BOUNDARY LAYER ASSOCIATED WITH THE DARCY-BRINKMAN-BOUSSINESQ MODEL FOR CONVECTION IN POROUS MEDIA

JAMES P. KELLIHER, ROGER TEMAM, AND XIAOMING WANG

Abstract. We study the asymptotic behavior of the infinite Darcy-Prandtl number Darcy-Brinkman-Boussinesq system for convection in porous media at small Brinkman-Darcy number. The existence of a boundary layer with thickness proportional to the square root of the Brinkman-Darcy number for the velocity field is established in both the $L^\infty(H^1)$ norm (in 2 and 3 d) and the $L^\infty(L^\infty)$ norm (in 2d). This improves in several respects an earlier result of Payne and Straughan [43] where the vanishing Brinkman-Darcy number limit is studied without resolving the boundary layer.

1. Introduction

In this article we investigate a singular perturbation problem for a convection system. Before we describe our results we will first comment on the mathematical and physical backgrounds of the problem.

On the mathematical side the context is the behavior at small viscosity of the incompressible Navier-Stokes equations when the boundary is characteristic. This problem is a major open problem in applied mathematics and theoretical fluid dynamics. It is still not solved, not even for a small interval of time, not even in the incompressible case in dimension two, although the existence and uniqueness of solutions are known for all time for both the Navier-Stokes and Euler equations. These singular perturbation boundary layer problems of fluid mechanics are usually difficult to handle even if the leading order Prandtl type equation is linear (see for instance the case of Ekman layer [34], and the boundary layer associated with the Navier-Stokes equations when the boundary is uniformly non-characteristic [58]) and can be difficult even when the system itself is linear (see, for instance, [67] for the case of the linearized compressible Navier-Stokes system).

There is an abundant literature on boundary layer associated with incompressible flows and the related question of vanishing viscosity (see for instance [2, 6, 46, 47, 9, 39, 14, 22, 45, 19, 20, 7, 27, 68, 30, 3, 4, 32, 29, 65, 1, 66, 21, 55, 56, 51, 53, 54, 59, 23, 8, 25, 18, 26, 18, 26, 37, 5, 15, 13, 59, 38,

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36, 35, 10, 11, 63, 64] among many others) and we will refrain from surveying the literature here, but emphasize that the boundary layer problem is still open and that there is a need to develop tools and methods to tackle them.

On the physical side, the context is that of convection phenomena in porous media which are relevant to a variety of science and engineering problems ranging from geothermal energy transport to fiberglass insulation [40]. Here we consider a Bénard like problem: convection in a porous media region, $\tilde{\Omega}$, bounded by two parallel planes saturated with fluids. The bottom plate is kept at temperature T_2 and the top plate is kept at temperature T_1 with $T_2 > T_1$.

For concreteness, we let $\tilde{\Omega} = (0, 2\pi h)^{d-1} \times (0, h)$, $d = 2$ or 3, be a d-dimensional channel, periodic in the x- or x- and y-directions. Most of our analysis is valid in three dimensions, but when we restrict ourselves to two dimensions we use (x, z)-coordinates, suppressing the y variable, so that the z variable is always in the direction normal to the boundary.

The setup is very much similar to the Rayleigh-Bénard convection and the model we start with is the following *Darcy-Brinkman-Oberbeck-Boussinesq system* in the non-dimensional form [40]:

$$\begin{aligned}
\gamma_a(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}) + \mathbf{v} - \tilde{Da}\Delta\mathbf{v} + \nabla p &= Ra_D\,\mathbf{k}T, \quad \operatorname{div}\mathbf{v} = 0,\\
\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T &= \Delta T,
\end{aligned}$$

where $\mathbf{k}$ is the unit normal vector directed upward (the positive z direction), $\mathbf{v}$ is the non-dimensional seepage velocity, p is the non-dimensional kinematic pressure, T is the non-dimensional temperature. The parameters in the system are given by the *Prandtl-Darcy number*, γ_a^{-1}, which is defined as

$$\gamma_a^{-1} = \frac{\nu h^2}{\kappa K},$$

where ν is the kinematic viscosity of the fluid, h is the distance between the top and bottom plates, κ is the thermal diffusivity and K is the permeability of the fluid; the *Brinkman-Darcy number*, $\tilde{Da}$, which is given by

$$\tilde{Da} = \frac{\nu_{eff} K}{\nu h^2},$$

where ν_{eff} is the effective kinematic viscosity of the porous media; and the *Rayleigh-Darcy number*, Ra_D, which takes the form

$$Ra_D = \frac{g\alpha(T_2 - T_1)Kh}{\nu\kappa}$$

where g is the gravitational acceleration constant, α is the thermal expansion coefficient. The parameter γ_a is also called the non-dimensional acceleration coefficient. The non-dimensionalized domain is given by $\Omega = (0, 2\pi)^{d-1} \times (0, 1)$.

The *classical Darcy number*, Da, is defined as

$$Da = \frac{K}{h^2}.$$

Therefore, the Prandtl-Darcy number is the ratio of the Prandtl number ($Pr = \frac{\nu}{\kappa}$) to the Darcy number; that is, $\gamma_a^{-1} = \frac{Pr}{Da}$.

The *Brinkman-Darcy number* is the product of the Darcy number and the ratio of the effective viscosity to viscosity; that is,

$$\tilde{Da} = \frac{\nu_{eff}}{\nu} Da. \tag{1.1}$$

The *Rayleigh-Darcy number* Ra_D is the product of the Rayleigh number, Ra, and the Darcy number, Da; that is, $Ra_D = (Ra)\,(Da)$.

The non-Darcy viscous term (the Brinkman correction), i.e., $\tilde{Da}\Delta\mathbf{v}$, is needed if the porosity of the media is large or the domain has a boundary so that the no-slip, no-penetration condition must be imposed [40, 28].

In many physically interesting settings the Prandtl-Darcy number, γ_a^{-1}, is large either because the Darcy number Da is small (relative smallness of the permeability K over h^2) or the Prandtl number is large [40]. Hence γ_a is a small parameter in general. The Brinkman-Darcy number $\tilde{Da}$ is also small in many cases since the effective viscosity ν_{eff} is usually of the same order as the viscosity ν while the the Darcy number Da is small in general [40]. Therefore, we have two small parameters γ_a and $\tilde{Da}$, and, in the present work, we will consider simplified models and their validity as these small parameters approach zero in different physical manners.

Since the Prandtl-Darcy number γ_a^{-1} is usually large [40], we formally set the Prandtl-Darcy number to infinity ($\gamma_a = 0$) in the Darcy-Brinkman-Boussinesq model to arrive at the following *infinite Prandtl-Darcy number Darcy-Brinkman-Boussinesq system* (IPDDBB):

$$\mathbf{v} - \tilde{Da}\Delta\mathbf{v} + \nabla p = Ra_D\,\mathbf{k}T, \quad \operatorname{div}\mathbf{v} = 0, \quad \mathbf{v}|_{z=0,1} = 0, \tag{1.2}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\cdot\nabla T = \Delta T, \quad T\big|_{z=0} = 1, \quad T\big|_{z=1} = 0. \tag{1.3}$$

Periodicity in the horizontal directions is assumed for simplicity.

Such a formal limit is very much the same as the infinite Prandtl number limit in the Rayleigh-Bénard convection (except for the harmless low order Darcy term $\mathbf{v}$) that one of us studied earlier [60, 61, 62] and we anticipate parallel results for this singular limit problem involving two time scales (of relaxation type). A related limit where one first drops the Brinkman correction and the inertial term, then taking the infinite Prandtl-Darcy limit can be found in [42].

Within the infinite Prandtl-Darcy number Darcy-Brinkman-Boussinesq system, (1.2), (1.3), we can consider the vanishing viscosity limit since the Brinkman-Darcy number, $\tilde{Da}$, is usually small [40]. Formally setting the

Brinkman-Darcy number to zero, we arrive at the following *infinite Prandtl-Darcy number Darcy-Boussinesq system* (IPDDB):

$$\mathbf{v}^0 + \nabla p^0 = Ra_D\,\mathbf{k}T^0, \quad \operatorname{div}\mathbf{v}^0 = 0, \quad v_3^0\big|_{z=0,1} = 0, \tag{1.4}$$

$$\frac{\partial T^0}{\partial t} + \mathbf{v}^0\cdot\nabla T^0 = \Delta T^0, \quad T^0\big|_{z=0} = 1, \quad T^0\big|_{z=1} = 0. \tag{1.5}$$

Periodicity in the horizontal directions is assumed again.

Our aim in this article is to investigate the relations between the problem (1.2), (1.3) and problem (1.4), (1.5). This is a singular limit involving a boundary layer (and hence multiple spatial scales) since the velocity field of the infinite Darcy-Prandtl number Darcy-Boussinesq equation (1.4) satisfies the no-penetration boundary condition while the velocity field of the Brinkman model (1.2) satisfies the no-slip, no-penetration boundary condition. This is very similar to the classical boundary layer problem for incompressible viscous fluids at small viscosity that we already recalled [48, 41, 52, 57, 58]. Indeed, following the original work of Prandtl [44], we can derive a Prandtl type equation for this Brinkman model which indicates the existence of a boundary layer in the velocity field of a width proportional to $\sqrt{\tilde{Da}}$ and with no boundary layer in the temperature field or pressure field (in the leading order).

In fact, as we shall demonstrate below, the Prandtl type equation for the boundary layer is linear even though the Darcy-Brinkman-Boussinesq model (1.2), (1.3) considered here is nonlinear through the nonlinear advection of temperature and buoyancy forcing. This is similar to the case of the boundary layer for the incompressible Navier-Stokes flows with non-characteristic boundary conditions [57, 58]. Physical considerations and numerical evidence [40] both support the existence of a stable laminar boundary layer. This suggests that we have a simple laminar boundary layer and that we should be able to analyze the zero Brinkman-Darcy number limit in detail.

Indeed, Payne and Straughan [43] have already established the convergence in L^2 on any finite time interval of the solutions of the infinite Prandtl-Darcy number Darcy-Brinkman-Boussinesq to those of the infinite Prandtl-Darcy number Darcy-Boussinesq without resolving the boundary layer. Here we are interested in information in higher Sobolev spaces (say in the space of vorticity) or L^∞ *where the boundary layer structure can be detected*. These new estimates are useful in justifying the convergence of physically important quantities such as the transport of heat in the vertical direction across a fixed plane, i.e. $\int v_3(x, y, z_0; t)T(x, y, z_0; t)dxdy$ [40].

The approach that we take is classical in the sense that we follow a Prandtl type approach. However, the technique that we employ to prove the uniform in space and time estimates is not classical. We would like to emphasize that the usual Sobolev imbeddings are not sufficient for our purpose, and we invoke and improve here specific results on anisotropic Sobolev imbeddings by two of the authors [52, 57, 58].

This paper is organized as follows. In Section 3, we derive the effective equation for the difference of the solutions of the Brinkman-Boussinesq model (1.2), (1.3) and the Darcy-Boussinesq model (1.4), (1.5). This approach differs slightly from Prandtl's original approach in the sense that we approximate the difference between the viscous and inviscid solutions instead of the viscous solution directly; that is, we derive the Prandtl type equation for $\boldsymbol{\theta}^\varepsilon \approx \mathbf{v}^\varepsilon - \mathbf{v}^0$ formally, where $\varepsilon = \tilde{Da}$ denotes the small Brinkman-Darcy number and $\mathbf{v}^\varepsilon$ denotes the solution to the infinite Prandtl-Darcy number Brinkman-Darcy-Boussinesq system (1.2), (1.3) with the Brinkman-Dracy number $\tilde{Da} = \varepsilon$. We construct approximate solutions to the Prandtl type equations in Section 4. It is this approximate solution that we use as a corrector to obtain a characterization of the boundary layer.

In Section 5 we give a precise statement of our main results, which we prove in Section 6, Section 7 and Section 8.

The rate of convergence presented in this work for the uniform in space and time estimate is not optimal. Improvements on this and other specific results is the subject of our future work.

The definitions of all of our function spaces reflect the fact that we are working in a domain that is periodic in the horizontal direction(s). Thus, for instance, $H^m = H^m_{\overline{per}}(\Omega)$, m a nonnegative integer, is the Sobolev space consisting of all functions on Ω whose derivatives up to order m are square integrable and whose derivatives up to order $m-1$ are periodic in the horizontal direction(s), with the usual norm. Equivalently, we can view such functions as being defined on $\mathbb{R}^{d-1} \times (0,1)$ with period 2π in the horizontal direction(s). Similarly, $H^1_{0,per}(\Omega)$ is the subspace of functions in $H^1_{\overline{per}}(\Omega)$ that vanish on $z = 0, 1$. We will use the classical function spaces of fluid mechanics,

$$
\begin{aligned}
H &= H(\Omega) = \left\{ \mathbf{v} \in (L^2_{\overline{per}}(\Omega))^d \colon \operatorname{div} \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } z = 0, 1 \right\}, \\
V &= V(\Omega) = \left\{ \mathbf{v} \in (H^1_{0,per}(\Omega))^d \colon \operatorname{div} \mathbf{v} = 0 \right\},
\end{aligned}
\tag{1.6}
$$

where $\mathbf{n}$ denotes the unit outer normal to $\partial\Omega$. We put the L^2-norm on H and the H^1-norm on V. Because of the Poincaré's inequality, we can equivalently use $\|u\|_V = \|\nabla u\|_{L^2}$.

The space H^m should not be confused with the m-fold product of the space H of (1.6).

We follow the convention that $\|\cdot\|$ is the L^2-norm.

2. Some preliminaries

Let $\mathbf{v}^\varepsilon, T^\varepsilon$ be the velocity and temperature of (1.2), (1.3) ($\varepsilon = \tilde{Da}$), fix $t^* > 0$, and write $\gamma = Ra_D$. Then

$$\begin{cases} -\varepsilon\Delta\mathbf{v}^\varepsilon + \mathbf{v}^\varepsilon + \nabla p^\varepsilon = \gamma T^\varepsilon \mathbf{k} & \text{on } (0,t^*)\times\Omega, \\ \operatorname{div}\mathbf{v}^\varepsilon = 0 & \text{on } (0,t^*)\times\Omega, \\ \partial_t T^\varepsilon + \mathbf{v}^\varepsilon\cdot\nabla T^\varepsilon = \Delta T^\varepsilon & \text{on } (0,t^*)\times\Omega, \\ \mathbf{v}^\varepsilon = 0,\ T^\varepsilon = f & \text{on } (0,t^*)\times\partial\Omega, \\ T^\varepsilon(0) = T_0 & \text{on } \{0\}\times\Omega. \end{cases} \tag{2.1}$$

For simplicity we assume that f lies in $C^\infty(\partial\Omega)$, though less regularity could be assumed.

The equations for $\mathbf{v}^0, T^0$ of (1.4), (1.5) are the same as for $\mathbf{v}^\varepsilon, T^\varepsilon$ with $\varepsilon = 0$, except for the boundary condition on $\mathbf{v}^0$:

$$\begin{cases} \mathbf{v}^0 + \nabla p^0 = \gamma T^0 \mathbf{k} & \text{on } (0,t^*)\times\Omega, \\ \operatorname{div}\mathbf{v}^0 = 0 & \text{on } (0,t^*)\times\Omega, \\ \partial_t T^0 + \mathbf{v}^0\cdot\nabla T^0 = \Delta T^0 & \text{on } (0,t^*)\times\Omega, \\ \mathbf{v}^0\cdot\mathbf{n} = 0,\ T^0 = f & \text{on } (0,t^*)\times\partial\Omega, \\ T^0(0) = T_0 & \text{on } \{0\}\times\Omega. \end{cases} \tag{2.2}$$

It is sufficient to specify the initial value of only the temperature in both (2.1) and (2.2), since the initial velocity can be fully recovered from the initial temperature.[1]

The well-posedness of (2.1) is standard and is formulated in Theorem 2.1. The well-posedness as well as further regularity of (2.2) was established in [12] and [33] for slightly different boundary conditions and smoother initial data.

For the purpose of studying the boundary layer, we need smooth solutions to (2.2) up to $t = 0$. This can be achieved if we assume certain regularity and compatibility conditions (see [49]) on the initial temperature, T_0, and the boundary temperature f. More specifically, we assume that the initial temperature is compatible with the boundary data, possibly because it has been "properly prepared" by being the solution to (2.2) for some positive time, and hence that for some $k \geq 6$

$$\mathbf{v}^0, T^0 \in C^k([0,t^*]\times\overline{\Omega}). \tag{2.3}$$

The proof of such a result under suitable compatibility conditions is a simple application of the technique developed in [49]. We leave the details to the interested reader.

As for $\mathbf{v}^\varepsilon, T^\varepsilon$, we need only weak solutions. For such solutions, we have the following basic existence and uniqueness result (which applies in both 2 and 3 dimensions):

[1]In fact, and for both variables, at each instant of time, $\mathbf{v}$ ($= \mathbf{v}^\varepsilon$ or $\mathbf{v}^0$) is a function (functional) of T ($= T^\varepsilon$, or T^0). In the language of climatology, the temperature T is a prognostic variable, $\mathbf{v}$ and p are diagnostic variables.

Theorem 2.1. *Fix $t^* > 0$ and assume that T_0 lies in L^2. Then there exists a unique weak solution to (2.1) with the temperature, T^ε, lying in $L^\infty(0,t^*;L^2)\cap L^2(0,t^*;H^1)$ and the components of the velocity, $\mathbf{v}^\varepsilon$, lying in $L^\infty(0,t^*;L^2)\cap L^\infty(0,t^*;H^1)$. Furthermore, there exists a constant C such that*

$$\begin{aligned}
\|T^\varepsilon\|_{L^\infty(0,t^*;L^2)} &\le C,\\
\|T^\varepsilon\|_{L^2(0,t^*;H^1)} &\le C,\\
\|\mathbf{v}^\varepsilon\|_{L^\infty(0,t^*;L^2)} &\le C,\\
\|\mathbf{v}^\varepsilon\|_{L^\infty(0,t^*;H^1)} &\le C\epsilon^{-\frac{1}{2}}.
\end{aligned} \tag{2.4}$$

The constant C in these equations depends upon T_0, f and t^* but is independent of ϵ.

Moreover, if $T_0 - f$ lies in H^1_0 then $\mathbf{v}^\varepsilon$ and $T^\varepsilon - f$ lie in $C([0,t^*];H^1_0)$ and T^ε belongs to $L^2(0,t^*;H^2)$.

The proof, which we omit, is straightforward.

3. Derivation of the Prandtl type equation

Because the boundary conditions for the temperature fields coincide for the infinite Prandtl Darcy number Darcy-Brinkman-Boussinesq model and the infinite Prandtl Darcy number Darcy-Boussinesq model we do not expect a boundary layer for the temperature field. On the other hand, there must exist a boundary layer for the velocity field due to the disparity between the no-slip, no-penetration boundary condition for the Brinkman-Boussinesq model (1.2) and the no-penetration boundary condition for the Darcy-Boussinesq model (1.4).

It is quite clear that the thickness of the boundary layer for the problem should be proportional to $\sqrt{\varepsilon}$ by our knowledge of the Stokes problem [52]. We can also derive this thickness easily by following Prandtl's original stretched coordinate argument.

Notice that $\boldsymbol{\theta}^\varepsilon = \mathbf{v}^\varepsilon - \mathbf{v}^0$ satisfies the following equation

$$-\varepsilon\Delta\boldsymbol{\theta}^\varepsilon + \boldsymbol{\theta}^\varepsilon + \nabla q^\varepsilon = \varepsilon\Delta\mathbf{v}^0, \quad \operatorname{div}\boldsymbol{\theta}^\varepsilon = 0, \quad \boldsymbol{\theta}^\varepsilon\big|_{z=0,1} = -\mathbf{v}^0\big|_{z=0,1}.$$

We have suppressed the term $\gamma(T^\varepsilon - T^0)\mathbf{k}$ on the right-hand side of the first equation because to first order we expect no boundary layer in the temperature. We will prove that the resulting corrector we derive in this section (or rather the approximation of it given in Section 4) is accurate to first order, thereby justifying dropping this term.

Concentrating on the boundary layer at $z = 0$ for the moment, we use the stretched coordinate, $Z = z\varepsilon^{-\alpha}$, and assume that

$$\boldsymbol{\theta}^\varepsilon(x,y,z;t) = \boldsymbol{\theta}(x,y,Z;t), q^\varepsilon(x,y,z;t) = q(x,y,Z;t).$$

We deduce for the horizontal velocities, after neglecting terms of the order of ε, that the components, θ_j, $j = 1, 2$, of $\boldsymbol{\theta}^\varepsilon$ satisfy

$$-\varepsilon^{1-2\alpha}\frac{\partial^2\theta_j}{\partial Z^2} + \theta_j + \frac{\partial q}{\partial x_j} = 0, \quad x_1 = x, x_2 = y.$$

Since the viscous term must be effective in the boundary layer, and since the other terms are of order one, we surmise that the only rational choice of α is

$$\alpha = \frac{1}{2}.$$

The convergence theorems below (Theorems 5.1 and 5.2) demonstrate that this choice of α is adequate. Pursuing this line of thought, we see that the incompressibility condition $\operatorname{div}\boldsymbol{\theta}^\varepsilon = 0$ implies that θ_3^ε must be of the order of $\sqrt{\varepsilon}$ since we have assumed that θ_1, θ_2 are of order one. Therefore, the Prandtl type equation for the corrector at $z = 0$ is given by

$$\begin{aligned}
-\frac{\partial^2\theta_1}{\partial Z^2} + \theta_1 + \frac{\partial q}{\partial x} &= 0, \\
-\frac{\partial^2\theta_2}{\partial Z^2} + \theta_2 + \frac{\partial q}{\partial y} &= 0, \\
\frac{\partial q}{\partial Z} &= 0, \\
\frac{\partial\theta_1}{\partial x} + \frac{\partial\theta_2}{\partial y} + \frac{1}{\sqrt{\varepsilon}}\frac{\partial\theta_3}{\partial Z} &= 0, \\
\boldsymbol{\theta}\big|_{Z=0} &= -\mathbf{v}^0\big|_{z=0}, \\
\theta_j\big|_{Z=\infty} &= 0, j = 1, 2, \\
q\big|_{Z=\infty} &= 0.
\end{aligned}$$

The last two boundary conditions at infinity come from the assumption that the viscous and inviscid solutions match each other far away from the boundary and hence the difference of the two should be approximately zero far away from the boundary. No far field boundary condition is imposed on θ_3 since only one boundary condition is needed for a first order equation (that is, the incompressibility condition).

This set of Prandtl type equations can be solved exactly, giving

$$\begin{aligned}
q_0 &\equiv 0, \\
\theta_{0,j}(x, y, Z; t) &= -v_j^0(x, y, 0; t)e^{-Z}, j = 1, 2, \\
\theta_{0,3}(x, y, Z; t) &= \sqrt{\varepsilon}\Big(\frac{\partial v_1^0(x, y, 0; t)}{\partial x} + \frac{\partial v_2^0(x, y, 0; t)}{\partial y}\Big)(1 - e^{-Z}).
\end{aligned}$$

Notice that θ_3 is of the order of $\sqrt{\varepsilon}$ for large Z. Hence, if this Prandtl type equation is valid (we will establish this in the next section), it implies that we need correction of the order of $\sqrt{\varepsilon}$ in the interior of the domain (not completely restricted to the boundary as for classical elliptic and parabolic equations [31]). This fact was noticed earlier for the Stokes problem [52].

Likewise, the corrector needed for the boundary at $z = 1$ can be approximated by

$$\begin{aligned}
q_1 &\equiv 0,\\
\theta^{\varepsilon}_{1,j}(x, y, z; t) &= -v^0_j(x, y, 1; t)e^{-\frac{1-z}{\sqrt{\varepsilon}}}, j = 1, 2,\\
\theta_{1,3}(x, y, z; t) &= \sqrt{\varepsilon}(\frac{\partial v^0_1(x, y, 1; t)}{\partial x} + \frac{\partial v^0_2(x, y, 1; t)}{\partial y})(1 - e^{-\frac{1-z}{\sqrt{\varepsilon}}}).
\end{aligned}$$

As stated above, we will now show that the proposed correctors encompass the H^1 singularity of $\mathbf{v}^\varepsilon - \mathbf{v}^0$ and $p^\varepsilon - p^0$, thus allowing us to establish convergence results in this space.

4. Approximate solution to the Prandtl type equation

The exact solution to the Prandtl type equation in Section 3 is not very useful for the convergence analysis since the corrector $\boldsymbol{\theta}^\varepsilon$ does not satisfy the boundary condition exactly. (The corrector at $z = 0$ does not match the boundary condition at $z = 1$, and the corrector at $z = 1$ does not match the boundary condition at $z = 0$.) As an alternative, we utilize a truncated version of the solution to the Prandtl type equation following a similar approach for the case of Stokes equations and Oseen type equations [52], as well as the Navier-Stokes equations with non-characteristic boundary conditions [57, 58]; see a different approach in [16, 17]. The truncation will be done at the stream function level in order to avoid the difficulty of estimating the pressure.

We illustrate the process here through the 2D case (suppressing the dependence on y and the second component). Let ρ be a cut-off function satisfying

$$\rho \in C^\infty[0, \infty), \quad \mathrm{supp}(\rho) \subset [0, \frac{1}{2}], \quad \rho(z) \equiv 1 \text{ for } z \in [0, \frac{1}{4}].$$

Letting

$$\psi^\varepsilon(x, z; t) = \sqrt{\varepsilon} v^0_1(x, 0; t)(1 - e^{-\frac{z}{\sqrt{\varepsilon}}})\rho(z),$$

we define an approximation, $\boldsymbol{\theta}^\varepsilon_0$, of the solution to the Prandtl type equation for the boundary layer at $z = 0$ by

$$\boldsymbol{\theta}^\varepsilon_0 = (\theta^\varepsilon_{0,1}, \theta^\varepsilon_{0,3}) = \nabla^\perp\psi^\varepsilon = (-\frac{\partial\psi^\varepsilon}{\partial z}, \frac{\partial\psi^\varepsilon}{\partial x}).$$

Thus,

$$\begin{aligned}
&\theta^\varepsilon_{0,1}(x, z; t) = a(t, x)e^{-z/\sqrt{\varepsilon}}\rho(z) - \sqrt{\varepsilon}a(t, x)(1 - e^{-z/\sqrt{\varepsilon}})\rho'(z),\\
&\theta^\varepsilon_{0,3}(x, z; t) = \sqrt{\varepsilon}\partial_x a(t, x)(1 - e^{-z/\sqrt{\varepsilon}})\rho(z),
\end{aligned} \tag{4.1}$$

where $a(t, x) = -v^0_1(x, 0; t)$. The approximate solution, $\boldsymbol{\theta}^\varepsilon_1$, to the Prandtl type equation for the boundary layer at $z = 1$ is entirely analogous.

Because $\boldsymbol{\theta}_0^\varepsilon$ and $\boldsymbol{\theta}_1^\varepsilon$ match the boundary condition, $\boldsymbol{\theta}^\varepsilon = -\mathbf{v}^0$, exactly at their respective boundaries and vanish in a fixed neighborhood of the opposite boundary, the sum of the two correctors,

$$\boldsymbol{\theta}^\varepsilon = \boldsymbol{\theta}_0^\varepsilon + \boldsymbol{\theta}_1^\varepsilon,$$

matches the boundary conditions exactly and is identical to the exact corrector in a boundary layer of fixed width 1/4. Moreover $\boldsymbol{\theta}^\varepsilon$ is an approximation throughout Ω of the order of $\sqrt{\varepsilon}$ in L^∞ of the exact solution to the Prandtl type equation.

We infer from (4.1) that $\boldsymbol{\theta}^\varepsilon$ satisfies the equation for $\mathbf{v}^\varepsilon - \mathbf{v}^0$ approximately in the sense that

$$\begin{cases} -\varepsilon\Delta\boldsymbol{\theta}^\varepsilon + \boldsymbol{\theta}^\varepsilon = \mathbf{f}^\varepsilon & \text{on } (0,t^*)\times\Omega, \\ \operatorname{div}\boldsymbol{\theta}^\varepsilon = 0 & \text{on } (0,t^*)\times\Omega, \\ \boldsymbol{\theta}^\varepsilon = -\mathbf{v}^0 & \text{on } (0,t^*)\times\partial\Omega. \end{cases} \tag{4.2}$$

Here, $\mathbf{f}^\varepsilon = \mathbf{f}_1^\varepsilon + \mathbf{f}_2^\varepsilon$, where $\mathbf{f}_1^\varepsilon = (g^\varepsilon, h^\varepsilon)$ with

$$\begin{aligned} g^\varepsilon e^{z/\sqrt{\varepsilon}} &= \varepsilon^{3/2}\Delta(a\rho')\left[e^{z/\sqrt{\varepsilon}} - 1\right] + \varepsilon\left[a\rho'' - \Delta(a\rho)\right] \\ &\qquad - \varepsilon^{1/2}a\rho'\left[e^{z/\sqrt{\varepsilon}} - 1\right], \\ h^\varepsilon e^{z/\sqrt{\varepsilon}} &= \varepsilon^{3/2}\Delta(\partial_x a\rho)\left[e^{z/\sqrt{\varepsilon}} - 1\right] + \varepsilon\partial_x a\rho' + \varepsilon^{1/2}\partial_x a\rho \end{aligned}$$

and $\mathbf{f}_2^\varepsilon$ has an analogous form. Letting $\boldsymbol{\tau}$ be the unit tangent vector, it follows that

$$\|\mathbf{f}^\varepsilon\| \le C\varepsilon^{\frac{1}{2}},$$

as long as $\partial_x^3(\mathbf{v}^0\cdot\boldsymbol{\tau})$ lies in $L^\infty([0,T];L^2(\partial\Omega))$, which follows from (2.3).

In fact, because derivatives in t and x act only on $v_1^0(x,0;t)$, it also follows that

$$\|\partial_t^m\partial_x^n\mathbf{f}^\varepsilon\|_{L^\infty(0,t^*;L^2)} \le C\varepsilon^{1/2}, \tag{4.3}$$

for all nonnegative integers m and n as long as $\partial_t^m\partial_x^{3+n}(\mathbf{v}^0\cdot\boldsymbol{\tau})$ lies in $L^\infty(0,t^*;L^2(\partial\Omega))$, which it does when we assume (2.3) and $m+n+3\le k$.

In particular, there is no pressure in (4.2). This equates to the approximate pressure being identically zero.

Although the expression for the approximate corrector, $\boldsymbol{\theta}^\varepsilon$, in (4.1) was derived using the stream function, it is clear that in three dimensions we can set the components of $\boldsymbol{\theta}_0^\varepsilon$ to

$$\begin{aligned} \theta_{0,1}^\varepsilon(x,y,z;t) &= a(t,x,y)e^{-z/\sqrt{\varepsilon}}\rho(z) - \sqrt{\varepsilon}a(t,x,y)(1-e^{-z/\sqrt{\varepsilon}})\rho'(z), \\ \theta_{0,2}^\varepsilon(x,y,z;t) &= b(t,x,y)e^{-z/\sqrt{\varepsilon}}\rho(z) - \sqrt{\varepsilon}b(t,x,y)(1-e^{-z/\sqrt{\varepsilon}})\rho'(z), \\ \theta_{0,3}^\varepsilon(x,y,z;t) &= \sqrt{\varepsilon}(\partial_x a(t,x,y) + \partial_y b(t,x,t))(1-e^{-z/\sqrt{\varepsilon}})\rho(z), \end{aligned}$$

where $a(t,x) = -v_1^0(x,y,0;t)$ and $b(t,x) = -v_2^0(x,y,0;t)$, with an analogous formula for $\boldsymbol{\theta}_1^\varepsilon$. Then $\operatorname{div}\boldsymbol{\theta}^\varepsilon = 0$ on Ω and $\boldsymbol{\theta}^\varepsilon = -(v_1^0, v_2^0, 0) = -\mathbf{v}^0$ on $\partial\Omega$, and it is easy to see that (4.2) continues to hold and that in place of (4.3) we have

$$\left\|\partial_t^m \partial_x^n \partial_y^l \mathbf{f}^\varepsilon\right\|_{L^\infty(0,t^*;L^2)} \leq C\varepsilon^{1/2}, \tag{4.4}$$

for all nonnegative integers m, n, and k as long as $\partial_t^m \partial_x^{l+n}(\mathbf{v}^0\cdot\boldsymbol{\tau})+\partial_t^m \partial_y^{n+l}(\mathbf{v}^0\cdot\boldsymbol{\tau})$ lies in $L^\infty(0,t^*;L^2(\partial\Omega))$ which follows from (2.3) provided $m+n+l+3 \leq k$.

5. Statement of main results

Let $\boldsymbol{\theta}^\varepsilon$ satisfying (4.1), (4.2) be the approximate solution to the Prandtl type equation derived in Section 4 and let

$$\mathbf{w}_v^\varepsilon = \mathbf{v}^\varepsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\varepsilon, \quad w_T^\varepsilon = T^\varepsilon - T^0. \tag{5.1}$$

We view $\boldsymbol{\theta}^\varepsilon$ as a "corrector" that, among other things, makes $\mathbf{w}_v^\varepsilon$ vanish on the boundary.

In the bounds below, we make the assumption that

$$\varepsilon \leq 1, \tag{5.2}$$

reducing the complexity of the expressions for some of the bounds that result. Since we only care about small ε, no significant information is lost.

In Theorem 5.1, we establish convergence of $\mathbf{w}_v^\varepsilon$ and w_T^ε in the classical energy space.

Theorem 5.1. *The assumptions are (2.3), (4.1), (4.2), and (5.1). Then*

$$\begin{aligned} \|\mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^2)} &\leq C\varepsilon^{1/2}, \\ \|\mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;H^1)} &\leq C, \\ \|w_T^\varepsilon\|_{L^\infty(0,t^*;L^2)} &\leq C\varepsilon^{1/2}, \\ \|w_T^\varepsilon\|_{L^2(0,t^*;H^1)} &\leq C\varepsilon^{1/2} \end{aligned} \tag{5.3}$$

and

$$\left\|\nabla p^\varepsilon - \nabla p^0\right\|_{L^\infty(0,t^*;L^2)} \leq C\varepsilon^{1/2}. \tag{5.4}$$

Each of the constants, C, in (5.3), (5.4) depends only on T_0, f, and t^*.

More precise information on the convergence of the velocity is given in Theorem 5.2, where we consider uniform convergence in $[0,T]\times\overline{\Omega}$.

Theorem 5.2. *The assumptions are as in Theorem 5.1 and $d=2$. Then*

$$\|w_T^\varepsilon\|_{L^\infty(0,t^*;H^1)}, \ \|\partial_t w_T^\varepsilon\|_{L^2(0,t^*;L^2)} \leq C\varepsilon^{1/4}, \tag{5.5}$$

and

$$\|\mathbf{w}_v^\varepsilon\|_{L^\infty((0,t^*)\times\Omega)} \leq C\varepsilon^{1/8}, \tag{5.6}$$

$$\|w_T^\varepsilon\|_{L^\infty((0,t^*)\times\Omega)} \leq C\varepsilon^{3/8}. \tag{5.7}$$

Each of the constants, C, in (5.5), (5.6) depends only on T_0, f, and t^*.

Remark 5.3. A useful observation that is universal to convergence with a good corrector is the optimal convergence rate at vanishing viscosity. Applying the triangle inequality to the expression, $\mathbf{v}^\varepsilon - \mathbf{v}^0 = \mathbf{w}_v^\varepsilon + \boldsymbol{\theta}^\varepsilon$, it follows by virtue of $(5.3)_1$ and a straightforward estimate on the corrector (4.1) that, as $\varepsilon \to 0$, there exist constants C_1 and C_2 depending only on T_0, f, and t^* such that

$$\begin{aligned} C_1\varepsilon^{1/4} &\leq \|\boldsymbol{\theta}^\varepsilon\|_{L^\infty(0,t^*;L^2)} - \|\mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^2)} \\ &\leq \left\|\mathbf{v}^\varepsilon - \mathbf{v}^0\right\|_{L^\infty(0,t^*;L^2)} \\ &\leq \|\boldsymbol{\theta}^\varepsilon\|_{L^\infty(0,t^*;L^2)} + \|\mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^2)} \\ &\leq C_2\varepsilon^{1/4} \end{aligned} \tag{5.8}$$

for all sufficiently small ε. This applies as long as $\mathbf{v}^0$ is not identically zero on $[0, t^*] \times \partial\Omega$ so that $\boldsymbol{\theta}^\varepsilon$ is not identically zero, and gives the optimal rate of convergence in the vanishing viscosity limit. (The upper bound in (5.8) was established in [43].)

Remark 5.4. Another observation that is universal to H^1 convergence with a boundary layer is the existence of a vortex sheet at vanishing viscosity. Indeed, let

$$\chi_\varepsilon(x;t) = \left[v_1^0(x,0;t)e^{-z/\sqrt{\varepsilon}} + v_1^0(x,1;t)e^{-(1-z)/\sqrt{\varepsilon}}\right]\varepsilon^{-1/2}, \tag{5.9}$$

which captures the leading order of the core of the curl of the corrector $\boldsymbol{\theta}^\varepsilon$ (4.1); we see that the vorticity $\omega(\mathbf{v}^\varepsilon) = \nabla^\perp \cdot \mathbf{v}^\varepsilon$, satisfies the following relation thanks to $(5.3)_2$, (4.1) and (5.9),

$$\|\omega(\mathbf{v}^\varepsilon) - \chi_\varepsilon\|_{L^\infty(0,t^*;L^2)} \leq C, \tag{5.10}$$

$$\chi_\varepsilon(x;t) \to v_1^0(x,0;t)\delta(z) + v_1^0(x,1;t)\delta(z-1) \text{ as } \varepsilon \to 0. \tag{5.11}$$

This proves the existence of a vortex sheet along the boundary. A similar observation was made in [24] for the Navier-Stokes system at vanishing viscosity. The same result holds for $d = 3$ with a more complicated expression for χ_ε. The constants in these expressions depend on T_0 and t^*.

We prove Theorem 5.1 in Section 6 and Theorem 5.2 in Section 7.

6. Convergence of $\mathbf{w}_v^\varepsilon$ and w_T^ε in the energy norm

In this section we prove Theorem 5.1. In Section 6.1 we establish the convergence of $\mathbf{w}_v^\varepsilon$ and w_T^ε as in (5.3) then, in Section 6.2, we establish convergence of the pressure.

6.1. Convergence of $\mathbf{w}_v^\varepsilon$ and w_T^ε.

Because $\mathbf{w}_v^\varepsilon$ and w_T^ε both vanish at $z = 0, 1$ and are periodic in the horizontal direction(s), the various boundary integrals that appear when integrating by parts disappear, making it much simpler to obtain energy bounds. The tradeoff is that extra terms appear because of the corrector $\boldsymbol{\theta}^\varepsilon$, and these must each be carefully bounded.

Subtracting the equations for $\mathbf{v}^0$ in (2.2) from that for $\mathbf{v}^\varepsilon$ in (2.1) gives

$$\mathbf{v}^\varepsilon - \mathbf{v}^0 - \varepsilon\Delta(\mathbf{v}^\varepsilon - \mathbf{v}^0) + \nabla p^\varepsilon - \nabla p^0 = \gamma \mathbf{k} w_T^\varepsilon + \varepsilon\Delta\mathbf{v}^0. \tag{6.1}$$

Define $\mathbf{w}_v^\varepsilon$ and w_T^ε as in (5.1), noting that $\mathbf{w}_v^\varepsilon = 0$ at $z = 0$ and $z = 1$ and is periodic in the x- and, when $d = 3$, y-directions. Subtracting $(4.2)_1$ from (6.1) gives

$$-\varepsilon\Delta\mathbf{w}_v^\varepsilon + \mathbf{w}_v^\varepsilon + \nabla r^\varepsilon = \gamma\mathbf{k}w_T^\varepsilon + \varepsilon\Delta\mathbf{v}^0 - \mathbf{f}^\varepsilon, \tag{6.2}$$

where $r^\varepsilon = p^\varepsilon - p^0$ and $\mathbf{f}^\varepsilon$ is as in $(4.2)_1$.

Similarly, subtracting the temperature equations in (2.1), (2.2) gives

$$\partial_t w_T^\varepsilon + \mathbf{v}^\varepsilon \cdot \nabla T^\varepsilon - \mathbf{v}^0 \cdot \nabla T^0 = \Delta w_T^\varepsilon.$$

Now, if we write

$$\begin{aligned}
\mathbf{v}^\varepsilon \cdot & \nabla T^\varepsilon - \mathbf{v}^0 \cdot \nabla T^0 \\
&= \mathbf{v}^\varepsilon \cdot \nabla T^0 - \boldsymbol{\theta}^\varepsilon \cdot \nabla T^0 - \mathbf{v}^0 \cdot \nabla T^0 + \mathbf{v}^\varepsilon \cdot \nabla T^\varepsilon - \mathbf{v}^\varepsilon \cdot \nabla T^0 + \boldsymbol{\theta}^\varepsilon \cdot \nabla T^0 \\
&= (\mathbf{v}^\varepsilon - \mathbf{v}^0 - \boldsymbol{\theta}^\varepsilon) \cdot \nabla T^0 + \boldsymbol{\theta}^\varepsilon \cdot \nabla T^0 + \mathbf{v}^\varepsilon \cdot \nabla w_T^\varepsilon \\
&= \mathbf{w}_v^\varepsilon \cdot \nabla T^0 + \boldsymbol{\theta}^\varepsilon \cdot \nabla T^0 + \mathbf{v}^\varepsilon \cdot \nabla w_T^\varepsilon,
\end{aligned}$$

we obtain

$$\partial_t w_T^\varepsilon + \mathbf{v}^\varepsilon \cdot \nabla w_T^\varepsilon + \mathbf{w}_v^\varepsilon \cdot \nabla T^0 = \Delta w_T^\varepsilon - \boldsymbol{\theta}^\varepsilon \cdot \nabla T^0. \tag{6.3}$$

Multiplying (6.2) by $\mathbf{w}_v^\varepsilon$ and integrating over the domain gives

$$\begin{aligned}
\varepsilon \|\nabla\mathbf{w}_v^\varepsilon\|^2 + \|\mathbf{w}_v^\varepsilon\|^2 &= \gamma(w_T^\varepsilon\mathbf{k}, \mathbf{w}_v^\varepsilon) - \varepsilon(\nabla\mathbf{v}^0, \nabla\mathbf{w}_v^\varepsilon) - (\mathbf{f}^\varepsilon, \mathbf{w}_v^\varepsilon) \\
&\le \gamma\|w_T^\varepsilon\|\|\mathbf{w}_v^\varepsilon\| + \varepsilon\left\|\nabla\mathbf{v}^0\right\|\|\nabla\mathbf{w}_v^\varepsilon\| + \|\mathbf{f}^\varepsilon\|\|\mathbf{w}_v^\varepsilon\| \\
&\le (\gamma\|w_T^\varepsilon\| + \|\mathbf{f}^\varepsilon\|)\|\mathbf{w}_v^\varepsilon\| + \frac{\varepsilon}{2}\left\|\nabla\mathbf{v}^0\right\|^2 + \frac{\varepsilon}{2}\|\nabla\mathbf{w}_v^\varepsilon\|^2,
\end{aligned}$$

leading to

$$\varepsilon\|\nabla\mathbf{w}_v^\varepsilon\|^2 + 2\|\mathbf{w}_v^\varepsilon\|^2 \le C\varepsilon + 2(\gamma\|w_T^\varepsilon\| + \|\mathbf{f}^\varepsilon\|)\|\mathbf{w}_v^\varepsilon\|.$$

Then

$$\varepsilon\|\nabla\mathbf{w}_v^\varepsilon\|^2 + 2\|\mathbf{w}_v^\varepsilon\|^2 \le C\varepsilon + C\|w_T^\varepsilon\|^2 + \frac{1}{2}\|\mathbf{w}_v^\varepsilon\|^2 + C\|\mathbf{f}^\varepsilon\|^2 + \frac{1}{2}\|\mathbf{w}_v^\varepsilon\|^2,$$

so, using (4.3) or (4.4),

$$\varepsilon\|\nabla\mathbf{w}_v^\varepsilon\|^2 + \|\mathbf{w}_v^\varepsilon\|^2 \le C\varepsilon + C\|w_T^\varepsilon\|^2. \tag{6.4}$$

We now use (6.3) to close the inequality in (6.4).

Multiplying (6.3) by w_T^ε, integrating over the domain, using the decomposition of $\boldsymbol{\theta}^\varepsilon$ in Lemma 6.1, below, and the following Lemma 6.3, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w_T^\varepsilon\|^2 + \|\nabla w_T^\varepsilon\|^2 &= -(\mathbf{w}_v^\varepsilon \cdot \nabla T^0, w_T^\varepsilon) - (\boldsymbol{\theta}^\varepsilon \cdot \nabla T^0, w_T^\varepsilon) \\
&\le \left\|\nabla T^0\right\|_{L^\infty} \|\mathbf{w}_v^\varepsilon\| \|w_T^\varepsilon\| + \left\|\nabla T^0\right\|_{L^\infty} \|\boldsymbol{\theta}_I^\varepsilon\| \|w_T^\varepsilon\| \\
&\qquad\qquad + \left\|\nabla T^0\right\|_{L^\infty} \|z\boldsymbol{\theta}_B^\varepsilon\| \|w_T^\varepsilon / z\| \\
&\le C\left(\|\mathbf{w}_v^\varepsilon\| + \varepsilon^{1/2}\right)\|w_T^\varepsilon\| + C\varepsilon^{1/2}\|\nabla w_T^\varepsilon\| \\
&\le C\left(\|\mathbf{w}_v^\varepsilon\|^2 + \|w_T^\varepsilon\|^2\right) + C\varepsilon + \frac{1}{2}\|\nabla w_T^\varepsilon\|^2 \\
&\le C\varepsilon + C\|w_T^\varepsilon\|^2 + \frac{1}{2}\|\nabla w_T^\varepsilon\|^2 .
\end{aligned} \tag{6.5}$$

The second inequality holds since T^0 lies in $L^\infty(0,T;C^1(\overline{\Omega}))$. In the last inequality we used (6.4). Applying Gronwall's inequality,

$$\|w_T^\varepsilon(t)\|^2 + \int_0^t \|\nabla w_T^\varepsilon\|^2 \le Cte^{Ct}\varepsilon.$$

Returning to (6.4) we then have

$$\varepsilon\|\nabla \mathbf{w}_v^\varepsilon\|^2 + \|\mathbf{w}_v^\varepsilon\|^2 \le C(1 + te^{Ct})\varepsilon.$$

We conclude that (5.3) holds.

Lemma 6.1. *There exists a decomposition* $\boldsymbol{\theta}^\varepsilon = \boldsymbol{\theta}_B^\varepsilon + \boldsymbol{\theta}_I^\varepsilon$ *such that*

$$\|z\boldsymbol{\theta}_B^\varepsilon\|_{L^\infty(0,t^*;L^2)}, \|\boldsymbol{\theta}_I^\varepsilon\|_{L^\infty(0,t^*;L^2)} \le C\varepsilon^{1/2}. \tag{6.6}$$

Proof. We assume that $d = 2$, the situation for $d = 3$ being quite similar. Let

$$\begin{aligned}
\boldsymbol{\theta}_{B_0}^\varepsilon &= (a(t,x)e^{-z/\sqrt{\varepsilon}}\rho(z), 0), \\
\boldsymbol{\theta}_{I_0}^\varepsilon &= (-\sqrt{\varepsilon}a(t,x)(1 - e^{-z/\sqrt{\varepsilon}})\rho'(z), \sqrt{\varepsilon}\partial_x a(t,x)(1 - e^{-z/\sqrt{\varepsilon}})\rho(z)).
\end{aligned}$$

It follows from (4.1) that $\boldsymbol{\theta}_0^\varepsilon = \boldsymbol{\theta}_{B_0}^\varepsilon + \boldsymbol{\theta}_{I_0}^\varepsilon$.

Now,

$$\left\|\boldsymbol{\theta}_{I_0}^\varepsilon\right\| \le C\left\|\boldsymbol{\theta}_{I_0}^\varepsilon\right\|_{L^\infty} \le C\sqrt{\varepsilon}, \tag{6.7}$$

and

$$\begin{aligned}
\left\|z\boldsymbol{\theta}_{B_0}^\varepsilon\right\| &\le C\|ze^{-z/\sqrt{\varepsilon}}\rho(z)\| \le C\,|\partial\Omega|^{1/2}\left(\int_0^{1/2} z^2 e^{-2z/\sqrt{\varepsilon}}\,dz\right)^{1/2} \\
&\le C\left(\varepsilon^{3/2}\right)^{1/2} = C\varepsilon^{3/4} \le C\varepsilon^{1/2}.
\end{aligned}$$

Defining $\boldsymbol{\theta}_{B_1}^\varepsilon$ and $\boldsymbol{\theta}_{I_1}^\varepsilon$ similarly and letting $\boldsymbol{\theta}_B^\varepsilon = \boldsymbol{\theta}_{B_0}^\varepsilon + \boldsymbol{\theta}_{B_1}^\varepsilon$, $\boldsymbol{\theta}_I^\varepsilon = \boldsymbol{\theta}_{I_0}^\varepsilon + \boldsymbol{\theta}_{I_1}^\varepsilon$, (6.6) follows. $\square$

Remark 6.2. It is easy to see that for all q in $[1, \infty)$,

$$C_1 \varepsilon^{1/2q} \leq \|\boldsymbol{\theta}^\varepsilon\|_{L^\infty(0,t^*;L^q)} \leq C_2 \varepsilon^{1/2q} \tag{6.8}$$

for some constant C_1 and C_2 depending on T_0 and (unless $f \equiv 0$) on t. The rate of $\varepsilon^{1/4}$ for $q = 2$ is insufficient, however, and led us to split $\boldsymbol{\theta}^\varepsilon$ into a boundary layer part and an interior part, as in Lemma 6.1. This same approach was taken in [58].

We used above the following version of Hardy's inequality:

Lemma 6.3 (Hardy's inequality)**.** *There exists a constant $C = C(\Omega)$ such that for all g in $H^1_{0,per}(\Omega)$,*

$$\|g/z\| + \|g/(1-z)\| \leq C \|\nabla g\|.$$

6.2. **Convergence of the pressure.** First, write (6.2) as

$$-\varepsilon \Delta \mathbf{w}_v^\varepsilon + \nabla r^\varepsilon = \mathbf{F},$$

where

$$\mathbf{F} = \mathbf{F}^\varepsilon = -\mathbf{w}_v^\varepsilon + \gamma \mathbf{k} w_T^\varepsilon + \varepsilon \Delta \mathbf{v}^0 - \mathbf{f}^\varepsilon.$$

From (5.3) with (4.3) or (4.4), we have

$$\|\mathbf{F}\|_{L^\infty(0,t^*;L^2)} \leq C\varepsilon^{1/2}.$$

Then for almost all t in $(0, t^*)$ from Proposition 2.2 in Chapter I of [50] and the remark following it (applied to $\varepsilon \mathbf{w}_v^\varepsilon$),

$$\varepsilon \|\mathbf{w}_v^\varepsilon(t)\|_{H^2} + \|\nabla r^\varepsilon(t)\| \leq c_0 \|\mathbf{F}(t)\|,$$

the constant c_0 depending only on Ω. We conclude that

$$\begin{aligned} \|\mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;H^2)} &\leq C\varepsilon^{-1/2}, \\ \|\nabla r^\varepsilon\|_{L^\infty(0,t^*;L^2)} &\leq C\varepsilon^{1/2}, \end{aligned} \tag{6.9}$$

giving the convergence of the pressure as in (5.4) and control on the blowup of $\Delta \mathbf{w}_v^\varepsilon$.

Because full spatial derivatives (specifically, those in the normal direction) of $\mathbf{f}^\varepsilon$ bring in a factor of $\varepsilon^{-1/2}$, we cannot apply Proposition 2.2 in Chapter I of [50] to higher derivatives to obtain stronger convergence of the pressure. (This is a limitation in obtaining stronger convergence of the velocity and temperature as well.)

It is interesting, however, that if we take the divergence of both sides of (6.1), we obtain

$$\Delta(p^\varepsilon - p^0) = \operatorname{div}(\mathbf{k} w_T^\varepsilon) = \gamma \nabla w_T^\varepsilon \cdot \mathbf{k}.$$

Then from (5.3),

$$\left\|\Delta(p^\varepsilon - p^0)\right\|_{L^2(0,t^*;L^2)} \leq C t^{1/2} \varepsilon^{1/2}.$$

This bound does not, however, give $p^\varepsilon \to p^0$ in $L^2(0,t^*;H^2)$. To obtain such a convergence, we would need to have convergence of the boundary values for p^ε and p^0. The problem with this latter approach is that while

$$\nabla p^0 \cdot \mathbf{n} = \gamma T^0 \mathbf{k} \cdot \mathbf{n} = \pm \gamma f \tag{6.10}$$

on the upper and lower boundaries,

$$\nabla p^\varepsilon \cdot \mathbf{n} = \pm \gamma f + \varepsilon \Delta \mathbf{v}^\varepsilon \cdot \mathbf{n}, \tag{6.11}$$

which brings in $\Delta \mathbf{v}^\varepsilon$, albeit with an ε in front of it.

7. Uniform convergence of $\mathbf{w}_v^\varepsilon$

In this section we prove Theorem 5.2. In Section 7.1 we obtain convergence of w_T^ε in $L^\infty([0,t];H^1)$ and use it in Section 7.2, along with the anisotropic embedding inequality of Lemma 7.2 that two of the authors developed earlier [52, 58], to obtain convergence of $\mathbf{w}_v^\varepsilon$ uniformly in time and space.

7.1. **Improved convergence of w_T^ε.**

Using the two-dimensional Agmon's inequality with (5.3), (6.9), we have

$$\begin{aligned}
\|\mathbf{w}_v^\varepsilon\|_{L^\infty((0,t^*)\times\Omega)} &\le C \|\mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^2)}^{1/2} \|\mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;H^2)}^{1/2} \\
&\le C \left(\varepsilon^{1/2}\right)^{1/2} \left(\varepsilon^{-1/2}\right)^{1/2} \le C.
\end{aligned}$$

Since $\mathbf{v}^0$ and $\boldsymbol{\theta}^\varepsilon$ are uniformly bounded in $L^\infty((0,t^*)\times\Omega)$, it follows that

$$\|\mathbf{v}^\varepsilon\|_{L^\infty((0,t^*)\times\Omega)} \le C. \tag{7.1}$$

Multiplying (6.3) by $\partial_t w_T^\varepsilon$ and integrating over the domain gives

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\nabla w_T^\varepsilon\|^2 + \|\partial_t w_T^\varepsilon\|^2 \\
&\qquad = -(\mathbf{v}^\varepsilon \cdot \nabla w_T^\varepsilon, \partial_t w_T^\varepsilon) - (\mathbf{w}_v^\varepsilon \cdot \nabla T^0, \partial_t w_T^\varepsilon) - (\boldsymbol{\theta}^\varepsilon \cdot \nabla T^0, \partial_t w_T^\varepsilon).
\end{aligned}$$

Integrating in time,

$$\begin{aligned}
&\frac{1}{2}\|\nabla w_T^\varepsilon(t)\|^2 + \int_0^t \|\partial_t w_T^\varepsilon\|^2 \\
&\quad \le \int_0^t \left\{ |(\mathbf{v}^\varepsilon \cdot \nabla w_T^\varepsilon, \partial_t w_T^\varepsilon)| + \left|(\mathbf{w}_v^\varepsilon \cdot \nabla T^0, \partial_t w_T^\varepsilon)\right| \right. \\
&\qquad\qquad \left. + \left|(\boldsymbol{\theta}^\varepsilon \cdot \nabla T^0, \partial_t w_T^\varepsilon)\right| \right\}.
\end{aligned} \tag{7.2}$$

In this integration we used the fact that T^ε lies in $C([0,t^*];H^1)$ by Theorem 2.1. Since this is also true of T^0 and $\boldsymbol{\theta}^\varepsilon$, it follows that $\nabla w_T^\varepsilon(t) \to \nabla w_T^\varepsilon(0) = 0$ in $L^2(\Omega)$ as $t \to 0$.

Using Hölder's and Young's inequalities,

$$\begin{aligned}
\int_0^t |(\mathbf{v}^\varepsilon \cdot \nabla w_T^\varepsilon, \partial_t w_T^\varepsilon)| &\le \int_0^t \|\mathbf{v}^\varepsilon\|_{L^\infty} \|\nabla w_T^\varepsilon\| \|\partial_t w_T^\varepsilon\| \\
&\le \frac{3}{2}\int_0^t \|\mathbf{v}^\varepsilon\|_{L^\infty}^2 \|\nabla w_T^\varepsilon\|^2 + \frac{1}{6}\int_0^t \|\partial_t w_T^\varepsilon\|^2.
\end{aligned} \tag{7.3}$$

Similarly,

$$\begin{aligned}\int_0^t \left|(\mathbf{w}_v^\varepsilon \cdot \nabla T^0, \partial_t w_T^\varepsilon)\right| &\le \int_0^t \left\|\nabla T^0\right\|_{L^\infty} \|\mathbf{w}_v^\varepsilon\| \, \|\partial_t w_T^\varepsilon\| \\ &\le C \int_0^t \|\mathbf{w}_v^\varepsilon\| \, \|\partial_t w_T^\varepsilon\| \\ &\le C \int_0^t \|\mathbf{w}_v^\varepsilon\|^2 + \frac{1}{6}\int_0^t \|\partial_t w_T^\varepsilon\|^2 \le C\varepsilon + \frac{1}{6}\int_0^t \|\partial_t w_T^\varepsilon\|^2 \,,\end{aligned} \tag{7.4}$$

and

$$\begin{aligned}\int_0^t \left|(\boldsymbol{\theta}^\varepsilon \cdot \nabla T^0, \partial_t w_T^\varepsilon)\right| &\le \int_0^t \left\|\nabla T^0\right\|_{L^\infty} \|\boldsymbol{\theta}^\varepsilon\| \, \|\partial_t w_T^\varepsilon\| \\ &\le C \int_0^t \|\boldsymbol{\theta}^\varepsilon\| \, \|\partial_t w_T^\varepsilon\| \\ &\le C \int_0^t \|\boldsymbol{\theta}^\varepsilon\|^2 + \frac{1}{6}\int_0^t \|\partial_t w_T^\varepsilon\|^2 \le C\varepsilon^{1/2} + \frac{1}{6}\int_0^t \|\partial_t w_T^\varepsilon\|^2 \,,\end{aligned}$$

where we used (5.3), (6.8).

Collecting these bounds gives

$$\|\nabla w_T^\varepsilon(t)\|^2 + \int_0^t \|\partial_t w_T^\varepsilon\|^2 \le C\varepsilon^{1/2} + \frac{3}{2}\int_0^t \|\mathbf{v}^\varepsilon\|_{L^\infty}^2 \|\nabla w_T^\varepsilon\|^2 \,.$$

Applying Gronwall's inequality and using (7.1) gives (5.5). The first of the bounds in (5.5) improves on the final bound in (5.3) (though the result is specific to two dimensions).

7.2. **Convergence of $\mathbf{w}_v^\varepsilon$.** Taking ∂_x of (6.2), multiplying by $\partial_x \mathbf{w}_v^\varepsilon$, and integrating over the domain gives

$$\begin{aligned}-(\varepsilon\Delta\partial_x \mathbf{w}_v^\varepsilon, \partial_x \mathbf{w}_v^\varepsilon) + (\partial_x \mathbf{w}_v^\varepsilon, \partial_x \mathbf{w}_v^\varepsilon) + (\nabla \partial_x r^\varepsilon, \partial_x \mathbf{w}_v^\varepsilon) \\ = (\gamma \mathbf{k} \partial_x w_T^\varepsilon, \partial_x \mathbf{w}_v^\varepsilon) + (\varepsilon \partial_x \Delta \mathbf{v}^0, \partial_x \mathbf{w}_v^\varepsilon) - (\partial_x \mathbf{f}^\varepsilon, \partial_x \mathbf{w}_v^\varepsilon).\end{aligned}$$

Integrating by parts,

$$\begin{aligned}&\varepsilon \|\nabla \partial_x \mathbf{w}_v^\varepsilon\|^2 + \|\partial_x \mathbf{w}_v^\varepsilon\|^2 \\ &\qquad \le \gamma \|\nabla w_T^\varepsilon\| \, \|\partial_x \mathbf{w}_v^\varepsilon\| + \varepsilon \|\mathbf{v}^0\|_{H^3} \|\partial_x \mathbf{w}_v^\varepsilon\| + \|\partial_x \mathbf{f}^\varepsilon\| \, \|\partial_x \mathbf{w}_v^\varepsilon\| \\ &\qquad \le C\left(\|\nabla w_T^\varepsilon\| + \varepsilon^{1/2}\right) \|\partial_x \mathbf{w}_v^\varepsilon\| \,.\end{aligned}$$

In the last inequality we used (5.2), (4.3).

Hence, using Young's inequality,

$$2\varepsilon \|\nabla \partial_x \mathbf{w}_v^\varepsilon\|^2 + \|\partial_x \mathbf{w}_v^\varepsilon\|^2 \le C\varepsilon + C \|\nabla w_T^\varepsilon\|^2 \,.$$

We conclude, using (5.5), that

$$\begin{aligned}\|\partial_x \mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^2)} &\le C\varepsilon^{1/4}, \\ \|\partial_x \mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;H^1)} &\le C\varepsilon^{-1/4}.\end{aligned} \tag{7.5}$$

Applying the anisotropic imbedding Lemma 7.2, we obtain

$$\begin{aligned}
\|\mathbf{w}_v^\varepsilon\|_{L^\infty((0,t^*)\times\Omega)} &\le C(\|\mathbf{w}_v^\varepsilon\|^{1/2}_{L^\infty(0,t^*;L^2)}\|\mathbf{w}_v^\varepsilon\|^{1/2}_{L^\infty(0,t^*;H^1)} \\
&\quad + \|\mathbf{w}_v^\varepsilon\|^{1/2}_{L^\infty(0,t^*;H^1)}\|\partial_x\mathbf{w}_v^\varepsilon\|^{1/2}_{L^\infty(0,t^*;L^2)} \\
&\quad + \|\mathbf{w}_v^\varepsilon\|^{1/2}_{L^\infty(0,t^*;L^2)}\|\nabla\partial_x\mathbf{w}_v^\varepsilon\|^{1/2}_{L^\infty(0,t^*;L^2)}) \\
&\le C\left(\left(\varepsilon^{\frac{1}{2}}\right)^{\frac{1}{2}} + \left(\varepsilon^{1/4}\right)^{1/2} + \left(\varepsilon^{1/2}\varepsilon^{-1/4}\right)^{1/2}\right) \\
&\le C\varepsilon^{1/8},
\end{aligned}$$

where we used (5.3), (7.5). This gives (5.6).

Remark 7.1. Heuristically, we suspect that the rate of convergence in (5.6) should be $\sqrt{\epsilon}$. Therefore the result we have derived here is sub-optimal. Optimal convergence rates can be derived via higher order correctors. There are two possibly different avenues in pursuing higher order correctors. One method is to follow the approach that we adopted here and look for higher order correctors that correct the error introduced by the truncation as well. A more systematic alternative approach is perhaps to consider the following modified Prandtl type equation (only the leading order corrector $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3)$ is presented here) that satisfies the desired boundary condition exactly:

$$\left\{\begin{aligned}
-\varepsilon\frac{\partial^2\eta_j}{\partial z^2} + \eta_j + \partial_j q &= 0, j = 1, 2, \\
\operatorname{div}\boldsymbol{\eta} &= 0, \\
\boldsymbol{\eta}|_{z=0,1} &= -\mathbf{v}^0|_{z=0,1}, \\
\partial_x\int_0^1 \eta_1(x,y,z)\,dz + \partial_y\int_0^1 \eta_2(x,y,z)\,dz &= 0 \text{ for all } x, y.
\end{aligned}\right. \tag{7.6}$$

This and other related topics will be studied elsewhere.

To state the anisotropic embedding inequality employed above, we define

$$\mathcal{K} = \{g = g(z) \colon g \in L^2(0,1)\}$$

which is a closed subspace of $L^2(\Omega)$. ($\mathcal{K}$ coincides with all elements of $L^2(\Omega)$ that are independent of x, or those elements in $L^2(\Omega)$ whose k^{th} Fourier coefficients in x are identically zero for $k \neq 0$.) Let $\mathcal{K}^\perp$ be the orthogonal complement of $\mathcal{K}$ in L^2. $\mathcal{K}^\perp$ is exactly all those elements in $L^2(\Omega)$ whose 0^{th} Fourier coefficient in x is identically zero (or zero horizontal average).

We first recall the following lemma which is proved in [52] (Remark 4.2):

Lemma 7.2. *For all u in $H^1_{0,per} \cap \mathcal{K}^\perp$,*

$$\|u\|_{L^\infty(\Omega)} \le C\|\partial_z u\|^{1/2}_{L^2}\|\partial_x u\|^{1/2}_{L^2} + C\|u\|^{1/2}_{L^2}\|\partial_x\partial_z u\|^{1/2}_{L^2},$$

where one or both sides of the inequality could be infinite.

This anisotropic embedding inequality is applicable to the velocity field since all components of our velocity field $\mathbf{v}^\varepsilon, \mathbf{v}^0$ belong to the space $\mathcal{K}^\perp$. Indeed, the horizontal velocity equations in (1.4) and (1.2) imply that the horizontal velocity belong to $\mathcal{K}^\perp$ (for the Darcy-Brinkman-Boussinesq (1.2) we need to utilise the no-slip boundary condition). Then the incompressibility condition together with the no penetration boundary condition implies that the vertical velocity also belongs to $\mathcal{K}^\perp$.

The above anisotropic embedding can be easily generalised to cover the case where the average in the horizontal direction may not vanish. For this purpose, we observe that the orthogonal projection in L^2 onto $\mathcal{K}$ is explicitly given by

$$(P_{\mathcal{K}}u)(z) = \frac{1}{2\pi}\int_0^{2\pi} u(x,z)\,dx.$$

It is then easy to verify that this projection is an orthogonal projection in various subspaces of $H^1_{0,per}$ as well, and hence we have, with the help of Fourier representation in x if necessary,

$$\begin{aligned}
\|P_{\mathcal{K}}u\|_{L^2} &\leq \|u\|_{L^2}\,,\\
\|u-P_{\mathcal{K}}u\|_{L^2} &\leq \|u\|_{L^2}\,,\\
\|\partial_z P_{\mathcal{K}}u\|_{L^2} &\leq \|\partial_z u\|_{L^2}\,,\\
\|\partial_z(u-P_{\mathcal{K}}u)\|_{L^2} &\leq \|\partial_z u\|_{L^2}\,.
\end{aligned}$$

Combining these estimates, the lemma above, and the triangle inequality we deduce

Corollary 7.3. *For all $u \in H^1_{0,per}(\Omega)$*

$$\|u\|_{L^\infty(\Omega)} \leq C(\|u\|_{L^2}^{\frac{1}{2}}\|\partial_z u\|_{L^2}^{\frac{1}{2}} + \|\partial_z u\|_{L^2}^{1/2}\|\partial_x u\|_{L^2}^{1/2} + \|u\|_{L^2}^{1/2}\|\partial_x\partial_z u\|_{L^2}^{1/2}),$$

where one or both sides of the inequality could be infinite.

Proof. We first notice that for $u \in H^1_{0,per}$, $P_{\mathcal{K}}u \in \mathcal{K}\cap H^1_{0,per}$, and hence $u - P_{\mathcal{K}}u \in \mathcal{K}^\perp \cap H^1_{0,per}$. Therefore

$$\begin{aligned}
\|u\|_{L^\infty(\Omega)} &\leq \|P_{\mathcal{K}}u\|_{L^\infty(\Omega)} + \|u-P_{\mathcal{K}}u\|_{L^\infty(\Omega)}\\
&\leq C\,\|P_{\mathcal{K}}u\|_{L^2}^{\frac{1}{2}}\|\partial_z P_{\mathcal{K}}u\|_{L^2}^{\frac{1}{2}}\\
&\quad +C(\|\partial_z(u-P_{\mathcal{K}}u)\|_{L^2}^{1/2}\|\partial_x(u-P_{\mathcal{K}}u)\|_{L^2}^{1/2}\\
&\quad +\|u-P_{\mathcal{K}}u\|_{L^2}^{1/2}\|\partial_x\partial_z(u-P_{\mathcal{K}}u)\|_{L^2}^{1/2})\\
&\leq C(\|u\|_{L^2}^{\frac{1}{2}}\|\partial_z u\|_{L^2}^{\frac{1}{2}} + \|\partial_z u\|_{L^2}^{1/2}\|\partial_x u\|_{L^2}^{1/2} + \|u\|_{L^2}^{1/2}\|\partial_x\partial_z u\|_{L^2}^{1/2}).
\end{aligned}$$

□

8. Uniform convergence of w_T^ε

In this section we prove the uniform convergence of the temperature field (5.7). The idea is very much the same as the proof of the convergence of the velocity field: anisotropic embedding plus estimates on tangential derivatives.

We recall that $\|\cdot\|$ denotes the L^2 norm unless otherwise specified.

For this purpose we differentiate (6.2) and we arrive at the following equation

$$-\varepsilon\Delta\partial_x\mathbf{w}_v^\varepsilon+\partial_x\mathbf{w}_v^\varepsilon+\nabla\partial_x r^\varepsilon=\gamma\mathbf{k}\partial_x w_T^\varepsilon+\varepsilon\Delta\partial_x\mathbf{v}^0-\partial_x\mathbf{f}^\varepsilon. \tag{8.1}$$

Multiplying this equation by $\partial_x\mathbf{w}_v^\varepsilon$, integrating over the domain and utlizing the estimate on $\mathbf{f}^\varepsilon$ we deduce

$$\varepsilon\left\|\nabla\partial_x\mathbf{w}_v^\varepsilon\right\|^2+\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|^2 \quad\leq\quad C\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|(\left\|\partial_x w_T^\varepsilon\right\|+\varepsilon^{\frac{1}{2}}), \tag{8.2}$$

much as in Section 7.2. After utilizing the $L^2(H^1)$ estimates on w_T^ε, this implies that,

$$\begin{aligned}\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|_{L^2(0,t^*;L^2)} &\leq C\varepsilon^{\frac{1}{2}}, &(8.3)\\ \left\|\partial_x\mathbf{w}_v^\varepsilon\right\|_{L^2(0,t^*;H^1)} &\leq C. &(8.4)\end{aligned}$$

Utilizing the first estimate above in equation (8.1) together with the estimates on $\mathbf{f}^\varepsilon$, the $L^2(H^1)$ estimate on w_T^ε and elliptic regularity for Stokes operator we have

$$\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|_{L^2(0,t^*;H^2)}\leq C\varepsilon^{-\frac{1}{2}}. \tag{8.5}$$

This further implies, when combined with the $L^2(L^2)$ estimate on $\partial_x\mathbf{w}_v^\varepsilon$ and Agmon's inequality

$$\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|_{L^2(0,t^*;L^\infty)}\leq C(\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|_{L^2(0,t^*;H^2)}\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|_{L^2(0,t^*;L^2)})^{\frac{1}{2}}\leq C. \tag{8.6}$$

Combining this with the estimate on the corrector $\boldsymbol{\theta}^\varepsilon$ we have

$$\begin{aligned}&\left\|\partial_x\mathbf{v}^\varepsilon\right\|_{L^2(0,t^*;L^\infty)}\\ \leq\quad&\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|_{L^2(0,t^*;L^\infty)}+\left\|\partial_x\mathbf{v}^0\right\|_{L^2(0,t^*;L^\infty)}+\left\|\partial_x\boldsymbol{\theta}^\varepsilon\right\|_{L^2(0,t^*;L^\infty)}\leq C.\end{aligned} \tag{8.7}$$

Next we focus on the temperature equation (6.3). We differentiate (6.3) in x and we arrive at the following equation

$$\begin{aligned}\partial_t\partial_x w_T^\varepsilon-\Delta\partial_x w_T^\varepsilon \quad=\quad&-\partial_x\mathbf{v}^\varepsilon\cdot\nabla w_T^\varepsilon-\mathbf{v}^\varepsilon\cdot\nabla\partial_x w_T^\varepsilon-\partial_x\mathbf{w}_v^\varepsilon\cdot\nabla T^0 &(8.8)\\ &-\mathbf{w}_v^\varepsilon\cdot\partial_x\nabla T^0-\partial_x\boldsymbol{\theta}^\varepsilon\cdot\nabla T^0-\boldsymbol{\theta}^\varepsilon\cdot\nabla\partial_x T^0.\end{aligned}$$

Multiplying this equation by $\partial_x w_T^\varepsilon$ and integrating over the domain we deduce

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\|\partial_x w_T^\varepsilon\right\|^2+\left\|\partial_x\nabla w_T^\varepsilon\right\|^2\\ \leq\quad&\left\|\partial_x\mathbf{v}^\varepsilon\right\|_{L^\infty}\left\|\nabla\partial_x w_T^\varepsilon\right\|\left\|w_T^\varepsilon\right\|+C\left\|\partial_x w_T^\varepsilon\right\|(\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|+\left\|\mathbf{w}_v^\varepsilon\right\|)\\ &+C(\left\|\boldsymbol{\theta}_I^\varepsilon\right\|\left\|\partial_x w_T^\varepsilon\right\|+\left\|z\boldsymbol{\theta}_{B,0}^\varepsilon\right\|\left\|\partial_x w_T^\varepsilon/z\right\|+\left\|(1-z)\boldsymbol{\theta}_{B,1}^\varepsilon\right\|\left\|\partial_x w_T^\varepsilon/(1-z)\right\|)\\ &+C(\left\|\partial_x\boldsymbol{\theta}_I^\varepsilon\right\|\left\|\partial_x w_T^\varepsilon\right\|+\left\|z\partial_x\boldsymbol{\theta}_{B,0}^\varepsilon\right\|\left\|\partial_x w_T^\varepsilon/z\right\|\\ &+\left\|(1-z)\partial_x\boldsymbol{\theta}_{B,1}^\varepsilon\right\|\left\|\partial_x w_T^\varepsilon/(1-z)\right\|)\\ \leq\quad&\frac{1}{2}\left\|\partial_x\nabla w_T^\varepsilon\right\|^2+C(\left\|\partial_x w_T^\varepsilon\right\|^2+\varepsilon\left\|\partial_x\mathbf{v}^\varepsilon\right\|_{L^\infty}^2+\left\|\partial_x\mathbf{w}_v^\varepsilon\right\|^2+\varepsilon),\end{aligned}$$

where we have utilized the smoothness of the "inviscid" temperature T^0, applied Hölder and Young's inequality, performed integration by parts and utilized the incompressibility of the velocity field, used the decomposition of the corrector $\boldsymbol{\theta}^\varepsilon$ into an interior and boundary parts, Hardy's inequality, the $L^\infty(L^2)$ estimate on w_T^ε and $\mathbf{w}_v^\varepsilon$, and explicit estimates on the corrector $\boldsymbol{\theta}^\varepsilon$.

A simple application of Gronwall's inequality leads to

$$\|\partial_x w_T^\varepsilon\|_{L^\infty(0,t^*;L^2)} \leq C\varepsilon^{\frac{1}{2}}, \tag{8.9}$$

$$\|\partial_x w_T^\varepsilon\|_{L^2(0,t^*;H^1)} \leq C\varepsilon^{\frac{1}{2}}, \tag{8.10}$$

where we have utilized the $L^2(L^\infty)$ estimate on $\partial_x\mathbf{v}^\varepsilon$ and then $L^2(L^2)$ estimate on $\partial_x\mathbf{w}_v^\varepsilon$.

This $L^\infty(L^2)$ estimate on $\partial_x w_T^\varepsilon$, together with the energy estimate (8.2) gives us

$$\|\partial_x \mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^2)} \leq C\varepsilon^{\frac{1}{2}}, \tag{8.11}$$

$$\|\partial_x \mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;H^1)} \leq C. \tag{8.12}$$

We combine the $\|\partial_x \mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^2)}$ and $\|\partial_x w_T^\varepsilon\|_{L^\infty(0,t^*;L^2)}$ estimates, together with (8.1), explicit estimate on $\mathbf{f}^\varepsilon$ and elliptic regularity for the Stokes and we obtain

$$\|\partial_x \mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;H^2)} \leq C\varepsilon^{-\frac{1}{2}}. \tag{8.13}$$

This further implies, when combined with the $L^\infty(L^2)$ estimate on $\partial_x\mathbf{w}_v^\varepsilon$ and Agmon's inequality

$$\|\partial_x \mathbf{w}_v^\varepsilon\|_{L^\infty(0,t^*;L^\infty)} \leq C. \tag{8.14}$$

Combining this with the estimate on the corrector $\boldsymbol{\theta}^\varepsilon$ together with the smoothness assumption on $\mathbf{v}^0$ we find

$$\|\partial_x \mathbf{v}^\varepsilon\|_{L^\infty(0,t^*;L^\infty)} \leq C. \tag{8.15}$$

Next, we multiply (8.9) by $\partial_t\partial_x w_T^\varepsilon$ and integrate over the domain. We deduce, after utilizing the explicit formula for the corrector (boundary layer function) $\boldsymbol{\theta}^\varepsilon$, and the smoothness of the "inviscid" solution $(T^0, \mathbf{v}^0)$,

$$\begin{aligned}
& \|\partial_t\partial_x w_T^\varepsilon\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\partial_x w_T^\varepsilon\|^2 \\
\leq\ & \|\partial_t\partial_x w_T^\varepsilon\| \big(\|\partial_x\mathbf{v}^\varepsilon\|_{L^\infty}\|\nabla w_T^\varepsilon\| + \|\mathbf{v}^\varepsilon\|_{L^\infty}\|\nabla\partial_x w_T^\varepsilon\| \\
& + \left\|\nabla T^0\right\|_{L^\infty}\|\partial_x\mathbf{w}_v^\varepsilon\| + \left\|\partial_x\nabla T^0\right\|_{L^\infty}\|\mathbf{w}_v^\varepsilon\| \\
& + \left\|\nabla T^0\right\|_{L^\infty}\|\partial_x\boldsymbol{\theta}^\varepsilon\| + \left\|\partial_x\nabla T^0\right\|_{L^\infty}\|\boldsymbol{\theta}^\varepsilon\|\big) \\
\leq\ & \frac{3}{4}\|\partial_t\partial_x w_T^\varepsilon\|^2 + \|\partial_x\mathbf{v}^\varepsilon\|_{L^\infty}^2\|\nabla w_T^\varepsilon\|^2 + \|\mathbf{v}^\varepsilon\|_{L^\infty}^2\|\nabla\partial_x w_T^\varepsilon\|^2 \\
& + C(\|\partial_x\mathbf{w}_v^\varepsilon\|^2 + \|\mathbf{w}_v^\varepsilon\|^2 + \varepsilon^{\frac{1}{2}}).
\end{aligned}$$

Applying Gronwall's inequality and utilizing the unfiorm $L^\infty(L^\infty)$ estimates on $\mathbf{v}^\varepsilon$ and $\partial_x\mathbf{v}^\varepsilon$, the $L^2(L^2)$ estimates on $w_T^\varepsilon, \partial_x w_T^\varepsilon, \partial_x\mathbf{w}_v^\varepsilon, \mathbf{w}_v^\varepsilon$, we deduce

$$\|\partial_x w_T^\varepsilon\|_{L^\infty(0,t^*;H^1)} \leq C\varepsilon^{\frac{1}{4}}. \tag{8.16}$$

Combining this with the $L^\infty(L^2)$ estimates on $w_T^\varepsilon, \partial_x w_T^\varepsilon, \nabla w_T^\varepsilon$ as well as the anisotropic embedding presented in corollary (7.3), we have the desired uniform convergence estimate of the temperature as stated in (5.7). This ends the proof of Theorem 5.2.

Many other miscellaneous estimates can be obtained as well, and we will not elaborate on these minor improvements.

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Department of Mathematics, University of California, Riverside, 900 University Ave., Riverside, CA 92521

E-mail address: kelliher@math.ucr.edu

The Institute for Scientic Computing and Applied Mathematics, Indiana University, 831 East Third Street, Bloomington, Indiana 47405, U.S.A

E-mail address: temam@indiana.edu

208 James J. Love Building, Department of Mathematics, Florida State University, 1017 Academic Way, Tallahassee, FL 32306-4510

E-mail address: wxm@math.fsu.edu