mathbb{T}^d)$ *for* $s > \frac{d}{2}+1$, $d = 2,3$. *Then there exists* $T > 0$ *such that the system* (1.1)–(1.4) *has a unique solution* (c, u) *in* $[0, T]$ *with*

$$c \in C([0,T]; H^s(\mathbb{T}^d)) \cap L^2(0,T; H^{s+2}(\mathbb{T}^d)), \quad u \in C([0,T]; H^{s-2}(\mathbb{T}^d)) \cap L^2(0,T; H^s(\mathbb{T}^d));$$

and satisfying the following energy estimate for $t \in [0, T]$*:*

$$\|c(t)\|_{H^s}^2 + \int_0^t \|c(\tau)\|_{H^{s+2}}^2 d\tau \le \|c_0\|_{H^s}^2 e^{\int_0^t G(\tau)d\tau}, \tag{2.1}$$

where

$$G(t) = \mathcal{F}(\|c\|_{L^\infty})(1+\|\nabla c\|_{L^\infty})^2(\|\nabla c\|_{L^\infty} + \|c\|_{H^3}^{\frac{d-2}{2}})^2(1+\|c\|_{H^2})^{2[2s]+2}, \tag{2.2}$$

and $\mathcal{F}$ *is an increasing function on* $\mathbb{R}^+$ *whose exact form depends on* s*.*

We have the following blow-up criterion of Beale-Kato-Majda type for system (1.1)–(1.4):

pBKM **Proposition 2.2.** *Let* $c_0(x) \in H^s(\mathbb{T}^d)$ *for* $s > \frac{d}{2}+1$*, and* (c, u) *be a solution of* (1.1)–(1.4) *stated in Theorem 2.1. Let* T^* *be the maximal existence time of the solution. If* $T^* < +\infty$*, then*

$$\int_0^{T^*} \|\nabla c(t)\|_{L^\infty}^4 dt = +\infty. \tag{2.3}$$ BKM

A typical property of the solutions to Cahn-Hilliard equation is the so-called *mass conservation.* Integrating (1.3) over $\mathbb{T}^d$ and using (1.2), we have

$$\frac{d}{dt}\int_{\mathbb{T}^d} c(t,\cdot)dx = 0, \tag{2.4}$$

which implies that

$$\int_{\mathbb{T}^d} c(t,\cdot)dx = \int_{\mathbb{T}^d} c_0(\cdot)dx, \quad \forall t \ge 0. \tag{2.5}$$

Another important property of system (1.1)-(1.4) is the following *basic energy law* (cf. [14, 22]). Let

$$E(c(t)) := \frac{\mathbf{C}}{2}\|\nabla c(t,\cdot)\|^2 + \int_{\mathbb{T}^d} f(c(t,x))dx. \tag{2.6}$$ E

Then $E(c(t))$ satisfies the following equality

$$\frac{d}{dt}E(c(t)) + \frac{1}{\mathbf{Pe}}\|\nabla\mu\|^2 + 12\mathbf{M}\int_{\mathbb{T}^d}\eta(c)|u|^2dx = 0, \quad \forall\ t \geq 0. \tag{2.7}$$

bel

An easy consequence of this energy law is the following estimates.

le **Lemma 2.1.** *We have the following uniform estimates of solutions to system* (1.1)-(1.4) *for* $t \geq 0$*:*

$$\|c(t,\cdot)\|_{H^1} \leq C, \tag{2.8}$$

$$\int_t^{t+1}\|c(\tau,\cdot)\|_{H^3}^2 d\tau \leq C\mathbf{Pe}, \tag{2.9}$$

where C *is a constant only depending on* $\|c_0\|_{H^1}$ *and possibly on the parameters* $\mathbf{M}, \mathbf{C}$.

Proof. It follows from the basic energy law (2.7) that

$$E(c(t)) + \int_0^t \left(\frac{1}{\mathbf{Pe}}\|\nabla\mu(\tau,\cdot)\|^2 + 12\mathbf{M}\int_{\mathbb{T}^d}\eta(c)|u(\tau,x)|^2dx\right)d\tau = E(0) \leq C(\|c_0\|_{H^1}). \tag{2.10}$$

er1

The above inequality easily yields that (2.8) and the following estimates:

$$\int_0^{+\infty}\|\nabla\mu(\tau,\cdot)\|^2 d\tau \leq \mathbf{Pe}E(0), \tag{2.11}$$

$$\int_0^{+\infty}\|u(\tau,\cdot)\|^2 d\tau \leq \frac{1}{12\mathbf{M}\underline{\eta}}E(0). \tag{2.12}$$

Since (cf. [22, pp. 10])

$$\begin{aligned}
\|c\|_{H^3} &\leq C(\|\nabla\Delta c\| + \|c\|_{H^2}) \leq C(\|\nabla\mu\| + \|f''(c)\nabla c\| + \|c\|_{H^2}) \\
&\leq C(\|\nabla\mu\| + \|c^2\nabla c\| + \|c\|_{H^2}) \\
&\leq C\|\nabla\mu\| + C(1 + \|c\|_{H^1}^2)\|c\|_{H^2} \\
&\leq C\|\nabla\mu\| + \frac{1}{2}\|c\|_{H^3} + C\|c\|_{H^1},
\end{aligned} \tag{2.13}$$

we infer from (2.13), (2.11) and (2.8) that (2.9) holds. □

We now address the global in time existence of weak solutions with initial data from H^1.

Theorem 2.1. *For any $c_0 \in H^1(\mathbb{T}^d), d = 2, 3$, system (1.1)–(1.4) admits at least one global weak solution (c, u) such that for any $T > 0$*

$$c \in L^\infty(0,T;H^1(\mathbb{T}^d)) \cap L^2(0,T;H^3(\mathbb{T}^d)) \cap H^1(0,T;H^{-1}(\mathbb{T}^d)), \quad u \in L^2(0,T;L^2(\mathbb{T}^d)). \tag{2.14}$$

`reg`

Proof. As in [22], we define the projection P_n by

$$P_n f(x) = \sum_{|k| \le n} f_k e^{2\pi i k\cdot x}, \quad \frac{1}{(2\pi)^d}\int_{\mathbf{T}^d} f(x) e^{-2\pi i k\cdot x dx} = f_k.$$

Consider the approximate problem

$$u_n = -\frac{1}{12\eta(c_n)}\left(\nabla p_n - \frac{1}{\mathbf{M}}(P_n\mu(c_n))\nabla c_n\right), \tag{2.15}$$

$$\nabla \cdot u_n = 0, \tag{2.16}$$

$$\partial_t c_n + P_n(u_n \cdot \nabla c_n) = \frac{1}{\mathbf{Pe}}\Delta P_n \mu(c_n), \tag{2.17}$$

$$c_n(0) = P_n c_0. \tag{2.18}$$

In analogy to [22], the above problem admits a unique smooth solution c_n on certain time interval $[0, T_n]$. Multiplying (2.17) by $P_n\mu(c_n)$, we can see that the approximate solution also satisfies the basic energy law

$$\frac{d}{dt}\left(\frac{\mathbf{C}}{2}\|\nabla c_n\|^2 + \int_{\mathbb{T}^d} f(c_n)dx\right) + \frac{1}{\mathbf{Pe}}\|\nabla P_n\mu(c_n)\|^2 + 12\mathbf{M}\int_{\mathbb{T}^d}\eta(c_n)|u_n|^2 dx = 0. \tag{2.19}$$

Integrating from 0 to T, we can see that c_n is uniformly bounded in $L^\infty(0,T;H^1)$, $\nabla P_n\mu(c_n)$ is uniformly bounded in $L^2(0,T;L^2)$, and by (A1), u_n is uniformly bounded in $L^2(0,T;L^2)$. Besides, since $|\int_{\mathbb{T}^d} P_n\mu(c_n)dx| = |\int_{\mathbb{T}^d} P_n f'(c_n)dx| \le C(\|c_n\| + \|c_n\|^3_{L^6})$, where C is independent of n, we know that $P_n\mu(c_n)$ is uniformly bounded in $L^2(0,T;H^1)$. This implies that c_n is uniformly bounded in $L^2(0,T;H^3)$. Then, by (2.17) we can obtain that $\partial_t c_n$ is uniformly bounded in $L^2(0,T;H^{-1})$. Summing up, using the well-known Aubin-Lions type compactness theorems, we can find a pair (c, u) satisfying (2.14) such that, up to subsequences,

$$u_n \to u, \quad \text{weakly in } L^2(0,T;L^2);$$

$$
\begin{aligned}
&c_n \to c, \quad \text{weakly-* in } L^\infty(0,T;H^1) \text{ and weakly in } L^2(0,T;H^3);\\
&c_n \to c, \quad \text{strongly in } C([0,T];H^{1-\varepsilon}) \text{ and } L^2(0,T;H^{3-\varepsilon});\\
&\partial_t c_n \to c_t, \quad \text{weakly in } L^2(0,T;H^{-1}).
\end{aligned}
$$

Consequently, we can pass to the limit in (2.15)-(2.17) that (c,u) solves (1.1)-(1.3) in the distributional sense. □

Before ending this preliminary section, we recall some useful lemma in the literature which will be used in our later proofs.

KP **Lemma 2.2.** (cf. [13]) *For $s \geq 0$, there holds*

$$
\|fg\|_{H^s} \leq C(\|f\|_{L^\infty}\|g\|_{H^s} + \|f\|_{H^s}\|g\|_{L^\infty}).
$$

WZle1 **Lemma 2.3.** (cf. [22, Lemma 6.2]) *For $s \geq 0$ and $\sigma \in (0, \frac{d}{2}]$, there holds*

$$
\|fg\|_{H^s} \leq C\left(\|f\|_{H^s}\|g\|_{L^\infty} + \|f\|_{H^{\frac{d}{2}-\sigma}}\|g\|_{H^{s+\sigma}}\right).
$$

WZle2 **Lemma 2.4.** (cf. [22, Lemma 6.4]) *Denote the Fourier multiplier $\langle D\rangle^s$ that*

$$
\langle D\rangle^s f(x) = \sum_{k\in\mathbb{Z}^d} (1+|k|^2)^{\frac{s}{2}} e^{2\pi i k\cdot x}\widehat{f}(k).
$$

For $s > 0$, there holds

$$
\|\langle D\rangle^s(fg) - f\langle D\rangle^s g\| \leq C\left(\|f\|_{H^{s+2}}\|g\| + \|f\|_{H^2}\|g\|_{H^{s-\frac{1}{2}}}\right).
$$

3 Long-time behavior of global solutions in 2D

The goal of this section is to demonstrate the convergence to stationary solution of the Cahn-Hilliard equation in the 2D case.

3.1 Uniform-in-time estimates

Based on Lemma 2.1 and Proposition 2.2, the global existence in $2D$ can be proved (cf. [22])

Proposition 3.1. *Let $c_0(x) \in H^s(\mathbb{T}^2)$ for $s > 2$, the unique local solution (c,u) for system (1.1)–(1.4) obtained in Theorem 2.1 is global.*

In what follows, we proceed to obtain some uniform-in-time estimates of solution (c, u), which is crucial in order to study the long-time behavior of global solutions to system (1.1)–(1.4). The estimate of the modified pressure p plays an important role in subsequent proof.

pes **Lemma 3.1.** cf. [22, Proposition 2.1] *Let $s \geq 0$. Assume that $c \in H^{s+2}(\mathbb{T}^d)$, and p is a smooth solution of the elliptic problem*

$$div\left(\frac{1}{\eta(c)}\nabla p\right) = div\left(\frac{1}{\mathbf{M}\eta(c)}\mu(c)\nabla c\right). \tag{3.1}$$ peq

If $s \in (\frac{k-2}{2}, \frac{k}{2}]$ for some $k \in \mathbb{N}$, then the solution p satisfies

$$\|\nabla p\|_{H^s} \leq \mathcal{F}(\|c\|_{L^\infty})(1 + \|\nabla c\|_{L^\infty})(1 + \|c\|_{H^2})^k \|c\|_{H^{s+2}}. \tag{3.2}$$ nphs

Here, $\mathcal{F}$ is an increasing function on $\mathbb{R}^+$ whose exact form depends on s. In particular, when $s = 0$, we have

$$\|\nabla p\| \leq C(\|\Delta c\| + \|c\|_{L^6}^3 + \|c\|)\|\nabla c\|_{L^\infty}. \tag{3.3}$$ npl2

ch22d **Lemma 3.2.** *In the 2D case ($d = 2$) the following estimates hold for the global solutions to system* (1.1)–(1.4) *for all $t \geq 0$:*

$$\|c(t, \cdot)\|_{H^2} \leq C, \tag{3.4}$$

$$\int_t^{t+1} \|c(\tau, \cdot)\|_{H^4}^2 d\tau \leq C, \tag{3.5}$$

where C is a constant only depending on $\|c_0\|_{H^2}$ and possibly on parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}$. *If $t \geq 1$, the constant C can be chosen so that it is a function of $\|c_0\|_{H^1}$ instead of $\|c_0\|_{H^2}$.*

Proof. A direct computation yields that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\Delta c\|^2 + \frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta^2 c\|^2 &= -(u \cdot \nabla c, \Delta^2 c) + \frac{1}{\mathbf{Pe}}(\Delta f'(c), \Delta^2 c) \\
&\leq \|u\|\|\nabla c\|_{L^\infty}\|\Delta^2 c\| + \frac{1}{\mathbf{Pe}}\|\Delta f'(c)\|\|\Delta^2 c\| \\
&:= I_1 + I_2. \qquad (3.6)
\end{aligned}$$

Using the Agmon inequality and (2.8), we have

$$\|c\|_{L^\infty} \leq C\|c\|_{H^2}^{\frac{1}{2}}\|c\|^{\frac{1}{2}} \leq C(1 + \|\Delta c\|^{\frac{1}{2}}), \tag{3.7}$$

$$\|\nabla c\|_{L^\infty} \leq C\|\nabla c\|_{H^2}^{\frac{1}{2}}\|\nabla c\|^{\frac{1}{2}} \leq C(1+\|\Delta c\|^{\frac{1}{2}}+\|\nabla\Delta c\|^{\frac{1}{2}}). \quad (3.8)$$

Keeping the above estimates in mind, we infer from (3.3) that

$$\begin{aligned}\|u\| &\leq C(\|\nabla p\|+\|\mu(c)\nabla c\|) \leq C(\|\mu\|+\|\Delta c\|+\|c\|_{L^6}^3+\|c\|)\|\nabla c\|_{L^\infty}\\ &\leq C(1+\|\Delta c\|)\|\nabla c\|_{L^\infty}, \quad (3.9)\end{aligned}$$

$$\|\Delta f'(c)\| \leq C(1+\|c\|_{L^\infty}^2)\|c\|_{H^2}+C(1+\|c\|_{L^6})\|\nabla c\|_{L^6}^2 \leq C(1+\|\Delta c\|^2). \quad (3.10)$$

Def2

As a consequence, we get

$$\begin{aligned}I_1 &\leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2+C\|u\|^2\|\nabla c\|_{L^\infty}^2\\ &\leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2+C(1+\|\Delta c\|^2)(1+\|\Delta c\|^{\frac{1}{2}}+\|\nabla\Delta c\|^{\frac{1}{2}})^4\\ &\leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2+C(1+\|\Delta c\|^2+\|\nabla\Delta c\|^2)\|\Delta c\|^2+C(1+\|\nabla\Delta c\|^2),\end{aligned}$$

$$I_2 \leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2+C\|\Delta f'(c)\|^2 \leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2+C(1+\|\Delta c\|^4), \quad (3.11)$$

which implies that

$$\frac{d}{dt}\|\Delta c\|^2+\frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta^2 c\|^2 \leq C(1+\|\Delta c\|^2+\|\nabla\Delta c\|^2)\|\Delta c\|^2+C(1+\|\nabla\Delta c\|^2). \quad (3.12)$$

dDc

Using the a priori estimate (2.9), for $t\in[0,1]$, we have

$$\begin{aligned}\|\Delta c(t)\|^2 &\leq e^{C\int_0^t(1+\|\Delta c(\tau)\|^2+\|\nabla\Delta c(\tau)\|^2)d\tau}\left(\|\Delta c_0\|^2+C\int_0^t(1+\|\nabla\Delta c(\tau)\|^2)d\tau\right)\\ &\leq C, \quad t\in[0,1].\end{aligned}$$

Besides, by (2.9), we can apply the uniform Gronwall inequality that for any $t\geq 0$,

$$\|\Delta c(t+1)\| \leq C, \quad t\geq 0.$$

Combining the above estimates, we arrive at (3.4). Then we integrate (3.12) with respect to time from t to $t+1$ ($t\geq 0$), it follows from (2.9) and (3.4) that

$$\int_t^{t+1}\|\Delta^2 c(\tau)\|^2 d\tau \leq C, \quad \forall t\geq 0,$$

which implies (3.5). The proof is complete. □

The above result can be strengthened to higher order Sobolev spaces in a straightforward fashion. Indeed, we have

chs2d **Proposition 3.2.** *In the 2D case ($d = 2$), for any $s \in (2k, 2k+2]$ ($k \in \mathbb{N}$), the following estimates hold for the global solutions to system* (1.1)–(1.4)*:*

$$\|c(t,\cdot)\|_{H^s} \le C, \quad \forall t \ge k, \tag{3.13}$$

$$\int_t^{t+1} \|c(\tau,\cdot)\|_{H^{s+2}}^2 d\tau \le C, \quad \forall t \ge k, \tag{3.14}$$

where C is a constant only depending on $\|c_0\|_{H^2}$ and possibly on the parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \underline{\eta}$. If $c_0 \in H^s$, then the above estimates hold for $t \ge 0$ with constant C depending on $\|c_0\|_{H^s}$ instead of $\|c_0\|_{H^2}$.

Proof. The following higher-order differential inequality can be obtained by using the pressure estimate Lemma 3.1 and commutator estimates (cf. [13] and [22, Appendix]). We refer to [22] for the details where the calculation was done for approximate solutions.

$$\frac{d}{dt}\|c\|_{H^s}^2 + \|c\|_{H^{s+2}}^2 \le G(t)\|c\|_{H^s}^2, \tag{3.15}$$ highchs

where

$$G(t) = \mathcal{F}(\|c\|_{L^\infty})(1 + \|\nabla c\|_{L^\infty})^2 \|\nabla c\|_{L^\infty}^2 (1 + \|c\|_{H^2})^{2[2s]+2},$$

and $\mathcal{F}$ is a certain increasing function on $\mathbb{R}^+$. It follows from (3.7), (3.8), the uniform estimates (3.4) and (2.9) (cf. Lemma 3.2) that

$$\int_t^{t+1} G(\tau) d\tau \le C(\|c\|_{H^2}) \int_t^{t+1} (1 + \|c(\tau)\|_{H^3}^2) d\tau \le C, \quad \forall t \ge 0,$$

where C is a constant only depending on $\|c_0\|_{H^2}$, and possibly on parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \underline{\eta}$. Next, we prove our conclusion by an easy iteration.

(i) $k = 1$. For any $s \in (2, 4]$, we infer from (3.5) that

$$\int_t^{t+1} \|c(\tau,\cdot)\|_{H^s}^2 d\tau \le \int_t^{t+1} \|c(\tau,\cdot)\|_{H^4}^2 d\tau \le C, \quad \forall t \ge k-1 = 0. \tag{3.16}$$ ichsa

By (3.15), (3.16) and (3.16), we are able to apply the uniform Gronwall inequality that (3.13) and (3.14) hold for $s \in (2, 4]$.

(ii) Suppose that (3.13) and (3.14) hold for $s = 2m+2$ with $k = m \in \mathbb{N}$. Then for $s \in (2m+2, 2m+4]$ with $k = m+1$, we have

$$\int_t^{t+1} \|c(\tau,\cdot)\|_{H^s}^2 d\tau \le \int_t^{t+1} \|c(\tau,\cdot)\|_{H^{2m+4}}^2 d\tau \le C, \quad t \ge m.$$

Using the uniform Gronwall inequality again, we have (3.13) and (3.14) hold for $s \in (2m+2, 2m+4]$ with $k = m+1$.

To complete the result to $t \ge 0$, we just notice that for any $s \in (2k, 2k+2]$, applying the standard Gronwall inequality to (3.15) and using the fact (3.16), we get

$$\|c(t)\|_{H^s}^2 + \int_0^t \|c(\tau)\|_{H^{s+2}}^2 d\tau \le \|c_0\|_{H^s}^2 e^{\int_0^t G(\tau)d\tau} \le \|c_0\|_{H^s}^2 e^{\int_0^k G(\tau)d\tau}, \quad \forall\, t \in [0,k).$$

□

hsrem **Remark 3.1.** *By a minor modification in the above proof, we can obtain the following result: for any $s > 2$ and arbitrary $\delta > 0$, the following estimates hold:*

$$\|c(t,\cdot)\|_{H^s} \le C, \quad \forall t \ge \delta, \tag{3.17}$$

$$\int_t^{t+1} \|c(\tau,\cdot)\|_{H^{s+2}}^2 d\tau \le C, \quad \forall t \ge \delta, \tag{3.18}$$

where C is a constant depending on $\|c_0\|_{H^2}$, δ and the parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \underline{\eta}$.

3.2 Decay of energy dissipation

conmuP **Proposition 3.3.**

$$\lim_{t\to+\infty} \|\nabla\mu(t)\| = 0. \tag{3.19}$$ conmu

Proof. Using integration by parts and equation (1.3), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 = -(\mu_t, \Delta\mu) \\
= \;& \mathbf{C}\left(\Delta\left(-u\cdot\nabla c + \frac{1}{\mathbf{Pe}}\Delta\mu\right), \Delta\mu\right) - \left(f''(c)\left(-u\cdot\nabla c + \frac{1}{\mathbf{Pe}}\Delta\mu\right), \Delta\mu\right) \\
= \;& -\frac{\mathbf{C}}{\mathbf{Pe}}\|\nabla\Delta\mu\|^2 + \mathbf{C}(\nabla(u\cdot\nabla c), \nabla\Delta\mu) + (f''(c)u\cdot\nabla c, \Delta\mu) - \frac{1}{\mathbf{Pe}}(f''(c)\Delta\mu, \Delta\mu)
\end{aligned}$$

$$:= \quad -\frac{\mathbf{C}}{\mathbf{Pe}}\|\nabla\Delta\mu\|^2 + I_3 + I_4 + I_5. \tag{3.20}$$

Note that

$$\begin{aligned} I_3 &\le C\|\nabla u\|\|\nabla c\|_{L^\infty}\|\nabla\Delta\mu\| + C\|u\|\|\Delta c\|_{L^\infty}\|\nabla\Delta\mu\|, \\ I_4 &\le C(1+\|c\|_{L^\infty}^2)\|u\|\|\nabla c\|_{L^\infty}\|\Delta\mu\|, \\ I_5 &\le C(1+\|c\|_{L^\infty}^2)\|\Delta\mu\|^2. \end{aligned}$$

According to Proposition 3.2, we have the uniform estimate that $\|c(t)\|_{H^5} \le C$ for $t \ge 2$. Then $I_3, ..., I_5$ are uniformly bounded and as a result,

$$\frac{d}{dt}\|\nabla\mu\|^2 \le C, \quad \forall t \ge 2,$$

where C is a constant only depending on $\|c_0\|_{H^2}$ and possibly on parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \underline{\eta}$. On the other hand, (2.11) implies that $\|\nabla\mu\|^2 \in L^1(0,+\infty)$. Our conclusion follows immediately. □

In order to study the decay property of the velocity field u, we first derive an estimate of ∇p_t.

pets **Lemma 3.3.** *Suppose that $c \in H^6(\mathbb{T}^2)$, we have the following estimate:*

$$\|\nabla p_t(t)\| \le C\|c\|_{H^6}(1+\|c\|_{H^4}^2) + C(1+\|c\|_{H^4}^5), \quad \forall\, t \ge 0, \tag{3.21}$$ pte1

where C is a constant depending on $\|c_0\|_{H^2}$, and possibly on the parameters $\mathbf{Pe}$, $\mathbf{M}$, $\mathbf{C}$, $\underline{\eta}$.

Proof. Since for the solution c to system (1.1)-(1.4), we are able to obtain its uniform H^2-norm estimate for all time (cf. Lemma 3.2), in what follows, we will absorb $\|c\|_{H^2}$ into the generic constant C for the sake of simplicity.

Differentiating (3.1) with respect to t, we can see that p_t satisfies the following elliptic problem

$$\begin{aligned} div\left(\frac{1}{\eta(c)}\nabla p_t\right) &= div\left[\frac{\eta'(c)}{\eta(c)^2}\left(\nabla p - \frac{1}{\mathbf{M}}\mu(c)\nabla c\right)c_t\right] - \frac{\mathbf{C}}{\mathbf{M}}div\left(\frac{1}{\eta(c)}\Delta c_t\nabla c\right) \\ &\quad + \frac{1}{\mathbf{M}}div\left(\frac{1}{\eta(c)}f''(c)c_t\nabla c\right) + \frac{1}{\mathbf{M}}div\left(\frac{1}{\eta(c)}\mu(c)\nabla c_t\right) \end{aligned} \tag{3.22}$$

Then we have

$$\begin{aligned}
\|\nabla p_t\| &\le \left\|\frac{\eta'(c)}{\eta(c)}\left(\nabla p - \frac{1}{\mathbf{M}}\mu(c)\nabla c\right)c_t\right\| + \frac{\mathbf{C}}{\mathbf{M}}\|\Delta c_t \nabla c\| \\
&\quad + \frac{1}{\mathbf{M}}\left\|f''(c)c_t\nabla c\right\| + \frac{1}{\mathbf{M}}\|\mu(c)\nabla c_t\| \\
&:= P_1 + P_2 + P_3 + P_4. \qquad (3.23)
\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}
&\sum_{i=1}^{4} P_i \\
\le\ & \left\|\frac{\eta'(c)}{\eta(c)}\right\|_{L^\infty}\left(\|\nabla p\|_{L^\infty} + \frac{1}{\mathbf{M}}\|\mu(c)\nabla c\|_{L^\infty}\right)\|c_t\| + \frac{\mathbf{C}}{\mathbf{M}}\|\Delta c_t\|\|\nabla c\|_{L^\infty} \\
& + \frac{1}{\mathbf{M}}\|f''(c)\|_{L^\infty}\|c_t\|\|\nabla c\|_{L^\infty} + \frac{1}{\mathbf{M}}\|\mu(c)\|_{L^\infty}\|\nabla c_t\| \\
\le\ & C[1 + \|\nabla p\|_{L^\infty} + \|c\|_{L^\infty}^3 + (1 + \|\mu\| + \|c\|_{L^\infty}^2)\|\nabla c\|_{L^\infty} + \|\Delta c\|_{L^\infty}]\|c_t\|_{H^2} \\
\le\ & C(1 + \|\nabla p\|_{L^\infty} + \|\nabla c\|_{L^\infty} + \|\Delta c\|_{L^\infty})\|c_t\|_{H^2} \\
\le\ & C(1 + \|\nabla p\|_{H^2} + \|c\|_{H^4})\|c_t\|_{H^2}. \qquad (3.24)
\end{aligned}$$

By Lemma 3.1, we have

$$\begin{aligned}
\|\nabla p\|_{H^2} &\le \mathcal{F}(\|c\|_{L^\infty})(1 + \|\nabla c\|_{L^\infty})(1 + \|c\|_{H^2})^4\|c\|_{H^4} \\
&\le C(1 + \|c\|_{H^3})\|c\|_{H^4}. \qquad (3.25)
\end{aligned}$$

It remains to estimate $\|c_t\|$, which involves the highest order derivative of c. First, we infer from equation (1.3) that

$$\|c_t\|_{H^2} \le C(\|\Delta\mu\|_{H^2} + \|u\cdot\nabla c\|_{H^2}).$$

By Lemma 2.2, we have

$$\begin{aligned}
\|\Delta\mu\|_{H^2} &\le C(\|\Delta^2 c\|_{H^2} + \|\Delta f'(c)\|_{H^2}) \le C\|c\|_{H^6} + C\|f'(c)\|_{H^4} \\
&\le C\|c\|_{H^6} + C(1 + \|c\|_{L^\infty}^2)\|c\|_{H^4} \\
&\le C(\|c\|_{H^6} + \|c\|_{H^4}). \qquad (3.26)
\end{aligned}$$

and

$$\|u\cdot\nabla c\|_{H^2} \le C(\|u\|_{H^2}\|\nabla c\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla c\|_{H^2}).$$

Using the equation (1.1), Lemma 2.2 and assumption (A1), we get

$$\begin{aligned}
\|u\|_{L^\infty} &\leq C\|u\|_{H^2} = C\left\|\frac{1}{12\eta(c)}\left(\nabla p - \frac{1}{\mathbf{M}}\mu\nabla c\right)\right\|_{H^2} \\
&\leq C\left\|\frac{1}{\eta(c)}\right\|_{L^\infty}(\|\nabla p\|_{H^2} + \|\mu\nabla c\|_{H^2}) + \left\|\frac{1}{\eta(c)}\right\|_{H^2}(\|\nabla p\|_{L^\infty} + \|\mu\nabla c\|_{L^\infty}) \\
&\leq C(\|\nabla p\|_{H^2} + \|\mu\|_{H^2}\|\nabla c\|_{L^\infty} + \|\mu\|_{L^\infty}\|\nabla c\|_{H^2}) \\
&\quad + C(1 + \|c\|_{H^2} + \|\nabla c\|_{L^4}^2)(\|\nabla p\|_{L^\infty} + \|\mu\|_{L^\infty}\|\nabla c\|_{L^\infty}) \\
&\leq C\|\nabla p\|_{H^2} + C(1 + \|c\|_{H^4})\|c\|_{H^3}.
\end{aligned}$$

As a consequence, we obtain that

$$\|u\cdot\nabla c\|_{H^2} \leq C\|c\|_{H^3}\|u\|_{H^2} \leq C\|c\|_{H^3}\|\nabla p\|_{H^2} + C(1 + \|c\|_{H^4})\|c\|_{H^3}^2,$$

which together with (3.26) yields

$$\|c_t\|_{H^2} \leq C(\|c\|_{H^6} + \|c\|_{H^3}\|\nabla p\|_{H^2} + \|c\|_{H^3}^2\|c\|_{H^4} + \|c\|_{H^4}^2 + 1). \quad (3.27)$$ ecth2

Combining (3.24), (3.25) and (3.27), we conclude that

$$\begin{aligned}
\|\nabla p_t\| &\leq C(1 + \|\nabla p\|_{H^2} + \|c\|_{H^4})(\|c\|_{H^6} + \|c\|_{H^3}\|\nabla p\|_{H^2} + \|c\|_{H^3}^2\|c\|_{H^4} + \|c\|_{H^4}^2 + 1) \\
&\leq C(1 + \|c\|_{H^4}^2)(\|c\|_{H^6} + \|c\|_{H^3}^2\|c\|_{H^4} + \|c\|_{H^4}^2 + 1) \\
&\leq C\|c\|_{H^6}(1 + \|c\|_{H^4}^2) + C(1 + \|c\|_{H^4}^5).
\end{aligned}$$

The proof is complete. □

conup **Proposition 3.4.**

$$\lim_{t\to+\infty}\|u(t)\| = 0. \quad (3.28)$$ conu

Proof. A direct calculation yields

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 = (u_t, u) \leq \|u_t\|\|u\|. \quad (3.29)$$

(3.9) implies that

$$\|u\| \leq C(1 + \|u\|_{H^2})\|u\|_{H^3}. \quad (3.30)$$ eu

We infer from the equation (1.1) that

$$u_t = \frac{\eta(c)'}{12\eta^2(c)}\left(\nabla p - \frac{1}{\mathbf{M}}\mu\nabla c\right)c_t - \frac{1}{12\eta(c)}\nabla p_t + \frac{1}{12\mathbf{M}\eta(c)}\mu(c)\nabla c.$$

Then it follows from Lemma 3.3 and (3.23), (3.24) that

$$\begin{aligned}\|u_t\| &\leq \frac{1}{12}\left\|\frac{1}{\eta(c)}\right\|_{L^\infty}\left[\|\nabla p_t\| + \left\|\frac{\eta'(c)}{\eta(c)}\left(\nabla p - \frac{1}{\mathbf{M}}\mu(c)\nabla c\right)c_t\right\| + \frac{\mathbf{C}}{\mathbf{M}}\|\Delta c_t \nabla c\|\right.\\ &\quad \left. + \frac{1}{\mathbf{M}}\|f''(c)c_t\nabla c\| + \frac{1}{\mathbf{M}}\|\mu(c)\nabla c_t\|\right]\\ &\leq \frac{C}{\underline{\eta}}\left(\|\nabla p_t\| + \sum_{i=1}^{4} P_i\right)\\ &\leq C\|c\|_{H^6}(1+\|c\|_{H^4}^2) + C(1+\|c\|_{H^4}^5). \qquad (3.31)\end{aligned}$$

By Proposition 3.2 (with $s=6$), we have $\|c(t)\|_{H^6} \leq C,\ \forall t \geq 2$. As a result, we infer from (3.30) and (3.31) that $\frac{d}{dt}\|u\|^2 \leq C$ for all $t \geq 2$. On the other hand, (2.12) implies that $\|u\|^2 \in L^1(\mathbb{R}^+)$. Therefore, our conclusion (3.28) follows. □

Remark 3.2. *Propositions 3.4, 3.3 and (A1) implies that the energy dissipation of system* (1.1)-(1.4) *(cf.* (2.7)*)* $\mathcal{D}(t) := \frac{1}{\mathbf{Pe}}\|\nabla\mu(t,\cdot)\|^2 + 12\mathbf{M}\int_{\mathbb{T}^d}\eta(c(t,\cdot))|u(t,\cdot)|^2dx$ *decays to* 0 *as time goes to infinity. This is expected since the total energy* E *is bounded below.*

3.3 Convergence to equilibria

Here we show the convergence of each trajectory to a certain stationary solution. We first recall the definition of the ω-limit set.

The ω-limit set of $(c_0) \in H^s$ is defined as follows:

$$\begin{aligned}\omega(c_0) = \{&(c_\infty(x)) \mid \text{ there exists } \{t_n\} \nearrow \infty \text{ such that}\\ &c(t_n) \to c_\infty \text{ in } H^s, \text{ as } t_n \to +\infty\}.\end{aligned}$$

And we define the set of stationary points associated with c_0 as

$$\mathcal{S} = \left\{\phi \in H^s \mid -\mathbf{C}\Delta\phi + f'(\phi) = \int_{\mathbb{T}^2} f'(\phi)dx, \text{ a.e. in } \mathbb{T}^2,\ \int_{\mathbb{T}^2}\phi dx = \int_{\mathbb{T}^2} c_0 dx\right\}.$$

esci **Proposition 3.5.** *Any* $\phi \in \mathcal{S}$ *is a* C^∞ *function and its* H^s*-norms* $(s \geq 0)$ *are bounded by a constant depending on* $|\int_{\mathbb{T}^2} c_0 dx|$ *and* $\mathbf{C}$.

Proof. The proof follows from classical elliptic regularity theory and bootstrap argument. We first notice that

$$
\begin{aligned}
& \mathbf{C}\|\nabla\phi\|^2 + 4\int_{\mathbb{T}^2}(\phi^4-\phi^2)dx = 4\int_{\mathbb{T}^2}(\phi^3-\phi)dx\int_{\mathbb{T}^2}\phi dx \\
= \;& 4\left(\int_{\mathbb{T}^2}\phi^3 dx - \int_{\mathbb{T}^2} c_0 dx\right)\int_{\mathbb{T}^2} c_0 dx \le 4\left|\int_{\mathbb{T}^2}\phi^3 dx\int_{\mathbb{T}^2} c_0 dx\right|.
\end{aligned}
$$

Using Young's inequality, we have

$$
\begin{aligned}
& \mathbf{C}\|\nabla\phi\|^2 + 4\int_{\mathbb{T}^2}\phi^4 dx \le 4\int_{\mathbb{T}^2}\phi^2 dx + 4\left|\int_{\mathbb{T}^2}\phi^3 dx\right|\left|\int_{\mathbb{T}^2} c_0 dx\right| \\
\le \;& 2\int_{\mathbb{T}^2}\phi^4 dx + 4 + 27\left|\int_{\mathbb{T}^2} c_0 dx\right|^4,
\end{aligned}
$$

which implies that

$$
\|\phi\|^2_{H^1} \le C\left(\mathbf{C}, \left|\int_{\mathbb{T}^2} c_0 dx\right|\right).
$$

By elliptic estimate and Sobolev embedding, we get

$$
\begin{aligned}
\|\phi\|_{H^2} &\le C\left(\|\Delta\phi\| + \|\phi\|\right) \le C\left(\|f'(\phi)\| + \left|\int_{\mathbb{T}^2} f'(\phi)dx\right| + \|\phi\|\right) \\
&\le C(\|\phi\|^3_{L^6} + \|\phi\|_{L^3} + \|\phi\|) \le C.
\end{aligned}
$$

Then for $s > 0$, by a classical result in [21] and the embedding $H^2 \hookrightarrow L^\infty$, we have

$$
\begin{aligned}
\|\phi\|_{H^{s+2}} &\le C\left(\|f'(\phi)\|_{H^s} + \left|\int_{\mathbb{T}^2} f'(\phi)dx\right| + \|\phi\|\right) \\
&\le C(1+\|\phi\|_{L^\infty})^{\lfloor s\rfloor+1}\|\phi\|_{H^s} + C\|\phi\|_{L^3} + C\|\phi\| \\
&\le C\|\phi\|_{H^s} + C. \qquad (3.32)
\end{aligned}
$$

Using (3.32), we can prove our conclusion by a simple induction. □

It is then easy to check the following relationship between the ω-limit set and the set of associated stationary points.

lim **Proposition 3.6.** *The ω-limit set of c_0 is a nonempty compact subset in H^s. Besides, all asymptotic limiting points c_∞ of problem* (1.1)–(1.4) *belong to $\mathcal{S}$, i.e., $\omega(c_0) \subset \mathcal{S}$.*

Proof. Due to Proposition 3.2 and the compact embedding $H^{s+1} \hookrightarrow H^s$, $\omega(c_0)$ is nonempty and compact. There exists a function $c_\infty \in H^s$ and an increasing unbounded sequence $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{t_n \to +\infty} \|c(t_n) - c_\infty\|_{H^s} = 0. \tag{3.33}$$ KKK

We infer from (3.19) and the Poincaré inequality that

$$\lim_{t\to+\infty} \left\| -\mathbf{C}\Delta c(t) + f'(c(t)) - \int_{\mathbb{T}^2} f'(c(t))dx \right\| = 0.$$

This and (3.33) yield that

$$\begin{aligned} & \left\| -\mathbf{C}\Delta c_\infty + f'(c_\infty) - \int_{\mathbb{T}^2} f'(c_\infty))dx \right\| \\ \leq \quad & \mathbf{C}\|\Delta(c_\infty - c(t_n))\| + \|f'(c_\infty) - f'(c(t_n))\| + \left\| \int_{\mathbb{T}^2} (f'(c_\infty) - f'(c(t_n)))dx \right\| \\ & + \left\| -\mathbf{C}\Delta c(t_n) + f'(c(t_n)) - \int_{\mathbb{T}^2} f'(c(t_n)))dx \right\| \\ \to \quad & 0, \quad \text{as } t_n \to +\infty. \end{aligned}$$

Therefore, $c_\infty \in \mathcal{S}$. □

Next, we demonstrate that the ω-limit set of each trajectory consists of one single stationary point.

For this purpose, we notice that thanks to Proposition 3.6, there exists an equilibrium $c_\infty \in \omega(c_0)$ and an increasing unbounded sequence $\{t_n\}_{n=1}^\infty$ such that

$$\lim_{t_n \to +\infty} \|c(t_n) - c_\infty\|_{H^s} = 0. \tag{3.34}$$ KKKa

We see from the basic energy law (2.7) that $E(c(t))$ is non-negative and decreasing in time. Moreover, $E(c(t)) \geq E(c_\infty)$, for all $t > 0$. As a result, it has a finite limit as time goes to infinity. (3.34) implies that

$$\lim_{t_n\to+\infty} E(c(t_n)) = E(c_\infty).$$

It follows from (1.3) and uniform estimate Proposition 3.2 that

$$\|c_t\|_{H^{-1}} \quad \leq \quad C(\|(u\cdot\nabla)c\|_{H^{-1}} + \|\Delta\mu\|_{H^{-1}}) \leq C(\|u\|\|\nabla c\|_{L^3} + \|\nabla\mu\|)$$

$$\leq \quad C\left(\|u\|+\|\nabla\mu\|\right). \tag{3.35}$$

In what follows, we shall apply the well-known Łojasiewicz-Simon approach to prove the convergence of whole trajectory $c(t)$. The procedure is standard and we only sketch the proof here.

First, we introduce the following Łojasiewicz–Simon type inequality. Let $\mathrm{P} : L^2 \mapsto L^2_0$ be a projection operator such that for any $\phi \in L^2$, $\mathrm{P}\phi = \phi - \int_{\mathbb{T}^2} \phi dx$. We have (cf. e.g., [8])

LS **Lemma 3.4** (Łojasiewicz–Simon Inequality)**.** *Let $c_\infty \in \mathcal{D}$ be a critical point of E. Then there exist constants $\theta \in (0, \frac{1}{2})$ and $\beta > 0$ depending on c_∞ such that for any $c \in \mathcal{D}$ satisfying $\|c - c_\infty\|_{H^2} < \beta$, such that*

$$\|\mathrm{P}(-\mathbf{C}\Delta c + f'(c))\| \;\geq\; |E(c) - E(c_\infty)|^{1-\theta}. \tag{3.36}$$ ls

First we consider the trivial case. If there is a $t_1 \in \mathbb{R}^+$ such that $E(c(t_1)) = E(c_\infty)$, then $\|u(t)\| = \|\nabla\mu(t)\| = 0$ for all $t \geq t_1$ by virtue of (2.7). Together with (3.35), it implies that c is independent of time for all $t \geq t_1$. Notice (3.34), we conclude that

$$\lim_{t\to+\infty} \|c(t) - c_\infty\|_{H^s} = 0. \tag{3.37}$$ c-con

Therefore, we only need to consider the nontrivial case that $E(c(t)) > E(c_\infty)$ for all $t \geq 0$. Due to the continuity $c \in C([0, +\infty), H^2)$, by a classical contradiction argument first introduced in [12], we can show that there is a time $t_0 > 0$ such that for all $t \geq t_0$, $c(t)$ satisfies the condition of Lemma 3.4, i.e., $\|c(t) - c_\infty\|_{H^2} < \beta$. Then for the constant $\theta \in (0, \frac{1}{2})$ in Lemma 3.4, using Lemma 3.4 and (2.7), we calculate that

$$\begin{aligned} & -\frac{d}{dt}(E(c(t)) - E(c_\infty))^\theta = -\theta(E(c(t)) - E(c_\infty))^{\theta-1}\frac{d}{dt}E(c(t)) \\ \geq\; & \frac{C\theta(\|u\|^2 + \|\nabla\mu\|^2)}{\|\nabla\mu\|} \geq C(\|u\| + \|\nabla\mu\|), \quad \forall\, t \geq t_0. \end{aligned} \tag{3.38}$$

Integrating from t_0 to ∞, we get

$$\int_{t_0}^\infty (\|u(\tau)\| + \|\nabla\mu(\tau)\|)d\tau < +\infty,$$

which together with (3.35) yields

$$\int_{t_0}^{\infty} \|c_t(\tau)\|_{H^{-1}} d\tau < +\infty.$$

Thus, we can conclude that $c(t)$ converges in H^{-1} as $t \to +\infty$. This fact together with the compactness of c in H^s and (3.34) indicates that (3.37) holds.

3.4 Rate of Convergence

Next, we study the rate of convergence. By Proposition 3.2 and Remark 3.5, we know that for any $s \geq 0$, the H^s-norms of c and c_∞ are bounded for $t \geq 1$.

The H^{-1}-estimate for $c - c_\infty$ follows from the classical argument in [11]. By (3.38) and Lemma 3.4, we have

$$\frac{d}{dt}(E(c(t)) - E(c_\infty)) + C(E(c(t)) - E(c_\infty))^{2(1-\theta)} \leq 0, \;\; t \geq t_0,$$

which implies

$$E(c(t)) - E(c_\infty) \leq C(1+t)^{-1/(1-2\theta)}, \;\; t \geq t_0.$$

Integrating (3.38) from t to ∞, ($t \geq t_0$), and using (3.35), we obtain

$$\int_{t}^{\infty} \|c_t\|_{H^{-1}} dt \leq C(1+t)^{-\theta/(1-2\theta)}, \;\; t \geq t_0,$$

which implies

$$\|c - c_\infty\|_{H^{-1}} \leq C(1+t)^{-\theta/(1-2\theta)}, \;\; t \geq 0.$$

Denote

$$r(t) = c(t) - c_\infty.$$

Since $\mu_\infty = -\mathbf{C}\Delta c_\infty + f'(c_\infty)$ is a constant, we infer from the equation (1.3) that

$$r_t + u \cdot \nabla(r + c_\infty) = \frac{1}{\mathbf{Pe}}\Delta(-\mathbf{C}\Delta r + g(c, c_\infty)r). \qquad (3.39) \quad \boxed{\texttt{d}}$$

where by the expression of f,

$$g(c, c_\infty)r := f'(c) - f'(c_\infty) = 4(c^2 + cc_\infty + c_\infty^2 - 1)r.$$

Next, we try to show the same convergence rate for arbitrary H^s-norm with $s \geq 0$.

For this aim, we need the following estimate of pressure difference whose proof is based on the argument for Lemma 3.1 (cf. [22]):

Lemma 3.5. *Assume that $c, c_\infty \in H^{s+2}$ $(s \geq 2)$, we have*

$$\|\nabla(p - p_\infty)\|_{H^s} \leq C\|r\|_{H^{s+2}}, \tag{3.40}$$ nphsi

where C is a constant depending on $\|c\|_{H^{s+2}}$, $\|c_\infty\|_{H^{s+2}}$ and the parameters $\mathbf{Pe}$, $\mathbf{C}, \mathbf{M}, \bar{\eta}, \underline{\eta}$.

Proof. It follows from (A1) and $u_\infty = 0 = -\frac{1}{12\eta(c_\infty)}\left(\nabla p_\infty - \frac{1}{\mathbf{M}}\mu_\infty \nabla c_\infty\right)$ that

$$\nabla p_\infty - \frac{1}{\mathbf{M}}\mu_\infty \nabla c_\infty = 0.$$

Then we can rewrite equation (1.1) as follows

$$u = -\frac{1}{12\eta(c)}\left[\nabla(p - p_\infty) - \frac{1}{\mathbf{M}}((\mu - \mu_\infty)\nabla(r + c_\infty) + \mu_\infty \nabla r)\right]. \tag{3.41}$$ diu

Applying *div* operator to (3.41), we get

$$div\left(\frac{1}{\eta(c)}\nabla(p - p_\infty)\right) = \frac{1}{\mathbf{M}} div\left[\frac{1}{\eta(c)}((\mu - \mu_\infty)\nabla c + \mu_\infty \nabla r)\right]. \tag{3.42}$$ dip

It easily follows from the energy estimate that

$$\begin{aligned} \|\nabla(p - p_\infty)\| &\leq \frac{\bar{\eta}}{\underline{\eta}\mathbf{M}}(\|(\mu - \mu_\infty)\nabla(r + c_\infty)\| + \|\mu_\infty \nabla r\|) \\ &\leq C(\|\Delta r\| + \|g(c, c_\infty)r\| + \|\nabla r\|) \\ &\leq C\|r\|_{H^2}. \end{aligned} \tag{3.43}$$

For higher-order estimate, we apply $\langle D\rangle^s$ to (3.42) and write the result in the following equivalent form

$$div\left(\frac{1}{\eta(c)}\nabla\langle D\rangle^s(p - p_\infty)\right)$$

$$
\begin{aligned}
&= div\langle D\rangle^s\left[\frac{1}{\mathbf{M}\eta(c)}((\mu-\mu_\infty)\nabla c+\mu_\infty\nabla r)\right]\\
&\quad -div\left[\langle D\rangle^s\left(\frac{1}{\eta(c)}\nabla(p-p_\infty)\right)-\left(\frac{1}{\eta(c)}\nabla\langle D\rangle^s(p-p_\infty)\right)\right]\\
&:= div(A+B),
\end{aligned}
$$

which yields that

$$
\|\nabla(p-p_\infty)\|_{H^s}\le\bar{\eta}(\|A\|+\|B\|). \tag{3.44}
$$

pAB

Since $\mu-\mu_\infty=-\mathbf{C}\Delta r+g(c,c_\infty)r$, we have

$$
\left\|\frac{1}{\eta(c)}(\mu-\mu_\infty)\nabla c\right\|_{H^s}\le\left\|\frac{\mathbf{C}}{\eta(c)}\Delta r\nabla c\right\|_{H^s}+\left\|\frac{1}{\eta(c)}(g(c,c_\infty)r\nabla c)\right\|_{H^s}.
$$

Moreover, by Lemma 2.2, we obtain

$$
\begin{aligned}
\left\|\frac{\mathbf{C}}{\eta(c)}\Delta r\nabla c\right\|_{H^s} &\le C\left\|\frac{1}{\eta(c)}\nabla c\right\|_{L^\infty}\|\Delta r\|_{H^s}+C\left\|\frac{1}{\eta(c)}\nabla c\right\|_{H^s}\|\Delta r\|_{L^\infty}\\
&\le C\|r\|_{H^{s+2}},
\end{aligned}
$$

$$
\begin{aligned}
&\left\|\frac{1}{\eta(c)}(g(c,c_\infty)r\nabla c)\right\|_{H^s}\\
\le\ & C\left\|\frac{1}{\eta(c)}\nabla c\right\|_{L^\infty}\|g(c,c_\infty)r\|_{H^s}+C\left\|\frac{1}{\eta(c)}\nabla c\right\|_{H^s}\|g(c,c_\infty)r\|_{L^\infty}\\
\le\ & C(\|g(c,c_\infty)\|_{L^\infty}\|r\|_{H^s}+\|g(c,c_\infty)\|_{H^s}\|r\|_{L^\infty}+\|g(c,c_\infty)\|_{L^\infty}\|r\|_{L^\infty})\\
\le\ & C\|r\|_{H^{s+2}}.
\end{aligned}
$$

$$
\left\|\frac{1}{\mathbf{M}\eta(c)}\mu_\infty\nabla r\right\|_{H^s}\le C\left\|\frac{1}{\eta(c)}\right\|_{L^\infty}\|\nabla r\|_{H^s}+C\left\|\frac{1}{\eta(c)}\right\|_{H^s}\|\nabla r\|_{L^\infty}\le C\|r\|_{H^{s+2}}.
$$

Therefore,

$$
\|A\|\le C\|r\|_{H^{s+2}}. \tag{3.45}
$$

pA

By the commutator estimate Lemma 2.4 and interpolation inequalities, we have

$$
\begin{aligned}
\|B\| &\le C\left(\left\|\frac{1}{\eta(c)}\right\|_{H^{s+2}}\|\nabla(p-p_\infty)\|+\left\|\frac{1}{\eta(c)}\right\|_{H^2}\|\nabla(p-p_\infty)\|_{H^{s-\frac{1}{2}}}\right)\\
&\le C\left(\|\nabla(p-p_\infty)\|+\|\nabla(p-p_\infty)\|_{H^s}^{\frac{2s-1}{2s}}\|\|\nabla(p-p_\infty)\|^{\frac{1}{2s}}\right)
\end{aligned}
$$

$$\leq \frac{1}{2\bar{\eta}}\|\nabla(p-p_\infty)\|_{H^s} + C\|\nabla(p-p_\infty)\|. \quad (3.46)$$

Finally, it follows from (3.44), (3.45), (3.46) and (3.43) that (3.40) holds. The proof is complete. □

We have known that for any $s \geq 2$, the following uniform estimates hold (cf. Remark 3.1):

$$\|c(t,\cdot)\|_{H^s} \leq C, \quad \forall t \geq 1, \quad (3.47)$$
$$\|c_\infty\|_{H^s} \leq C,$$

where C is a constant depending on $\|c_0\|_{H^2}$, and the parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \underline{\eta}$.

Taking the H^s inner product of equation (3.39) with r, we obtain

$$\frac{1}{2}\frac{d}{dt}\|r\|_{H^s}^2 + \frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta r\|_{H^s}^2 = (\Delta g(c,c_\infty)r, r)_{H^s} - (u\cdot\nabla(r+c_\infty), r)_{H^s}. (3.48)$$

Using integration by parts and Lemma 2.2, we can see that

$$\begin{aligned} & |(\Delta g(c,c_\infty)r, r)_{H^s}| \leq \|g(c,c_\infty)r\|_{H^s}\|\Delta r\|_{H^s} \\ \leq\ & C(\|g(c,c_\infty)\|_{L^\infty}\|r\|_{H^s} + \|g(c,c_\infty)\|_{H^s}\|r\|_{L^\infty})\|\Delta r\|_{H^s} \\ \leq\ & \frac{\mathbf{C}}{2\mathbf{Pe}}\|\Delta r\|_{H^s}^2 + C\|r\|_{H^s}^2. \end{aligned} \quad (3.49)$$

$$\begin{aligned} & |(u\cdot\nabla(r+c_\infty), r)_{H^s}| \leq \|u\cdot\nabla c\|_{H^s}\|r\|_{H^s} \\ \leq\ & C(\|u\|_{H^s}\|\nabla c\|_{L^\infty} + \|u\|_{L^\infty}\|\nabla c\|_{H^s})\|r\|_{H^s} \\ \leq\ & C\|u\|_{H^s}\|r\|_{H^s}. \end{aligned} \quad (3.50)$$

It remains to estimate $\|u\|_{H^s}$. We infer from (3.41), (3.40) that

$$\begin{aligned} \|u\|_{H^s} \leq\ & C\left\|\frac{1}{\eta(c)}\nabla(p-p_\infty)\right\|_{H^s} + C\left\|\frac{1}{\eta(c)}(\mu-\mu_\infty)\nabla c\right\|_{H^s} + C\left\|\frac{1}{\eta(c)}\mu_\infty\nabla r\right\|_{H^s} \\ \leq\ & C(\|\nabla(p-p_\infty)\|_{L^\infty} + \|\nabla(p-p_\infty)\|_{H^s}) + C(\|\mu-\mu_\infty\|_{H^s} + \|\mu-\mu_\infty\|_{L^\infty}) \\ & + C|\mu_\infty|(\|\nabla r\|_{H^s} + \|\nabla r\|_{L^\infty}) \\ \leq\ & C(\|\nabla(p-p_\infty)\|_{H^s} + \|\mu-\mu_\infty\|_{H^s} + \|r\|_{H^{s+1}}) \\ \leq\ & C\|r\|_{H^{s+2}}. \end{aligned} \quad (3.51)$$

Since

$$\|r\|_{H^{s+2}} \le C(\|\Delta r\|_{H^s} + \|r\|_{H^s}),$$
$$\|r\|_{H^s} \le C\|r\|_{H^{s+2}}^{\frac{s+1}{s+3}}\|r\|_{H^{-1}}^{\frac{2}{s+3}} \le \epsilon\|r\|_{H^{s+2}} + C_\epsilon\|r\|_{H^{-1}},$$

then combining (3.48)-(3.51) and taking ϵ sufficiently small, we obtain

$$\frac{d}{dt}\|r\|_{H^s}^2 + C\|r\|_{H^{s+2}}^2 \le C\|r\|_{H^{-1}}^2, \quad t \ge 1.$$

This and (3.39) implies that

$$\frac{d}{dt}\|r\|_{H^s}^2 + C\|r\|_{H^s}^2 \le C(1+t)^{-2\theta/(1-2\theta)}, \tag{3.52}$$

which yields our conclusion (1.6) (cf. [23]).

4 Global wellposedness and long-time behavior in 3D

In this section we deal with the 3D case. We first show global in time well-posedness of the Hele-Shaw-Cahn-Hilliard system under the assumption that either (1) the Péclet number is small, or (2) the initial data is close to a local minimizer of the energy. We then deduce the eventual regularity of all trajectories as well as their convergence to stationary solutions.

uni3d **Lemma 4.1.** *For the 3D case ($d = 3$), if the diffusion Péclet number* **Pe** *is sufficiently small (cf. (4.11)), then we have the uniform estimate*

$$\|c(t)\|_{H^2} \le C, \quad \forall\ t \ge 0, \tag{4.1}$$ H23d

$$\int_t^{t+1} \|c(\tau)\|_{H^4}^2 d\tau \le C, \quad t \ge 0. \tag{4.2}$$ iH43d

Proof. Using the Agmon inequality and (2.8), we have

$$\|c\|_{L^\infty} \le C\|c\|_{H^2}^{\frac12}\|c\|_{H^1}^{\frac12} \le C(1+\|\Delta c\|^{\frac12}), \tag{4.3}$$
$$\|\nabla c\|_{L^\infty} \le C\|\nabla c\|_{H^2}^{\frac12}\|\nabla c\|_{H^1}^{\frac12} \le C(1+\|\Delta c\|+\|\nabla\Delta c\|)^{\frac12}(1+\|\Delta c\|)^{\frac12} \tag{4.4}$$

On the other hand, since $\|\Delta c\| = \|\nabla c\|^{\frac{1}{2}}\|\nabla\Delta c\|^{\frac{1}{2}}$, by Young's inequality and the uniform H^1 estimate, we have

$$\begin{aligned}
\|\nabla\Delta c\| &\leq \frac{1}{\mathbf{C}}(\|\nabla\mu\| + \|f''(c)\nabla c\|) \leq \frac{1}{\mathbf{C}}\|\nabla\mu\| + C(\|c\|_{L^\infty}^2 + 1)\|\nabla c\| \\
&\leq \frac{1}{\mathbf{C}}\|\nabla\mu\| + C(\|\Delta c\| + 1) \leq \frac{1}{\mathbf{C}}\|\nabla\mu\| + \frac{1}{2}\|\nabla\Delta c\| + C. \qquad (4.5)
\end{aligned}$$

Keeping the above estimates in mind, we infer from (3.3) that

$$\begin{aligned}
\|u\| &\leq C(\|\nabla p\| + \|\mu(c)\nabla c\|) \leq C(\|\mu\| + \|\Delta c\| + \|c\|_{L^6}^3 + \|c\|)\|\nabla c\|_{L^\infty} \\
&\leq C(1 + \|\Delta c\|)\|\nabla c\|_{L^\infty}, \qquad (4.6) \\
\|\Delta f'(c)\| &\leq C(1 + \|c\|_{L^\infty}^2)\|c\|_{H^2} + C(1 + \|c\|_{L^6})\|\nabla c\|_{L^6}^2 \leq C(1 + \|\Delta c\|^2), \qquad (4.7) \\
\|\Delta c\| &\leq C\|\nabla\mu\|^{\frac{1}{2}} + C. \qquad (4.8)
\end{aligned}$$

Assume the diffusion Péclet number $\mathbf{Pe}$ satisfy $0 < \mathbf{Pe} \leq 1$. We re-estimate the right-hand side of (3.6).

$$\begin{aligned}
I_1 &\leq C(1 + \|\Delta c\|)^2(1 + \|\Delta c\| + \|\nabla\Delta c\|)\|\Delta^2 c\| \\
&\leq C(1 + \|\Delta c\|^3 + \|\Delta c\|^{\frac{5}{2}}\|\Delta^2 c\|^{\frac{1}{2}} + \|\Delta c\|^{\frac{1}{2}}\|\Delta^2 c\|^{\frac{1}{2}})\|\Delta^2 c\| \\
&\leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2 + \frac{1}{4}\left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{\frac{1}{2}}\|\Delta c\|^2\|\Delta^2 c\|^2 + C\left[\left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{-\frac{1}{2}} + \left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{-\frac{3}{2}}\right]\|\Delta c\|^4 \\
&\quad + C\mathbf{Pe}(1 + \|\Delta c\|^2) \\
&\leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2 + \left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{\frac{1}{2}}\|\Delta c\|^2\|\Delta^2 c\|^2 + C(1 + \|\nabla\mu\|^2).
\end{aligned}$$

$$\begin{aligned}
I_2 &\leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2 + C\frac{\mathbf{Pe}}{\mathbf{C}}\|\Delta f'(c)\|^2 \leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2 + C\mathbf{Pe}(1 + \|\Delta c\|^4) \\
&\leq \frac{\mathbf{C}}{4\mathbf{Pe}}\|\Delta^2 c\|^2 + C(1 + \|\nabla\mu\|^2).
\end{aligned}$$

which implies that

$$\frac{d}{dt}\|\Delta c\|^2 + \left[\frac{\mathbf{C}}{\mathbf{Pe}} - \frac{1}{2}\left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{\frac{1}{2}}\|\Delta c\|^2\right]\|\Delta^2 c\|^2 \leq M_1(1 + \|\nabla\mu\|^2). \qquad (4.9)$$

dDc3

It follows from (2.11) that for any $t \geq 0$,

$$\int_t^{t+1}\|\nabla\mu(\tau)\|^2 d\tau \leq \mathbf{Pe}E(0) \leq E(0) := M_2.$$

By the Poincaré inequality, we have

$$\|\Delta c\| \leq \frac{1}{\mathbf{C}}(\|\mu\|+\|f'(c)\|) \leq C\|\nabla\mu\|+C\left|\int_{\mathbb{T}^3}\mu dx\right|+\frac{1}{\mathbf{C}}\|f'(c)\| \leq M_3\|\nabla\mu\|+M_4,$$

which implies

$$\int_t^{t+1}\|\Delta c(\tau)\|^2 d\tau \leq 2M_4^2+2M_3^2M_2 := M_5, \quad \forall t\geq 0. \tag{4.10}$$

inDc

We note that $M_1, ..., M_5$ are positive constants independent of $\mathbf{Pe}$. Finally, we assume that $\mathbf{Pe}$ satisfies the following relation

$$0<\mathbf{Pe}\leq \min\left\{1, \frac{\mathbf{C}}{\|\Delta c_0\|^4+M_1^2(1+M_2)^2+4M_5^2}\right\}. \tag{4.11}$$

Pe

Once can check that $\frac{\mathbf{C}}{\mathbf{Pe}}-\left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{\frac{1}{2}}\|\Delta c_0\|^2>0$, which implies there exists a $T_0>0$ such that for $t\in[0,T_0]$,

$$\frac{\mathbf{C}}{\mathbf{Pe}}-\left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{\frac{1}{2}}\|\Delta c(t)\|^2\geq 0. \tag{4.12}$$

ddb

As a result, on $[0,T_0]$, we infer from (4.9) that

$$\frac{d}{dt}\|\Delta c\|^2+\frac{1}{2}\frac{\mathbf{C}}{\mathbf{Pe}}\|\Delta^2 c\|^2\leq M_1(1+\|\nabla\mu\|^2). \tag{4.13}$$

dDc3a

Let $T=\sup T_0$. First, we show that $T\geq 1$. If this is not true, it follows from the above inequality that

$$\|\Delta c(T)\|^2\leq\|\Delta c_0\|^2+M_1\int_0^T(1+\|\nabla\mu(t)\|^2)dt\leq\|\Delta c_0\|^2+M_1(1+M_2).$$

Thus, we have

$$\frac{\mathbf{C}}{\mathbf{Pe}}-\left(\frac{\mathbf{C}}{\mathbf{Pe}}\right)^{\frac{1}{2}}\|\Delta c(T)\|^2>0, \tag{4.14}$$

gg

which contradicts the definition of T. Besides, if $T<+\infty$, then it follows from (4.10) that there exists $t^*\in[T-\frac{1}{2},T]$ such that $\|\Delta c(t^*)\|^2\leq 2M_5$. Then we have

$$\|\Delta c(T)\|^2\leq\|\Delta c(t^*)\|^2+M_1\int_{t^*}^T(1+\|\nabla\mu(t)\|^2)dt\leq 2M_5+M_1(1+M_2),$$

which again yields (4.14). Summing up, we can conclude that for all $t\geq 0$, (4.12) holds, namely, the H^2-norm of c is uniformly bounded in time. Then integrating (4.13) with respect to time we obtain (4.2). □

Once we obtained the uniform estimate of $\|c\|_{H^2}$, in analogy to Lemma 3.2, we are able to obtain uniform estimates on H^s-norms and prove the existence of global in time strong solution.

chs3d **Theorem 4.1.** *Let $d = 3$. For any $c_0(x) \in H^s(\mathbb{T}^d)$ for $s > \frac{5}{2}$, if the assumption of Lemma 4.1 are satisfied, system* (1.1)–(1.4) *admits a unique global solutions. For $s \in (2k, 2k+2]$ $(k \in \mathbb{N})$, the following estimates hold for :*

$$\|c(t,\cdot)\|_{H^s} \leq C, \quad \forall t \geq k, \tag{4.15}$$

$$\int_t^{t+1} \|c(\tau,\cdot)\|^2_{H^{s+2}} d\tau \leq C, \quad \forall t \geq k, \tag{4.16}$$

where C is a constant only depending on $\|c_0\|_{H^2}$ and possibly on the parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \underline{\eta}$. If $c_0 \in H^s$, then the above estimates hold for $t \geq 0$ with constant C depending on $\|c_0\|_{H^s}$ instead of $\|c_0\|_{H^2}$.

Proof. We remark that (3.15) still holds with a different G such that

$$G(t) = \mathcal{F}(\|c\|_{L^\infty})(1 + \|\nabla c\|_{L^\infty})^2 \left(\|\nabla c\|_{L^\infty} + \|c\|^{\frac{1}{2}}_{H^3}\right)^2 (1 + \|c\|_{H^2})^{2[2s]+2},$$

and $\mathcal{F}$ is a certain increasing function on $\mathbb{R}^+$. Then it follows from (4.3), (4.4), the uniform estimates (4.1) and (2.9) that for all $t \geq 0$, $\int_t^{t+1} G(\tau) d\tau \leq C(\|c\|_{H^2}) \int_t^{t+1} (1 + \|c(\tau)\|^2_{H^3}) d\tau \leq C$, where C is a constant only depending on $\|c_0\|_{H^2}$, and possibly on parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \underline{\eta}$. using Lemma 4.1, we can prove our conclusion by the same iteration argument as in Proposition 3.2. □

The next result is an alternative provided that the gradient of the chemical potential is bounded initially: either we have a global in time classical solution, or the energy can be decreased by a certain fixed amount along the trajectory. This result further lead to the eventual regularity result.

sm **Theorem 4.2.** *Suppose $d = 3$, $c_0 \in H^3$. For any $R \in (0, \infty)$, whenever $\|\nabla \mu(0)\|^2 \leq R$, there is a small constant $\varepsilon_0 \in (0,1)$ depending on R and coefficients of the system such that either (i) Problem* (1.1)–(1.4) *has a unique global classical solution (c, u), or (ii) there is a $T_* \in (0, +\infty)$ such that $E(c(T_*)) < E(c_0) - \varepsilon_0$.*

Proof. The key step is to establish the following higher-order energy inequality.

Lemma 4.2. *Let c be the classical solution to the system* (1.1)-(1.4)*, then there holds*

$$\frac{d}{dt}\|\nabla\mu\|^2 + \frac{\mathbf{C}}{\mathbf{Pe}}\|\nabla\Delta\mu\|^2 \le K(1+\|\nabla\mu\|^6), \tag{4.17}$$

where K is a constant only depending on $\|c_0\|_{H^1}$ and possibly on parameters $\mathbf{Pe}, \mathbf{M}, \mathbf{C}, \bar{\eta}, \underline{\eta}$.

Proof. We revisit the equality (3.20). By the Agmon inequality and (2.8), we have

$$\|\Delta c\|_{L^\infty} \le C\|\Delta c\|_{H^1}^{\frac12}\|\Delta c\|_{H^2}^{\frac12} \le C(1+\|\Delta c\|+\|\nabla\Delta c\|+\|\Delta^2 c\|)^{\frac12}(1+\|\Delta c\|+\|\nabla\Delta c\|)^{\frac12}.$$

Besides, it follows from the Sobolev embedding theorem and (2.8) that

$$\begin{aligned}\|\Delta^2 c\| &\le \frac{1}{\mathbf{C}}(\|\Delta\mu\| + \|\Delta f'(c)\|) \le \frac{1}{\mathbf{C}}\|\Delta\mu\| + C(1+\|\Delta c\|^2)\\ &\le \frac{1}{\mathbf{C}}\|\nabla\Delta\mu\|^{\frac12}\|\nabla\mu\|^{\frac12} + C(1+\|\nabla\mu\|).\end{aligned} \tag{4.18}$$

By equation (1.1), assumption (A1) and we have

$$\begin{aligned}\|\nabla u\| &\le \left\|\frac{\eta'(c)}{\eta(c)}\right\|_{L^\infty}\|\nabla c\|_{L^\infty}\|u\| + \frac{1}{12}\left\|\frac{1}{\eta(c)}\right\|_{L^\infty}\|\nabla p\|_{H^1}\\ &\quad+\frac{1}{12\mathbf{M}}\left\|\frac{1}{\eta(c)}\right\|_{L^\infty}(\|\nabla\mu\|\|\nabla c\|_{L^\infty} + \|\mu\|_{L^6}\|c\|_{W^{2,3}})\\ &\le C\|\nabla c\|_{L^\infty}\|u\| + C\|\nabla p\|_{H^1} + C(\|\nabla\mu\|\|\nabla c\|_{L^\infty} + \|\mu\|_{L^6}\|c\|_{W^{2,3}})\\ &:= J_1 + J_2 + J_3.\end{aligned} \tag{4.19}$$

By (4.4)-(4.6), we have

$$\begin{aligned}J_1 &\le C(1+\|\Delta c\| + \|\nabla\Delta c\|)(1+\|\Delta c\|)^2 \le C(1+\|\nabla\mu\|^2)\\ J_3 &\le C\|\nabla\mu\|(1+\|\nabla\mu\|) + C(1+\|\nabla\mu\|)(\|\nabla\Delta c\|^{\frac12}\|\Delta c\|^{\frac12} + \|\Delta c\| + 1)\\ &\le C(1+\|\nabla\mu\|^2).\end{aligned} \tag{4.20}$$

Since the function $\mathcal{F}$ in Lemma 3.1 is not given in an explicit form and at this stage we are not able to bound $\|c\|_{L^\infty}$, we use a direct way to estimate

J_2. Using the periodic boundary condition, we have $\|\nabla p\|_{H^1}^2 = \|\nabla\nabla p\|^2 + \|\nabla p\|^2 = \|\Delta p\|^2 + \|\nabla p\|^2$. Then using the equation (3.1), (A1), (4.6) and (4.20), we obtain

$$\begin{aligned}
J_2 &\leq C\|\Delta p\| + C\|\nabla p\| \\
&\leq C\bar{\eta}\left[\left\|\nabla\left(\frac{1}{\eta(c)}\right)\cdot\nabla p\right\| + \left\|div\left(\frac{1}{\mathbf{M}\eta(c)}\mu(c)\nabla c\right)\right\|\right] + C\|\nabla p\| \\
&\leq C(\|\nabla c\|_{L^\infty}+1)\|\nabla p\| + C\|\mu\|_{L^6}\|c\|_{W^{2,3}} + C\|\nabla\mu\|\|\nabla c\|_{L^\infty} \\
&\leq C(1+\|\nabla\mu\|^2). \qquad (4.21)
\end{aligned}$$

Therefore, we have

$$\|\nabla u\| \leq C(1+\|\nabla\mu\|^2). \qquad (4.22)$$

Now we are able to re-estimate the terms $I_3, ..., I_5$ on the right-hand side of (3.20).

$$\begin{aligned}
I_3 &\leq C\|\nabla u\|\|\nabla c\|_{L^\infty}\|\nabla\Delta\mu\| + C\|u\|\|\Delta c\|_{L^\infty}\|\nabla\Delta\mu\| \\
&\leq \frac{\mathbf{C}}{12\mathbf{Pe}}\|\nabla\Delta\mu\|^2 + C\|\nabla u\|^2\|\nabla c\|_{L^\infty}^2 + C\|u\|^2\|\Delta c\|_{L^\infty}^2 \\
&\leq \frac{\mathbf{C}}{12\mathbf{Pe}}\|\nabla\Delta\mu\|^2 + C(1+\|\nabla\mu\|^2)^2(1+\|\nabla\mu\|)^2 \\
&\quad + C(1+\|\nabla\mu\|^{\frac{1}{2}})^2(1+\|\nabla\mu\|)^2(1+\|\nabla\mu\|+\|\nabla\Delta\mu\|^{\frac{1}{2}}\|\nabla\mu\|^{\frac{1}{2}})(1+\|\nabla\mu\|) \\
&\leq \frac{\mathbf{C}}{6\mathbf{Pe}}\|\nabla\Delta\mu\|^2 + C(1+\|\nabla\mu\|^6). \qquad (4.23)
\end{aligned}$$

$$\begin{aligned}
I_4 &\leq C(1+\|c\|_{L^\infty}^2)\|u\|\|\nabla c\|_{L^\infty}\|\Delta\mu\| \leq C(1+\|\Delta c\|)^3(1+\|\Delta c\|+\|\nabla\Delta c\|)\|\nabla\Delta\mu\|^{\frac{1}{2}}\|\nabla\mu\|^{\frac{1}{2}} \\
&\leq C(1+\|\nabla\mu\|^3)\|\nabla\Delta\mu\|^{\frac{1}{2}} \leq \frac{\mathbf{C}}{6\mathbf{Pe}}\|\nabla\Delta\mu\|^2 + C(1+\|\nabla\mu\|^4), \qquad (4.24)
\end{aligned}$$

$$\begin{aligned}
I_5 &\leq C(1+\|c\|_{L^\infty}^2)\|\Delta\mu\|^2 \leq C(1+\|\nabla\mu\|^{\frac{1}{2}})\|\nabla\Delta\mu\|\|\nabla\mu\| \\
&\leq \frac{\mathbf{C}}{6\mathbf{Pe}}\|\nabla\Delta\mu\|^2 + C(1+\|\nabla\mu\|^3). \qquad (4.25)
\end{aligned}$$

Inserting (4.23)-(4.25) into (3.20) and using the Young's inequality we can see that (4.17) holds. □

Now we turn to the proof of Theorem 4.2. The proof follows from the idea in [15] and can be performed as in [24, Theorem 4.3]. For the sake of convenience, we sketch it here. Comparing (4.17), we consider the following ODE problem:

$$\frac{d}{dt}Y(t) = K(1 + Y(t)^3), \quad Y(0) = R.$$

We denote by $I = [0, T_{max})$ the maximum existence interval of $Y(t)$ such that $\lim_{t\to T_{max}^-} Y(t) = +\infty$. Take $t_0 = \frac{1}{2}T_{max}(R,K) > 0$ and $\varepsilon_0 = \frac{Rt_0}{2\mathbf{Pe}}$. We can see that $\|\nabla\mu(t)\|$ is uniformly bounded on $[0, t_0]$ by a constant depending on R, K, t_0. If (ii) is not true, we have $E(c(t)) \geq E(c_0) - \varepsilon_0$, for all $t \geq 0$. Then from the basic energy law (2.7), there exists a $t_* \in [\frac{t_0}{2}, t_0]$ such that $\|\nabla\mu(t_*)\|^2 \leq \frac{2\mathbf{Pe}\varepsilon_0}{t_0} = R$. Taking t_* as the initial time, we infer from the above argument that $\|\nabla\mu(t)\|$ is bounded at least on $[0, \frac{3t_0}{2}] \subset [0, t_* + t_0]$. Moreover, its bound remains the same as that on $[0, t_0]$. By iteration, we can show that $\|\nabla\mu(t)\|^2$ is uniformly bounded for $t \geq 0$. Then our conclusion follows from a similar argument as in Theorem 4.1. $\square$

A direct consequence of the above result is the eventually regularity of weak solutions:

Corollary 4.1. *Let $d = 3$ and (c, u) be a weak solution to problem (1.1)-(1.4) on $[0, +\infty)$. Then there is some $T^* > 0$ such that (c, u) is regular after T^*.*

Proof. Let $R = 1$ in the proof of Theorem 4.2. Then we can fix $t_0 = \frac{1}{2}T_{max}(1, K) > 0$ and $\varepsilon_0 = \frac{t_0}{2}$. There exists $\bar{t}(\varepsilon_0) \geq 1$ such that $\int_{\bar{t}}^{+\infty} \|\nabla\mu\|^2 dt \leq \varepsilon_0$. Hence, there exists $T^* \geq \bar{t}$ that $\|\nabla\mu(T^*)\|^2 \leq 1$ and $E(c(t)) - E(c_0) \geq -\int_{T^*}^{+\infty} \|\nabla\mu\|^2 dt \geq -\varepsilon_0$ for $t \geq T^*$. Then we can apply Theorem 4.2. $\square$

The last result we present is global existence of classical solution for initial data close to local minimizer of the energy.

3dlom **Theorem 4.3.** *Suppose $d = 3$. Let $c^* \in H^1(\mathbb{T}^3)$ be a local minimizer of $E(c)$ in the sense that there exists a $\delta > 0$ such that $E(c^*) \leq E(c)$ for all $c \in H^1(\mathbb{T}^3)$ satisfying $\int_{\mathbb{T}^3} c dx = \int_{\mathbb{T}^3} c^* dx$ and $\|c - c^*\|_{H^1} < \delta$. Then there exists a constant $\sigma \in (0, 1]$ which may depend on c^*, δ and coefficients of the system such that for any $c_0 \in H^3$ satisfying $\int_{\mathbb{T}^3} c_0 dx = \int_{\mathbb{T}^3} c^* dx$ and $\|c_0 - c^*\|_{H^3} \leq 1$, and $\|c_0 - c^*\|_{H^2} \leq \sigma$, the problem (1.1)–(1.4) must admit a unique global classical solution.*

Proof. It is easy to see that Proposition 3.5 also holds in $3D$ case. Since c^* is a local minimizer of E, then we can see that c^* is smooth and its H^s-norms only depend on $\int_{\mathbb{T}^3} c^* dx$ and $\mathbf{C}$. Thus $\|c_0\|_{H^3} \leq \|c^*\|_{H^3} + 1$ only depends on c^*. In the subsequent proof, we denote by C_i, $i = 1, 2, ...$ constants that only depend on c^* and coefficients of the problem. It follows from Lemma 2.1 that $\|c(t)\|_{H^1} \leq C_1$ for $t \geq 0$. By Sobolev embedding, $E(c_0) - E(c(t)) \leq C_2 \|c_0 - c(t)\|_{H^1}$ for $t \geq 0$ with C_2 depending only on c^*.

Since $\|\nabla \mu(0)\|^2 \leq C(\|c_0\|_{H^3}) := R$, then as in the proof of Theorem 4.2, we can subsequently fix t_0 and ε_0. All those three quantities depend only on c^* and coefficients of the problem. Furthermore, we see that on $[0, t_0]$, $\|\nabla \mu(t)\|$ is uniformly bounded by a constant only depending on R, K, t_0 (thus on c^*). Since c^* is a critical point of E, Lemma 3.4 holds with c_∞ replaced by c^* in (3.36) and the constants β, θ are determined by c^*. Set

$$\varpi = \min\left\{\frac{1}{2}\beta, \delta, \frac{2\varepsilon_0}{3C_2}\right\}.$$

For $\sigma \leq \frac{1}{2}\varpi$, let $t_\sigma > 0$ be the smallest and finite time for which $\|c(t_\sigma) - c^*\|_{H^2} \geq \varpi$. We first show that there exists σ such that $t_\sigma > t_0$ by a contradiction argument. Applying Lemma 3.4, similar to (3.38), we infer from (3.35) that the following inequality holds on the interval $[0, t_\sigma] \subset [0, t_0]$,

$$-\frac{d}{dt}(E(c(t)) - E(c^*))^\theta \geq C_3(\|u\| + \|\nabla \mu\|) \geq C_4 \|c_t\|_{H^{-1}}, \tag{4.26}$$

Therefore, we have

$$\int_0^{t_\sigma} \|c_t\|_{H^{-1}} dt \leq C_4^{-1}(E(c_0) - E(c^*))^\theta \leq C_5 \|c_0 - c^*\|_{H^2}^\theta, \tag{4.27}$$

which implies that

$$\begin{aligned}
& \|c(t_\sigma) - c^*\|_{H^2} \\
\leq\ & \|c_0 - c^*\|_{H^2} + \|c(t_\sigma) - c_0\|_{H^2} \leq \|c_0 - c^*\|_{H^2} + C_6 \|c(t_\sigma) - c_0\|_{H^3}^{\frac{3}{4}} \|c(t_\sigma) - c_0\|_{H^{-1}}^{\frac{1}{4}} \\
\leq\ & \|c_0 - c^*\|_{H^2} + C_7 \left(\int_0^{t_\sigma} \|c_t\|_{H^{-1}} dt\right)^{\frac{1}{4}} \\
\leq\ & \|c_0 - c^*\|_{H^2} + C_8 \|c_0 - c^*\|_{H^2}^{\frac{\theta}{4}}.
\end{aligned} \tag{4.28}$$

Choosing

$$\sigma = \min\left\{\frac{1}{2}\varpi, \left(\frac{\varpi}{4C_8}\right)^{\frac{4}{\theta}}\right\},$$

we have $\|c(t_\sigma) - c^*\|_{H^2} \le \frac{3}{4}\varpi < \varpi$, which yields a contradiction with the definition of t_σ. Hence, for such σ, we have $\|c(t) - c^*\|_{H^2} \le \varpi$ for $t \in [0, t_0]$, which implies that $\|c(t) - c_0\|_{H^2} \le \|c(t) - c^*\|_{H^2} + \|c_0 - c^*\|_{H^2} \le \frac{3}{2}\varpi \le \frac{\varepsilon_0}{C_2}$. As a consequence, $E(c(t)) - E(c_0) \ge -\varepsilon_0$ on $[0, t_0]$. Then we can find $t_1 \in [\frac{1}{2}t_0, t_0]$ such that $\|\nabla\mu(t_1)\|^2 \le R$. Starting from t_1, we can actually extend our classical solution c to interval $[0, t_1 + t_0]$ with the same estimates as on $[0, t_0]$. Then repeating the above argument, we can show that $E(c(t)) - E(c_0) \ge -\varepsilon_0$ on $[0, t_1 + t_0]$. By iteration, we have $E(c(t)) - E(c_0) \ge -\varepsilon_0$ for $t \ge 0$. Our conclusion follows from Theorem 4.2. □

Remark 4.1. *By the eventual regularity of the weak solution, we only have to consider the long-time behavior of global classical solution. For the classical solution obtained in the above cases, one can argue as in Section 3 to obtain the same result like in* $2D$. *Besides, for the case that the initial datum is near a local minimizer* c^* *of* E. *We can easily see that the asymptotic limit point* c_∞ *has the property that* $E(c_\infty) = E(c^*)$. *From the proof of Theorem 4.3, we can see that* $\|c_\infty - c^*\| < \beta$ *and then by Lemma* (3.4) *(with* $c := c_\infty$ *and* $c_\infty := c^*$ *in* (3.36)*), it holds* $|E(c_\infty) - E(c^*)|^{1-\theta} \le \|P(-\Delta c_\infty + f(c_\infty))\| = 0$. *Thus,* c_∞ *is also a local minimizer of* E. *If* c^* *is an isolated minimizer, we have* $c_\infty = c^*$ *and our result provide the stability of* c^* *in this case.*

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