CONFORMAL TILINGS II:
LOCAL ISOMORPHISM, HIERARCHY, AND CONFORMAL TYPE

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Abstract. This is the second in a series of papers on conformal tilings. The overriding themes of this paper are local isomorphisms, hierarchical structures, and the type problem in the context of conformally regular tilings, a class of tilings introduced first by the authors in 1997 with an example of a conformally regular pentagonal tiling of the plane [2]. We prove that when a conformal tiling has a combinatorial hierarchy for which the subdivision operator is expansive and conformal, then the tiling is parabolic and tiles the complex plane $\mathbb{C}$. This is used to examine type across local isomorphism classes of tilings and to show that any conformal tiling of bounded degree that is locally isomorphic to a tiling obtained as an expansion complex of a shrinking and dihedrally symmetric subdivision operator with one polygonal type is parabolic.

Introduction

We continue the development of conformal tilings begun in earnest in [4]. This work draws its inspiration from two sources, the first from Cannon, Floyd, and Parry's articulation of finite subdivision rules in a series of papers over the last two decades, and the second from the study of aperiodic hierarchical tilings begun in the seventies, and developed into a mature discipline over the last decade and a half. In the first paper of the series [4], the authors laid out a general development of **conformal tilings**, tilings by conformally regular curvilinear polygons with a reflective structure. There we uncovered the beautiful combinatorial and conformal substructures that encode the dual and quad tilings associated to a conformal tiling, and we examined conditions that guarantee that a *combinatorial* subdivision of a tiling yields a *conformal* subdivision of the tiling. All of this builds upon the earlier work of the authors in [2] and [3]. The results needed in the present work from the presentation of [4] are reviewed, repackaged, and expanded in the first section of this paper, so that this paper is self-contained.

Recently connections between conformal tilings and traditional aperiodic tilings have emerged, first through the doctoral dissertation of Maria Ramirez-Solano [9]. This classical discipline of aperiodic hierarchical tilings exerts its influence immediately. In that discipline, a tiling T determines a tiling space, the continuous hull of the tiling, in which every tiling locally isomorphic to T has natural residence and on which lives a canonical dynamical

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system. The canonical transversal is a Cantor set slice of the hull that serves as the local isomorphism class of pointed tilings. A subsequent paper will detail a parallel theory in the setting of conformal tilings, but for now we borrow from the canonical transversal and in a standard way describe a metric space of planar polygonal complexes and, for a given complex K, a local isomorphism class of rooted complexes, denoted as $\mathsf{LI}(K)$. This is developed in the second section to an extent beyond our immediate use, but serves to place conformal tilings in a fairly general setting where one can illuminate some properties of K that will mirror those in the traditional setting of aperiodic tilings. In particular, we will see the importance of finite local complexity and repetitiveness, two important features of aperiodic tilings, in establishing pertinent facts about conformal tilings. Though the broad ideas of this section promise to be old hat to the traditional tiling experts, the particulars will be new, most certainly to those in discrete conformal geometry, and for completeness are developed in some detail.

The overriding theme of the remainder of the paper centers on the type problem in the setting of conformal tilings. The classical **type problem** is that of determining whether a concretely given, non-compact, simply connected Riemann surface is conformally equivalent to the complex plane $\mathbb{C}$ or the unit disk $\mathbb{D}$. In the former case, the surface is said to have **parabolic type**, and in the latter, **hyperbolic type**. The terminology is descriptive as the plane $\mathbb{C}$ with its usual metric serves as a standard model of parabolic, or Euclidean, geometry, while the disk $\mathbb{D}$ with its Poincaré metric serves as a standard model of hyperbolic, or Lobachevski, geometry. In the present context, the type problem immediately arises when a planar polygonal complex is endowed with a piecewise flat metric in which each polygon is realized as a convex Euclidean polygon. Such a metric determines a natural maximal complex atlas, presenting one with a concretely given, non-compact, simply connected Riemann surface. Is this surface parabolic or hyperbolic? This construction produces a locally finite tiling by curvilinear polygons filling the complex plane $\mathbb{C}$ when the type is parabolic, and filling the disk $\mathbb{D}$ when the type is hyperbolic.

To focus the lens a bit, we start with a combinatorial object K, a decomposition of the plane into combinatorial polygons, and our interest is in conformal structures on K in which each polygon is realized as a "regular" polygon. It turns out that among the uncountably many ways to endow K with a conformal structure so that each polygon is realized as a conformally regular polygon, precisely one of these—the one derived from the β-equilateral structure of the first paper [4]—yields a tiling $T = T_K$ that contains within its detailed structure a reflective tiling $T^\dagger$ in the pattern of the dual complex $K^\dagger$, as well as a reflective tiling $T^\diamond$ in the pattern of an associated 4-gon complex $K^\diamond$. The β-equilateral structure was introduced originally by the authors in [2] in building their pentagonal tiling, used further in [3] in the setting of their study of dessins d'enfants, and modified by Cannon, Floyd, and Parry in [5, 6] in their study of expansion complexes of finite subdivision rules. Our attention is focused on the conformal type of the Riemann surface that carries this conformal structure determined by a given planar polygonal complex, and the chief tool developed is inspired by examples of traditional aperiodic tilings and the hierarchical structures one perceives in examining them closely.

There seems to be no formal definition of the term *hierarchical tiling*, and we certainly refrain from suggesting one that would fit all occasions of its use in the traditional tiling community, or even in this new world of conformal tilings. Nonetheless in the third section of this paper, we give a precise definition of what it means for a planar polygonal complex K to exhibit a **combinatorial hierarchy**. We quickly restrict our attention to hierarchies whose subdivisions are rotationally symmetric and we formalize this by the use of a rotationally symmetric combinatorial subdivision operator τ. We then describe what it means for a conformal tiling to exhibit a **conformal hierarchy**, and we ask when does the existence of a *combinatorial* hierarchy for K imply the existence of a *conformal* hierarchy for the conformal tiling T_K. We find that this implication prevails when the simple subdivision operator τ is not only rotationally symmetric, but in addition is dihedrally symmetric, which guarantees that the reflective structure of T_K is compatible with the reflective structure of $T_{\tau K}$. After defining and examining an important class of hierarchies known as **expansive** hierarchies, we prove that a conformal tiling of bounded degree that exhibits an expansive conformal hierarchy must be parabolic and, therefore, find its existence in the complex plane $\mathbb{C}$ rather than in the Poincaré disk $\mathbb{D}$. In addition, a corollary implies that type is constantly parabolic across the local isomorphism class $\mathsf{LI}(K)$ when the hierarchy is *strongly* expansive, a strengthening of the expansive property. In this section, we also give a simple property—the **shrinking** property—on a subdivision operator that guarantees that the hierarchy is strongly expansive. We close this section with an examination of conformal hierarchies that have a **supersymmetry**, a loxodromic Möbius transformation that generates the conformal hierarchy.

The authors introduced the notion of **expansion complex** for the pentagonal subdivision rule in their construction of the pentagonal tiling of [2]. The pentagonal subdivision rule along with many other finite subdivision rules had been introduced by Cannon in his study of what is now termed the **Cannon Conjecture**, that every negatively curved group with 2-sphere Gromov boundary is, essentially, a cocompact Kleinian group. Cannon, Floyd, and Parry in two papers [5, 6] formalized the notion of expansion complex, further refined and developed the idea of finite subdivision rule, and worked out many specific examples, including one of importance to us that is examined in a later paper in this series. In the fourth and final section of this paper, we define expansion complexes in the setting of planar n-gon complexes and a rotationally symmetric subdivision of an n-gon into n-gons. This is inspired by traditional aperiodic tilings as well as the work of Cannon, Floyd, and Parry on one-tile rotationally invariant finite subdivision rules. We should point out, though, that our definition appears in a broader context than that of Cannon, Floyd, and Parry, which exploits their extensive machinery of finite subdivision rules in all their subtlety, machinery to a large extent that we avoid. Our interest in expansion complexes is that they provide a stable of ready-made examples of conformal tilings that exhibit three important properties, namely, combinatorial finite local complexity, combinatorial repetitiveness, and combinatorial hierarchy, that traditional aperiodic tilings exhibit in a stronger sense—not combinatorial but in a rigid Euclidean sense. If the subdivision τ that generates the expansion complex is shrinking and, in addition, is dihedrally symmetric, then it is conformal and the combinatorial hierarchy of the complex K translates to an expansive conformal hierarchy of the tiling T_K. In particular, applications of the main theorems of the third section imply that any conformal tiling locally isomorphic to a tiling obtained from the expansion complex K of a dihedrally symmetric and shrinking subdivision operator with a single polygonal type is parabolic, so type is constantly parabolic across the local isomorphism class of K. This answers positively a question of Maria Ramirez-Solano as to whether any conformal tiling combinatorially locally isomorphic to the conformally regular pentagonal tiling of [2] is parabolic.

The final topic—**supersymmetric expansion complexes**—is explored in some detail to close the fourth section, as well as the paper. These are those expansion complexes that exhibit a combinatorial version of the conformal supersymmetry introduced at the end of the third section. These include the expansion complexes defined and studied by Cannon, Floyd, and Parry that give rise to conformal expansion maps, but also include generalizations of theirs. The exploration proceeds through the lens of an action $\hat{\tau}$ induced by the subdivision operator τ that is defined on the local isomorphism class (K) of the expansion complex K. Our study attempts to understand how common are the supersymmetric expansion complexes among all the expansion complexes for τ and pays off by discovering how to identify and construct all possible supersymmetric ones, which turn out to be quite rare—a countable family among the uncountable set (K)—whenever K is plural.

Acknowledgement. This paper arose from thinking about the type problem in the context of locally isomorphic tilings, a concept new to the authors when introduced by Maria Ramirez-Solano on a visit to Dane Mayhook and the two authors at the University of Tennessee in May of 2012. It was pleasing to learn that the pentagonal tiling that we had introduced in 1997 had garnered the interest of a small group in the tiling community and that Maria's doctoral thesis would dissect the pentagonal tiling into finer slices than the architects of the example ever imagined possible. The initial impetus for this work arose from the invitation by Maria, and by her co-organizer Jean Savinien, to the workshop, *Non standard hierarchical tilings*, held at the Center for Symmetry and Deformation at the University of Copenhagen from 20–22 September, 2012. This may be counted as the beginning of what is expected to be a very fruitful conversation between the traditional discipline of aperiodic hierarchical tilings and that of discrete conformal geometry and nonstandard conformal tilings. The authors extend their thanks to Maria and Jean, from whom they have learned a great deal about traditional aperiodic tilings, and to Natalie Frank, Chaim Goodman-Strauss, and Lorenzo Sadun, traditional aperiodic tiling experts whose work and conversations have been an inspiration to the authors.

Contents

1. Conformal Tilings

We review in this section only the basic material from the paper [4] that is needed to discuss the conformal type problem for conformal tilings. A fuller explanation may be found in that paper where, in particular, the subtleties of conformal tilings are examined and a defense for choosing the β-equilateral structure as the conformal structure most deserving of study and development is launched, we think successfully. The ingredients of the theory that we choose to review are planar polygonal complexes, the β-equilateral structure on these complexes, the corresponding conformally regular and reflective polygonal tilings of either the plane $\mathbb{C}$ or the disk $\mathbb{D}$, associated 4-gon complexes and their quad tilings, and combinatorial and conformal subdivision rules. In particular, we review just enough so that the following theorem, proved at the end of this section and used in the proof of the corollaries of the main theorem of Section 3, is understandable to the reader.

Theorem 1.1. *If τ is a dihedrally symmetric simple subdivision operator and K is a planar polygonal complex, then τK, the subdivision of K wherein each face of K is subdivided by τ, is a conformal subdivision of K, so that the conformal tiling $T_{\tau K}$ may be realized as a dihedrally symmetric, conformal subdivision of the conformal tiling T_K.*

A brief caution is in order for those who are familiar with paper [4]. There we abused notation and used the same symbol to denote a tiling with several associated structures: the combinatorial complex underlying the tiling, the piecewise Euclidean metric space derived from this combinatorial tiling, the resulting Riemann surface, and, finally, the conformal tiling itself. In this paper we find that we need to take more care in delineating these structures and, accordingly, we refine the notation of paper [4] and use differing notation for the various structures: K for the combinatorial complex, $|K^\beta|_{\mathrm{eq}}$ for the metric space, $\mathcal{S}_K$ for the Riemann surface, and T_K for the tiling. We may consider a *maximal* tiling $\mathcal{T}$ of [4] as a 4-tuple $\mathcal{T} = (K, |K^\beta|_{\mathrm{eq}}, \mathcal{S}_K, T_K)$.[1]

To describe a conformal tiling, we begin with the combinatorial object that underlies the tiling. Informally, a **planar polygonal complex** is a decomposition of the plane into curvilinear polygons that meet along vertices and edges of their boundaries. More precisely, a planar polygonal complex is an oriented 2-dimensional regular CW-decomposition K of the plane whose attaching maps are homeomorphisms. The 0-skeleton $K^{(0)}$ is a countable discrete collection of vertices, each edge of the 1-skeleton spans two distinct vertices, and each face is the image of an attaching map that is a homeomorphism of the 2-cell boundary onto a finite union of edges forming a cycle of length greater than or equal to two. The complex is locally finite, meaning, of course, at most finitely many edges emanate from any given vertex. While in the first paper [4] these complexes are defined in greater generality than here—for example, the complexes may be defined on more general surfaces and may allow loops, dangling edges, and non-embedded polygons—in this paper we restrict our attention to planar complexes where the polygons are embedded. If the boundary of a face f has n edges, we say that f is a **combinatorial n-gon** and we define the **polygonal type** of f to be n. The complex K decomposes the plane into combinatorial polygons of possibly varying polygonal types. Each edge lies in exactly two faces and two faces meet along a union of common boundary edges and vertices, not necessarily contiguous to one another. We find it convenient to think of the cells of K—the edges e and the faces f—as closed cells, and when we mean the corresponding open cell we will use the notation e° and f°. The **barycentric subdivision** of K, denoted as K^β, is defined by introducing a new vertex to each open face and one to each open edge, and then adding new edges connecting the added **face barycenter** to both the original boundary vertices of that face as well as to the added **edge barycenters**. This subdivides each edge into two edges and each face f of K of polygonal type n into $2n$ triangles with a common vertex in the open face f°. If c is the vertex added to the open face f°, a typical triangle of K^β has vertices a, b, and c, where b is an edge barycenter of an edge e of f and a is a vertex of e. This complex K^β is a simplicial decomposition of the plane into combinatorial triangles. The polygonal complex K has **degree at most** d, where d is a positive integer, if each face of K has polygonal type at most d and at most d edges meet at any vertex of K, and K has **bounded degree** if it has degree at most d for some value of d. The **degree** of the planar polygonal complex K of bounded degree is the smallest integer d for which K has degree

[1]The conformal tilings of [4] are even more general in that they also carry with them a conformal tiling map that allows for a much greater variety of concrete realizations of conformal tilings with fixed combinatorics. Essentially in this paper, we restrict our attention only to the maximal tilings of [4] as it is this setting that encompasses the question of conformal type.

at most d. Note that the degree d is greater than or equal to 4 for any planar polygonal complex of bounded degree. For a fixed positive integer $n \geq 3$, the complex K is a **planar n-gon complex** if each face of K is a combinatorial n-gon. We are interested particularly in planar n-gon complexes of bounded degree, but this restriction is not enforced strictly until Section 4.

We now endow K with the **β-equilateral metric** where first K has been barycentrically subdivided to yield the triangular complex K^β, and then each edge has been given a unit length and each triangular face of K^β has been identified as a unit equilateral triangle. Observe that two faces meeting along an edge e isometrically reflect across e to one another. The resulting metic space $|K^\beta|_{\text{eq}}$ is piecewise flat with cone type singularities at the vertices of K^β. There is a canonical maximal complex atlas $\mathcal{A}$, called the **β-equilateral conformal structure**, where the cone type singularities are resolved by local power mappings that define vertex charts, and edge charts are defined by mapping two contiguous faces isometrically to the union of two equilateral triangles in the plane meeting along a common unit edge. The resulting surface $\mathcal{S}_K = (|K^\beta|_{\text{eq}}, \mathcal{A})$ is a non-compact, simply connected Riemann surface and the classical Uniformization Theorem implies that it is conformally equivalent to one of the plane $\mathbb{C}$ or the disk $\mathbb{D}$. We refer the reader to the first paper [4] for details of this procedure. The conformal isomorphism of $\mathcal{S}_K$ to either $\mathbb{C}$ or $\mathbb{D}$ is unique up to Möbius equivalence. The set T_K that consists of the images of the polygonal faces of the original complex K under such an isomorphism provides a **conformal tiling** of the appropriate plane—Euclidean or hyperbolic—by curvilinear polygons that are conformally regular and meet in the pattern of the complex K. We of course do not distinguish between tilings that are conformally equivalent, and so we consider that this procedure identifies a unique tiling T_K associated to the planar polygonal complex K. The tiling T_K is locally finite in its geometry, either $\mathbb{C}$ or $\mathbb{D}$, and when the degree of a vertex is d, each tile meeting that vertex has angle $2\pi/d$ at that vertex. When $\mathcal{S}_K$ is parabolic and T_K tiles the complex plane, the tiling is unique up to orientation-preserving Euclidean similarities, or complex affine transformations, and when $\mathcal{S}_K$ is hyperbolic and T_K tiles the Poincaré disk, the tiling is unique up to orientation-preserving hyperbolic isometries. The type problem now is manifest. Given K, is $\mathcal{S}_K$ parabolic or hyperbolic? Does T_K tile the complex plane $\mathbb{C}$ or the Poincaré disk $\mathbb{D}$? In general, this is very difficult to resolve.

Each tile t of T_K of polygonal type n is homeomorphic to a regular Euclidean n-gon by a homeomorphism that preserves vertices and is conformal on the interior of t. It is in this sense that t is a conformally regular n-gon. This follows from the fact that the dihedral group of order $2n$ acts on t as a group of conformal isomorphisms that preserves and is transitive on vertices, which is a direct consequence of the equilateral metric on $|K^\beta|_{\text{eq}}$. Again, the reader is refered to [4] for details. Now there are uncountably many pairwise non-isomorphic ways to realize K as a planar tiling by conformally regular polygons in the pattern of K—what is special about T_K? The answer is in the **reflective structure** that T_K uniquely possesses among all conformally regular tiings in the pattern of K, and in the rich and beautiful substructures that uniquely appear in the tiling T_K as a result.

By reflective structure we mean that the isometric reflection of two contiguous faces of K^β about their common edge in the metric space $|K^\beta|_{\text{eq}}$ descends to a conformal reflection[2] in the tiling T_K. From this we see that each edge e of T_K is an analytic arc incident to two tiles, say s and t, and serves as the fixed point set of a conformal reflection that exchanges the conformal centers of s and t and is defined on and exchanges at least the triangular faces of s and t that meet along e that are the images of the equilateral faces of $|K^\beta|_{\text{eq}}$ under the uniformization mapping. Two important consequences of this reflective property follow. First, each triangular subtile t^β of a tile t of T_K, where t^β is the image of one of the equilateral faces of $|K^\beta|_{\text{eq}}$ under the uniformization mapping, encodes all the combinatorial and conformal data of the tiling T_K. By this we mean that, starting with t^β and nothing else, we may recover the whole of the tiling T_K, and so the whole of the combinatorial structure of K as well as the conformal type of T_K, by iterated conformal reflection through the edges of t^β and its reflected iterates. Second, the tiling T_K determines an **associated quad tiling** $T_K^\diamond$ as follows. Each edge e of K determines a tile s_e of $T_K^\diamond$ that is defined as the union of the images of the four equilateral faces of $|K^\beta|_{\text{eq}}$ that meet at the barycenter of e under the uniformization mapping. This tiling actually is conformally equivalent to the tiling $T_{K^\diamond}$ with the β-equilateral structure, where $K^\diamond$, the **associated 4-gon complex**, is the planar 4-gon complex encoding the combinatorics of the tiling $T_K^\diamond$. In particular, each tile s_e is a conformal square that conformally reflects to any neighboring tile with which it shares an edge, and so $T_K^\diamond$ is a reflective tiling by conformally regular 4-gons. The important point is that this quad tiling determines two orthogonal conformal tilings simultaneously, the tiling T_K as well as the dual tiling $T_K^\dagger$ in the following way. Each tile s of $T_K^\diamond$ has two conformal diagonals e and $e^\dagger$ that form the fixed point sets of conformal reflections and meet orthogonally at the conformal center of the conformal square s. The diagonals e cut out the tiling T_K while the diagonals $e^\dagger$ cut out the dual tiling $T_K^\dagger$. The dual tiling $T_K^\dagger$ is conformally equivalent to the tiling $T_{K^\dagger}$ with the β-equilateral structure, where $K^\dagger$ is the planar polygonal complex dual to K. Moreover, the barycentric tiling T_K^β whose triangular tiles are obtained from the conformal square tiles of $T_K^\diamond$ by cutting each along its conformal diagonals into four triangles, is conformally equivalent to the tiling T_{K^β}. It is only in the tiling determined by the β-equilateral structure that all four tilings T_K, $T_K^\dagger = T_{K^\dagger}$, $T_K^\diamond = T_{K^\diamond}$, and $T_K^\beta = T_{K^\beta}$ live simultaneously as reflective and conformally regular tilings. This beautiful structure is explored with greater precision and in more detail in [4].

So far we have examined the migration from the combinatorial to the geometric and have noted the importance of reflectivity. Turning the discussion around a bit, we now start with the geometric—a locally finite tiling T by conformally regular polygons—and ask for an appropriate definition that the tiling be reflective. The dihedral group D_{2n} of order $2n$ acts as a group of conformal isomorphisms on each tile t of polygonal type n. The action

[2]This more properly is called an *anti-conformal reflection* since the term *conformal* indicates not only the preservation of angular measure, but also the preservation of orientation; nonetheless, we will drop the modifier *anti-* in the sequel and let the term *reflection* carry the weight of the modifier.

preserves and is transitive on vertices and the fixed point, interior to t, is the **conformal center** of t. The elements of D_{2n} that preserve orientation act as a conformal rotation group of order n and the ones that reverse orientation act as conformal reflections whose fixed point sets—the **axes** of the action—are analytic arcs spanning across two vertices or two edges of t if n is even, and across vertex-edge pairs if n is odd. The intersections of the axes with the edges are the **conformal centers of the edges determined by the tile** t. By slicing each tile along subarcs of its axes from the conformal center of the tile to the vertices of the tile and the conformal centers of the edges, one obtains a tiling T^{β}. Two conditions must be met to call the tiling T **reflective**. First, whenever two tiles s and t of T meet along an edge e, the conformal centers of the edge e determined by s and t must agree. Second, each edge of T is the fixed point set of a conformal reflection that exchanges the triangular tiles of T^{β} that meet along e. When T is reflective, the tiling T^{β} is combinatorially a tiling in the pattern of the barycentric subdivision K_T^{β}, where K_T is the planar polygonal complex in the pattern of T. Also, the associated quad tiling $T^{\diamond}$ whose tiles are the unions of the four tiles of T^{β} that meet at the conformal centers of the edges of T is a reflective tiling by conformally regular 4-gons, or conformal squares, and each 4-gon tile conformally reflects across any of its edges to its 4-gon tile neighbor contiguous along that edge. Moreover, the conformal diagonals of the conformally square tiles of $T^{\diamond}$ that are not edges of T are precisely the edges of the reflective tiling $T^{\dagger}$ dual to T. So the reflective property of the tiling T guarantees the existence of four conformally regular and reflective tilings, T, $T^{\dagger}$, $T^{\diamond}$, and T^{β} that exists simultaneously. Indeed, the reflectivity of any of the four tilings guarantees the existence and reflectivity of the remaining three.[3]

The next theorem is the primary uniqueness result for conformal tilings and summarizes the importance of the reflective property. It implies that the dual requirements of conformal regularity of the tiles and of a reflective structure on the tiling uniquely define the tiling up to conformal automorphism, and when starting with a planar polygonal complex K, uniquely identifies T_K among the uncountably many tilings by conformally regular polygons in the pattern of K.

Theorem 1.2. *If S and T are combinatorially equivalent tilings by conformally regular polygons both of which are reflective, then S and T are of the same type and are conformally equivalent by a conformal isomorphism induced from a combinatorial isomorphism of K_S and K_T. In particular, T is conformally equivalent to the tiling T_K determined by the β-equilateral structure on the Riemann surface $\mathcal{S}_K$, where $K = K_T$.*

The proof is a consequence of the Schwarz Reflection Principle applied to each pair of contiguous tiles in S^{β} and T^{β} and appears in the first paper [4].

Because of the rich structure that arises from conformally regular and reflective tilings as well as the uniqueness of the theorem, we henceforth will mean by the term **conformal**

[3] There are beautiful graphics illustrating these coexisting conformal tilings in the first paper of the series [4] where the tilings are differentiated by colored edges and generated by iterated conformal reflections through the sides of edge-colored triangles. The authors highly recommend them to the reader.

tiling a tiling by conformally regular polygons that has a reflective structure that fills the Euclidean plane or the Poincaré disk, meaning that the union of the tiles is the whole of $\mathbb{C}$ or $\mathbb{D}$.[4] In particular, any conformal tiling T_K induced from a planar polygonal complex K will have the β-equilateral conformal structure. A conformal tiling T always will have hidden in its reflective structure the dual tiling $T^\dagger$, the associated quad tiling $T^\diamond$, and the barycentric tiling T^β, all conformally regular and reflective and any of which may be made manifest and exploited for our use.

The final review topic we need for understanding the statement and proof of Theorem 1.1 is that of subdivision of planar polygonal complexes and tilings. There are two points of view on subdivisions of conformal tilings by conformally regular polygons. The one is combinatorial in which the subdivision takes place at the level of the planar polygonal complex K, the other is geometric in which the subdivision takes place at the level of the conformal tiling T. The combinatorial subdivision always leads to a new conformal tiling by conformally regular polygons, but one that may not mesh so well with the original tiling. The geometric subdivision always preserves the combinatorics of the original tiling in that the original can be recovered by aggregation of tiles, and its reflective structure meshes well with the reflective structure of the original tiling. The former is useful for producing new tilings while the latter is useful for exploring a fixed tiling, and in particular the type problem for the local isomorphism class of a tiling. The former is simple and transparent, the latter is more subtle upon reflection than it appears at first sight. The former is called a **combinatorial subdivision** while the latter is a **conformal subdivision**. These were introduced and studied rather carefully in [4], and here we will offer an expanded review with the aim of proving Theorem 1.1, which states that a dihedrally symmetric *combinatorial* subdivision of a planar polygonal complex induces a *conformal* subdivision of its associated conformal tiling.

A **combinatorial subdivision**, or just a **subdivision** for short, of the planar polygonal complex K is a planar polygonal complex L for which each open cell (vertex, edge, or face) of L is contained in an open cell of K. In particular, each closed face of K is the union of finitely many closed faces of L, and each closed edge of K is a union of finitely many closed edges of L. Moreover, each open cell of L is contained in a unique open cell of K, and each closed face of L is contained in a unique closed face of K. When L is a subdivision of K, we call K an **aggregate** of L. A subdivision is **totally nontrivial** if every face of K actually gets subdivided, meaning that each face of K is a union of at least two faces of L. When L subdivides K we will write $L \le K$.

Subdivisions often are obtained by applying a **subdivision rule** that partitions the faces of K and then describes a precise algorithm for subdividing the faces in each partition set. The finite subdivision rules introduced by Canon, Floyd and Parry using model complexes and subdivision maps are sophisticated examples of these. Our needs are a bit more pedestrian as we primarily are concerned with rotationally symmetric subdivision rules that may

[4]These are the **maximal tilings** of paper [4].

be encoded by a collection of oriented regular CW-decompositions of combinatorial polygons of differing polygonal types. To be precise, for each $n \geq 2$, let τ_n be a rotationally symmetric, oriented regular CW-decomposition of the standard n-gon Δ_n. We assume that the collection $\tau = \{\tau_n\}$ is **edge-compatible**, meaning that there is an integer N such that each **seed** τ_n subdivides each edge of Δ_n into N sub-edges. This ensures that we may subdivide K by subdividing each face of polygonal type n by the pattern of τ_n.[5] This subdivision of K is denoted by τK and the subdivision rule τ is called a (rotationally invariant) **subdivision operator**. Whereas a particular subdivision rule generally is defined only for a *restricted* class of planar polygonal complexes, a subdivision operator may be applied to *any and every* planar polygonal complex. When τ is a subdivision rule for K and F is a subcomplex of K, τF has the obvious meaning as the subdivision of F induced by τ, and is a subcomplex of τK.

The reader is referred to the first paper [4] for examples of subdivision rules; however, we do want to review some specific subdivision rules, starting with illustrations of seeds of some specific ones—the star, barycentric, hex, delta, quad, pentagonal, and twisted pentagonal, along with others. These may be combined in various ways to describe subdivision rules and operators; for example, one may declare that all the 4-gon faces are subdivided by the pentagonal rule, and all other faces by the quad rule, and since each subdivides an edge into two edges, this describes an edge compatible subdivision operator. We have a special interest in rotationally symmetric rules in which just one type of subdivision algorithm is used on each face—all quad subdivisions on all faces regardless of polygonal type, or all pentagonal, etc. A subdivision rule of this type in which there is one type of subdivision wherein each face gets subdivided by the same algorithm is said to be **simple**. The seed examples of Fig.'s 1 and 2, when each face of K no matter what the polygonal type is subdivided by the same rotationally invariant algorithm, give examples of combinatorial subdivisions by simple, edge compatible subdivision operators—so we have for example the star, barycentric, hex, delta, quad, pentagonal, and twisted pentagonal simple subdivision operators that may be applied to an arbitrary planar polygonal complex. To be a bit more precise, a subdivision rule τ defined on K is **simple** if the rule partitions the faces of K by polygonal type and all the seeds of the rule are rotationally symmetric and agree on triangles forming fundamental domains of the appropriate rotation groups acting on the seeds. The seeds of such rules may be encoded in a triangular diagram that shows the subdivision of the triangular fundamental region of the action of the rotation group on any face, as in the middle columns of the figures. There, the lower, horizontal side of the triangle represents an edge of any face to which the rule is applied and the apex of the triangle represents the fixed point of the rotation action on any polygon. Vertices of the subdivision appear in yellow and edges appear as solid segments. By identifying the two non-horizontal sides, this information may be encoded by subdividing a 1-gon,

[5]If the seed subdivisions are rotationally symmetric, then there is a unique way to subdivide each face of K, up to isomorphism. This uses the fact that K as well as the seed subdivisions are oriented, and the application of the subdivision is assumed to preserve orientation. Edge-compatibility ensures that the subdivisions of contiguous faces may be matched up along their shared edges.

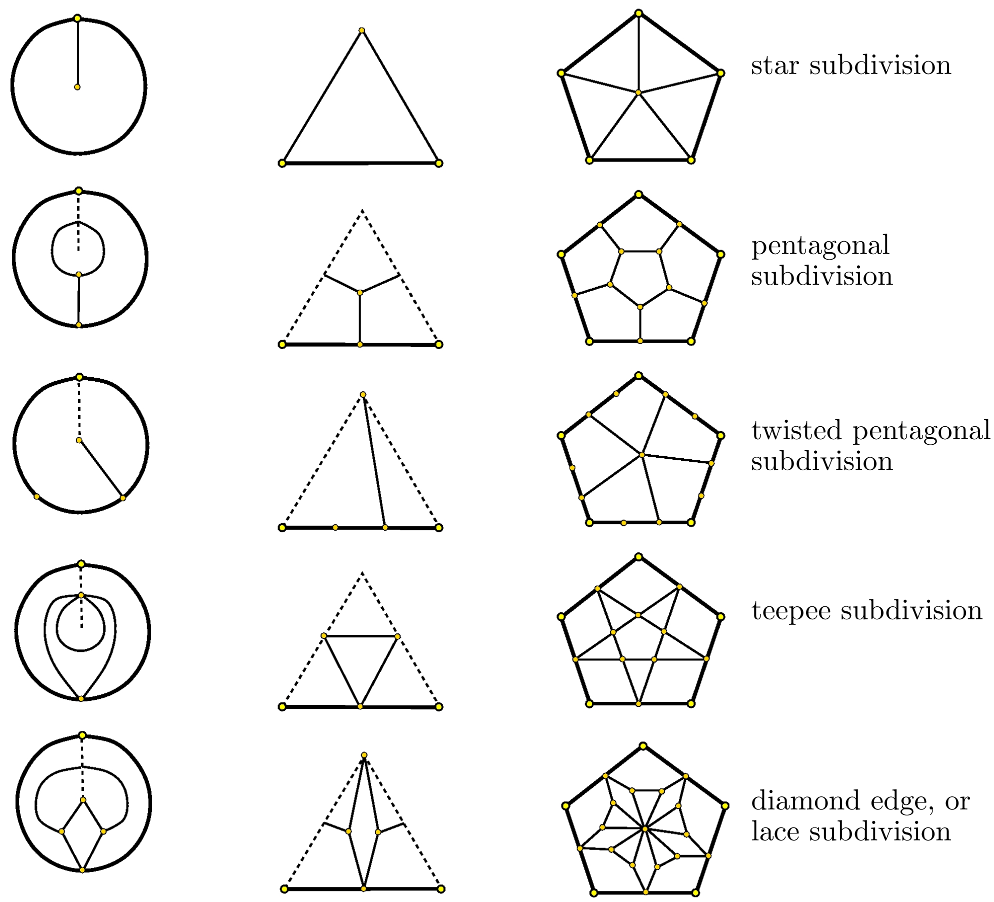

FIGURE 1. Seeds of simple rotationally symmetric subdivision rules, with all but the twisted pentagonal dihedrally symmetric. Each rule applies to any n-gon for any $n \geq 2$. We have encoded each rule using a 1-gon (left-hand column) and a triangle (central column), and have illustrated its application to a 5-gon.

a cell complex with a single vertex, a single edge, and a single face, as in the left-hand columns of the figures. The right hand columns show how the seed applies to subdivide a 5-gon in Fig. 1 and a 4-gon in Fig. 2, but of course the seed can be used to subdivide any n-gon.

We now move on to conformal subdivisions of conformal tilings. Let T be a conformal tiling, a tiling of either the plane $\mathbb{C}$ or the disk $\mathbb{D}$ by conformally regular polygons that

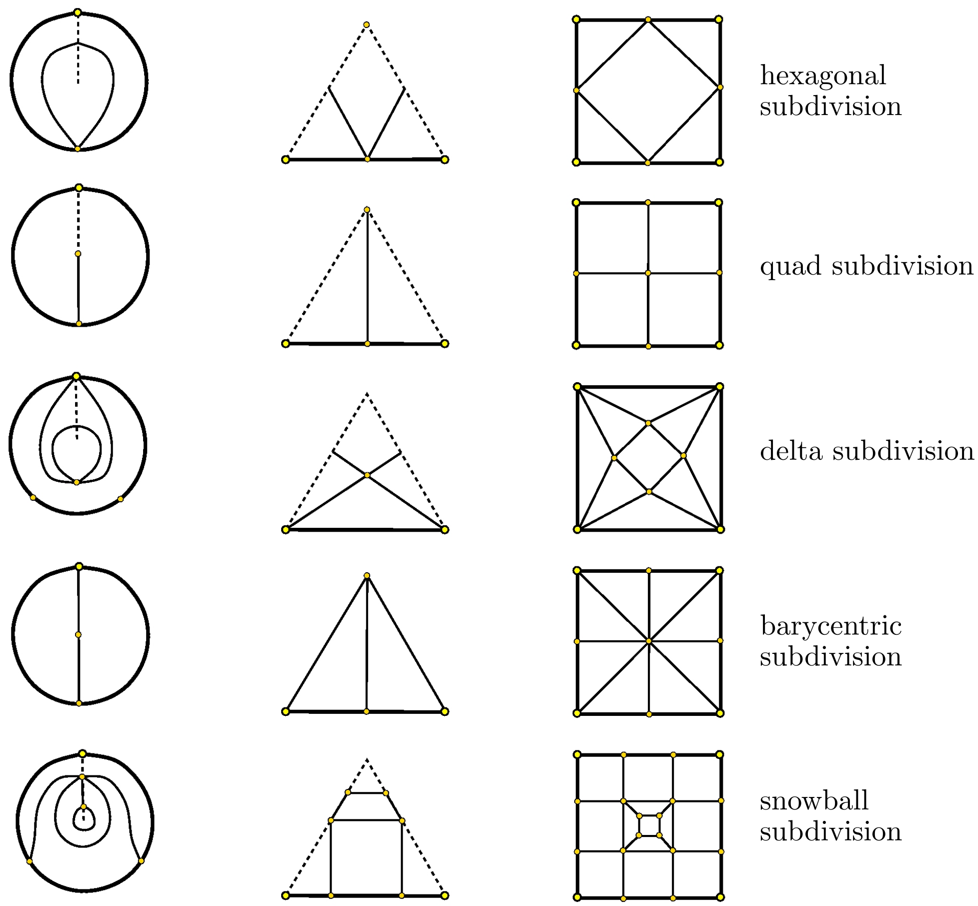

Figure 2. Seeds of simple dihedrally symmetric subdivision rules. Each rule applies to any n-gon for any $n \geq 2$. We have encoded each rule using a 1-gon (left-hand column) and a triangle (central column), and have illustrated its application to a 4-gon.

is conformally reflective. A **polygonal subdivision** of T is a tiling S for which each tile of S is a combinatorial polygon—a closed Jordan domain with finitely many distinguished boundary points—contained in a tile of T. Moreover, each open edge of S is contained in either an open edge or an open tile of T. When e is an edge of T contiguous to the tiles t and t', the polygonal subdivision S when restricted to t and t' induces a common subdivision of e and, of course, any two tiles of S meet along a set of full edges and vertices. In particular, the planar polygonal complex K_S determined by S is a combinatorial subdivision of K_T. Our interest is in polygonal subdivisions whose properties mirror those of T. In particular, the polygonal subdivision S is a **conformal subdivision** of T if S shares with T the properties of conformal regularity of its tiles as well as conformal reflectivity of the tiling. Alternately said, S is a conformal subdivision of T if S is a conformal tiling that polygonally subdivides T. Though a conformal subdivision S of T determines a combinatorial subdivision K_S of K_T, in general, this does not go in the reverse direction. When it does, i.e., when a combinatorial subdivision L of K will produce a reflective tiling T_L that conformally subdivides T_K, we say that L is a **conformal subdivision** of K. When the subdivision L is obtained by the application of a combinatorial subdivision rule τ, we say that τ is a **conformal subdivision rule** for K. The content of Theorem 1.1 is that every dihedrally symmetric, simple subdivision operator is a conformal subdivision rule for any planar polygonal complex K, which we now prove.

Proof of Theorem 1.1. Apply the subdivision rule to the standard n-gon Δ_n to obtain the complex $\tau_n = \tau\Delta_n$. If the tile t of T_K has polygonal type n, let $h_t : |\tau_n^\beta|_{\mathrm{eq}} \to t$ be any homeomorphism from the equilateral surface $|\tau_n^\beta|_{\mathrm{eq}}$ onto the conformally regular n-gon tile t of T_K that is conformal on the interior and takes vertices to vertices. These conformal isomorphisms h_t, for all tiles t of T_K, exist since the dihedral symmetry of τ translates to isometric, and therefore conformal, dihedral symmetry of the equilateral surface $|\tau_n^\beta|_{\mathrm{eq}}$, making this n-gon into a conformally regular one. Now the isomorphism h_t carries the subdivision τ_n of Δ_n onto a subdivision τt of t that decomposes t into conformally regular polygons and, moreover, endows t with a reflective structure in which the images of any of the triangular faces of $|\tau_n^\beta|_{\mathrm{eq}}$ under h_t conformally reflects to any other. *Since τ is simple, has dihedral symmetry, and the tiling T_K is reflective*, two facts emerge: first, the tiling τt is independent of the choice of h_t, and second, (*) the intersection pattern of the tiles of the subdivision τt with a triangular tile of T_K^β contained in the tile t that meets the tile t' of T_K along the edge e conformally reflects to the intersection pattern of the tiles of the subdivision $\tau t'$ with the triangular tile of T_K^β contained in the tile t' that meets t along e. In particular, the edge subdivisions induced by h_t and $h_{t'}$ match up along the common edge of any two contiguous tiles t and t'. The subdivisions τt for all the tiles t of T_K induce a polygonal subdivision τT_K of T_K in the pattern of the subdivided complex τK that is reflective across any edge of τT_K whose open edge is interior to a tile of T_K. But the dihedral symmetry of τ and observation (*) imply that the subdivision tiling τT_K is reflective across any edge of τT_K that is contained in an edge of T_K, and so the polygonal subdivision τT_K of T_K by conformally regular polygons is, in fact, reflective. This verifies that τT_K is a conformal subdivision of T_K. Since the combinatorially equivalent tilings τT_K and $T_{\tau K}$ are both conformally regular and reflective, Theorem 1.2 implies that they are conformally equivalent tilings. It follows that τ is a conformal subdivision rule for K and that the conformal tiling $T_{\tau K}$ may be realized as a conformal subdivision of T_K. □

Since we focus on the type problem in this paper our interest is in infinite planar polygonal complexes; however, the discussion of this first section applies also to finite complexes. In fact, in the first paper of this series [4] we do not restrict our attention to planar polygonal complexes, but allow for polygonal decompositions of essentially arbitrary surfaces. Theorem 1.1 is beautifully illustrated by applying the snowball subdivision of Fig. 2 to the 2-sphere decomposed initially as a combinatorial cube, six 4-gons glued together along their edges where three meet at each vertex. Four stages are illustrated in the striking Fig. 3 with the original cube in bold black, followed by blue, green, red, and black successive snowball subdivisions. These are (approximations to the) conformal square tilings that are reflective and the subdivisions are conformal subdivisions and can be continued ad infinitum. There is a discussion of the topics of this first section of the present paper in [4] with many examples and no restriction to infinite complexes. The tangent planes to this **Sierpiński snowball** example may be realized as conformal tilings of $\mathbb{C}$ associated to expansion complexes derived from the snowball subdivision rule, all of which are locally isomorphic to one another though there are uncountably many up to global combinatorial isomorphism.

To close this section, let T_L denote the collection of tiles in T_K that correspond to the faces of L, where L is a connected subcomplex of K in which every cell of L is contained in a closed face. T_L is called a **patch** of the tiling T_K, and L is called a **combinatorial patch** of K. Patches generally will be finite, but for the moment we will not enforce this restriction. Two patches, T_L of the tiling T and $T'_{L'}$ of the tiling T', are conformally equivalent if there is a conformal isomorphism of $|T_L| = \cup\{t : t \in T_L\}$ onto $|T'_{L'}|$ that takes tiles to tiles. Obviously conformally equivalent patches have combinatorially equivalent subcomplexes L and L'. We say that this conformal equivalence of T_L and $T'_{L'}$ **realizes** the combinatorial equivalence of L and L', and the next theorem provides a converse.

Theorem 1.3. *If L and L' are combinatorial patches of K and K', respectively, that are combinatorially equivalent, then T_L and $T'_{L'}$ are conformally equivalent patches, where $T = T_K$ and $T' = T_{K'}$.*

Proof. The construction of the equilateral metrics implies that when L and L' are isomorphic subcomplexes, the metricized versions, $|L^\beta|_{\mathrm{eq}}$ and $|L'^\beta|_{\mathrm{eq}}$, are isometric by an isometry that takes the equilateral faces of $|L^\beta|_{\mathrm{eq}}$ onto the equilateral faces of $|L'^\beta|_{\mathrm{eq}}$ that correspond under a combinatorial isomorphism of L^β to L'^β. This translates to a conformal isomorphism of the interiors of the corresponding subsets of $\mathcal{S}_K$ and $\mathcal{S}_{K'}$ that preserves faces, and finally to a conformal equivalence of the patches T_L and $T'_{L'}$. $\square$

From now on we enforce the finiteness restriction on patches, so that the combinatorial patch L is a finite subcomplex of K and T_L is a finite set of tiles.

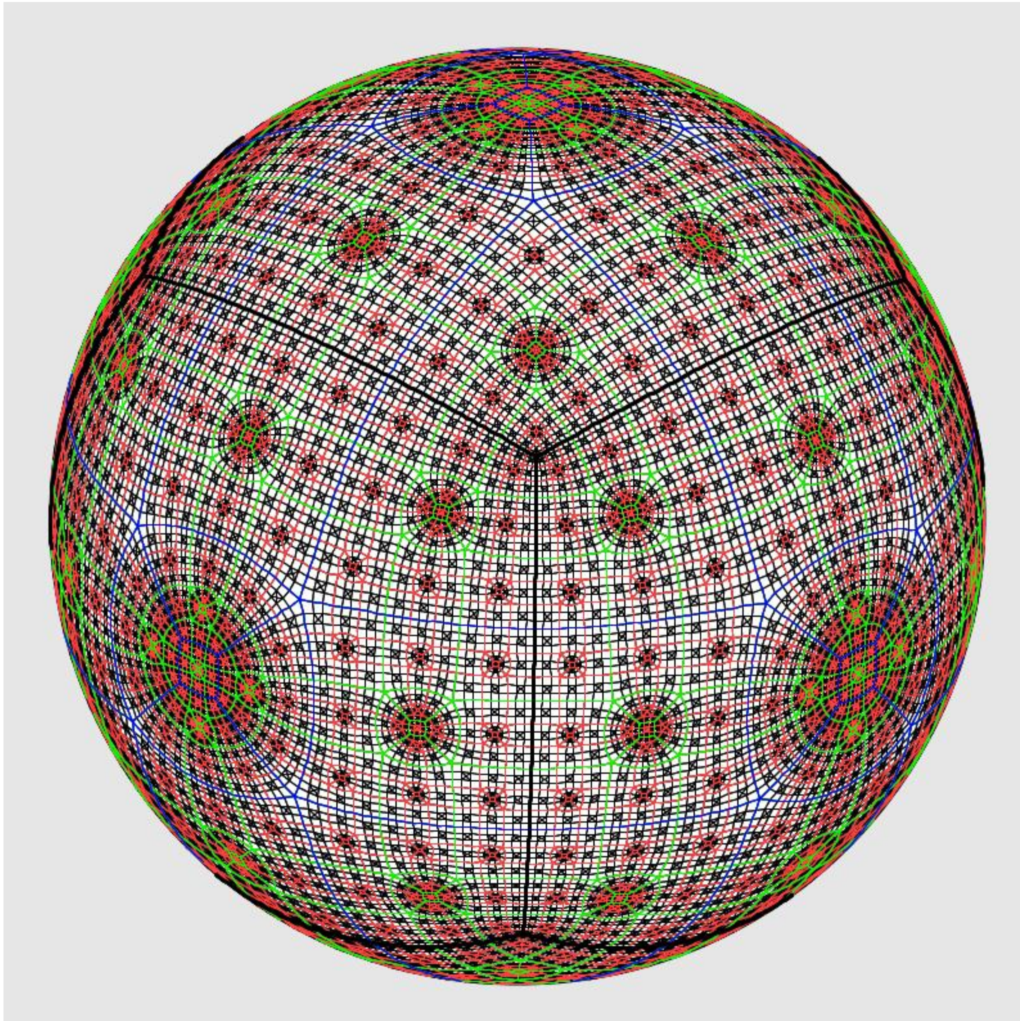

FIGURE 3. Snowball subdivisions of a cube. This figure represents four successive conformal subdivisions of a cube by the dihedrally symmetric snowball subdivision rule. The blue subdivides the bold black-edged cube, the green then subdivides the blue, the red the green, and the black the red. This represents a conformally correct picture of the first four stages of the Sierpiński snowball complex and may be continued ad infinitum.

2. Parameterizing Structure: the Space of Complexes and the Local Isomorphism Class

In the traditional world of aperiodic Euclidean tilings, an appropriate tiling determines a tiling space, the so-called **continuous hull** of the tiling, an action of the translation group on the hull, and a **canonical transversal** of the action that encodes pointed tilings that are locally isomorphic to the original tiling. Much of the effort in traditional tiling theory is exerted in understanding the topological structure of the hull and the dynamical system determined by the action. The conformal tilings of this paper generally fail to have the local Euclidean properties needed to produce this much structure, but do have some combinatorial features that allow one to recover some aspects of the traditional setting.[6] Our goals in this section are modest—to define the space of rooted polygonal complexes to serve as a universe for conformal tilings and to identify therein the topological properties of the local isomorphism classes. The combinatorial analogues of the properties of traditional hierarchical aperiodic tilings force desirable topological properties on these local isomorphism classes. Traditional tiling experts will see mirrored here several of the structures that appear in their traditional constructions, but only those that emerge from the combinatorics of tilings. Those that emerge from the Euclidean geometry of traditional tilings are, without the benefit of additional geometric structure, absent in this combinatorial and conformal setting.

2.1. **The space of rooted planar polygonal complexes.** We begin by describing a metric space parameterizing rooted, oriented planar polygonal complexes that is modeled after similar metrics used with graphs and traditional tilings. The incarnation herein of this space in which the combinatorics of our tilings are compared and contrasted takes a form tailored for our uses that stresses the combinatorial structure of our tilings rather than the geometric structure. The definition we choose is a little more complicated than the perhaps more natural one modeled after the *big ball* metric of traditional tilings. The big ball metric in the setting of traditional tilings is complete, but a straightforward generalization of this to the combinatorial setting that is based on combinatorial neighborhoods of the root face is not complete. The completion of this metric involves including in the space of complexes those planar complexes that fail to be simply connected, so that they fill arbitrary planar domains rather than filling the whole plane. We modify the big ball metric to obtain a complete metric on the space of rooted, oriented planar polygonal complexes by replacing combinatorial neighborhoods of the root face by **combinatorial filled neighborhoods**, which are defined presently. The use of these combinatorial filled neighborhoods ensures that any limit point of planar polygonal complexes determines a *simply connected* combinatorial complex, guaranteeing that the limit complex is a planar polygonal complex per our definition.

The elements of the **space of rooted planar polygonal complexes RC** are the rooted, orientation-preserving isomorphism classes of pairs (K, f), where K is an oriented planar polygonal complex and f is a fixed **root face** of K. In this setting, a rooted isomorphism

[6] Maria Ramirez-Solano's doctoral thesis [9] generalizes some of the tools of the traditional theory and succeeds in defining a continuous hull, an appropriate action, and a canonical transversal for certain conformal tilings. The present authors in a later paper of this series will introduce further geometric structure on conformal tilings that will allow for a theory of conformal tiling more or less parallel to the theory of aperiodic tilings.

of (K, f) to (L, g) is an orientation-preserving cellular isomorphism of K to L that takes f to g; i.e., an orientation-preserving homeomorphism α of the plane that is cellular with respect to the CW-decompositions K and L and such that $\alpha(f) = g$. Generally we will drop the adjective *orientation-preserving*, but we ask the reader to keep in mind that all our complexes are oriented and *isomorphism* means *orientation-preserving isomorphism.* Note that the root faces f and g must be of the same polygonal type if (K, f) is rooted isomorphic to (L, g). We will employ the notational use of sans serif letters to denote elements of RC, so we may write $(K, f) \in \mathsf{k} \in \mathsf{RC}$. There is in this setting a map $* : \mathsf{RC} \to \mathsf{C}$ to the set of isomorphism classes of planar polygonal complexes C that forgets the root face and that is countable-to-one. Thus, for example, $\mathsf{k}^* = K$ whenever $(K, f) \in \mathsf{k}$. Notice that we do not distinguish between isomorphic planar polygonal complexes, writing $K = L$ whenever $K \cong L$.

A tool that will prove useful subsequently is the **combinatorial distance** between two faces f and g of a planar polygonal complex K. This distance, denoted as $d_K(f, g)$, is defined as follows. We say that $d_K(f, g) \le n$, for an integer $n \ge 0$, if there exists a **chain** of faces of **length** n from f to g, i.e., a finite sequence $f = f_0, \dots, f_n = g$ such that f_{i-1} and f_i share at least one edge for $i = 1, \dots, n$; further, $d_K(f, g) = n$ if $d_K(f, g) \le n$ and $d_K(f, g) \not\le n - 1$. The function d_K is a metric on the collection of faces of K and merely measures efficiently how many steps it takes to get from f to g by stepping along contiguous faces of K. Alternately, this metric may be described as the edge metric of the dual 1-skeleton of K, as in the first paper [4] of the series.

We now describe, for the non-negative integer n, the **filled n-neighborhood** of the face f of the planar polygonal complex K. There are various equivalent ways to do this, and perhaps it is easiest to describe this **filled neighborhood of radius** n as the smallest *simply connected* subcomplex $B_K(f, n)$ of K that contains the **combinatorial n-neighborhood** $C_K(f, n)$ of f; the faces of the combinatorial n-neighborhood $C_K(f, n)$ in turn are precisely all those faces g of K with $d_K(f, g) \le n$. A slightly differing description of filled neighborhoods is that $B_K(f, n)$ is the smallest combinatorial disk of K that contains $C_K(f, n)$ or, equivalently, $B_K(f, n)$ is the subcomplex of K whose underlying space is the complement of the unbounded complementary domain of $C_K(f, n)$. The point is that a combinatorial n-neighborhood of f may fail to be simply connected. As such its open complementary domain is a disjoint union of finitely many open disks in the plane and a single unbounded open component. $B_K(f, n)$ is obtained as the smallest subcomplex that contains $C_K(f, n)$ as well as the faces of K that meet the open disk components of the complementary domain of this combinatorial n-neighborhood. The combinatorial boundary of $B_K(f, n)$ is a simple closed edge path that separates f from infinity and is called the **outer sphere** of **radius** n centered on f. Note that each face g of K incident to an edge of the boundary of $C_K(f, n)$ satisfies $n \le d_K(f, g) \le n + 1$. Each $B_K(f, n)$ is a combinatorial disk whose boundary is its corresponding outer sphere, and K is the union of the nested sequence

$$B_K(f, 1) \subset B_K(f, 2) \subset \cdots \subset B_K(f, n) \subset B_K(f, n + 1) \subset \cdots$$

of combinatorial disks. Moreover, each $B_K(f,n)$ is contained in the interior of the disk $B_K(f,N)$ for some $N > n$. Another characterization of filled neighborhoods is useful: a face g of K fails to be in $B_K(f,n)$ if and only if there is an infinite chain $g = g_1, g_2, \ldots$ of *pairwise distinct* faces in K for which, for all i, g_i and g_{i+1} share an edge, and no g_i is contained in the combinatorial n-neighborhood of f. Colloquially, g is not a face of $B_K(f,n)$ if and only if there is a unbounded path of faces starting at g that misses the combinatorial n-neighborhood of f. Define $B_K(f,\infty)$ to be the whole complex K.

Given two rooted complexes $\mathsf{k} = (K,f)$ and $\mathsf{l} = (L,g)$, let

$$\rho(\mathsf{k},\mathsf{l}) = \rho((K,f),(L,g)) = e^{-n}$$

where, either n is the largest integer, or the symbol ∞, for which the rooted complexes $(B_K(f,n),f)$ and $(B_L(g,n),g)$ are rooted isomorphic, or $n = -1$, this whenever f and g have differing polygonal types. This defines a metric on $\mathbf{RC}$ that satisfies the property that, whenever $\mathsf{j},\mathsf{k},\mathsf{l} \in \mathbf{RC}$, then either

$$\rho(\mathsf{j},\mathsf{l}) \leq \rho(\mathsf{j},\mathsf{k}) = \rho(\mathsf{k},\mathsf{l})$$

holds, or one of the permutations of this relation holds when the arguments are permuted. This is equivalent to the condition that

$$\rho(\mathsf{j},\mathsf{l}) \leq \max\left\{\rho(\mathsf{j},\mathsf{k}),\rho(\mathsf{k},\mathsf{l})\right\},$$

which is the defining triangle inequality for an ultrametric, a topic of significant application in p-adic analysis. See, for example, Section 2.1 of [10] for basic properties of ultrametrics.

Theorem 2.1. *The metric ρ on the space of rooted planar polygonal complexes $\mathbf{RC}$ is bounded and complete, and the metric space $(\mathbf{RC},\rho)$ is totally disconnected.*

Proof. Obviously ρ is bounded since $n \geq -1$ in the definition of ρ so that, in fact, $\operatorname{diam}_\rho \mathbf{RC} = e$. Let k_i, $i = 1,2,3,\ldots$, be a ρ-Cauchy sequence in $\mathbf{RC}$. By passing to a subsequence if necessary, we may assume that for each i, $\rho(\mathsf{k}_i,\mathsf{k}_{i+1}) \leq e^{-i}$. Choose representatives $(K_i,f_i) \in \mathsf{k}_i$, and observe that our definition of the metric ρ implies that for each i, the filled neighborhood $B_{K_i}(f_i,i)$ is rooted isomorphic to the filled neighborhood $B_{K_{i+1}}(f_{i+1},i)$. Let $h_i : B_{K_i}(f_i,i) \to B_{K_{i+1}}(f_{i+1},i)$ be a rooted isomorphism and enlarge the codomain of h_i to $B_{K_{i+1}}(f_{i+1},i+1)$ to obtain the directed sequence

$$B_{K_1}(f_1,1) \xrightarrow{h_1} B_{K_2}(f_2,2) \xrightarrow{h_2} B_{K_3}(f_3,3) \xrightarrow{h_3} \cdots \quad ,$$

where each h_i is an isomorphic embedding with $h_i(f_i) = f_{i+1}$. The direct limit complex $\varinjlim B_{K_i}(f_i,i)$ is simply connected since each $B_{K_i}(f_i,i)$ is a combinatorial disk, and this implies, since in addition the image of $B_{K_i}(f_i, i)$ under $h_{N-1} \circ \cdots \circ h_i$ is contained in the interior of a disk $B_{K_N}(f_N, N)$ for some $N > i$, that the direct limit complex is a planar polygonal complex. Let K denote the planar polygonal complex $\varinjlim B_{K_i}(f_i, i)$ and f the face of K corresponding to f_1. Finally, let k denote the rooted isomorphism class of the rooted planar polygonal complex (K, f). It is easy to show that $\mathsf{k}_i \to \mathsf{k}$ as $i \to \infty$, and so ρ is complete. We mention that under the convention that we do not distinguish between rooted isomorphic complexes, this direct sequence is a sequence of set containments and the direct limit is just the union.

Every ultrametric space is totally disconnected, but we can see this in the present context as follows. For any point $\mathsf{k} \in \mathbf{RC}$ and positive integer n, the metric ball $B_\rho(\mathsf{k}, e^{-n}/2)$ of radius $e^{-n}/2$ is both open and closed, since the metric takes on only the values in the countable set $\{0\} \cup \{e^{-k} : k = -1, 0, 1, \dots\}$. This implies that $(\mathbf{RC}, \rho)$ is totally disconnected. □

For each integer $n \geq 3$, let $\mathbf{RC}^n$ be the subspace of $\mathbf{RC}$ of rooted isomorphism classes of planar polygonal complexes of degree bounded by n,[7] and $\mathbf{RC}^\omega = \cup_{n=3}^{\infty} \mathbf{RC}^n$ the space of bounded degree, rooted planar polygonal complexes. The subsets $\mathbf{C}^n$ and $\mathbf{C}^\omega$ of $\mathbf{C}$ are defined in the obvious manners as the images of the respective spaces $\mathbf{RC}^n$ and $\mathbf{RC}^\omega$ under the map $*$.

Theorem 2.2. *For each $n \geq 3$, the subset $\mathbf{RC}^n$ is a closed, nowhere dense subspace of $\mathbf{RC}$.*

Proof. Let $\mathsf{k}_i \to \mathsf{k}$ where $\mathsf{k}_i \in \mathbf{RC}^n$ and $\mathsf{k} \in \mathbf{RC}$. To verify that $\mathbf{RC}^n$ is closed in $\mathbf{RC}$, it suffices to show that $\mathsf{k} \in \mathbf{RC}^n$. But this is quite easy since the convergence of the sequence k_i to k implies that an arbitrarily large finite filled neighborhood of the root face of k^* is isomorphic to a filled neighborhood of the root of k_i^* for large enough i, and this guarantees that the degree of k^* is at most n so that $\mathsf{k} \in \mathbf{RC}^n$. It follows that $\mathbf{RC}^n$ is closed in $\mathbf{RC}$.

$\mathbf{RC}^n$ is nowhere dense in $\mathbf{RC}$ since every element k of $\mathbf{RC}^n$ is the limit of a sequence k_i, each term with at least one vertex of degree greater than or equal to $n+1$, as the reader may construct rather easily. □

Corollary 2.3. *The space $\mathbf{RC}^\omega$ of bounded degree, rooted planar polygonal complexes is a dense F_σ subspace of $\mathbf{RC}$, and the space $\mathbf{RC} - \mathbf{RC}^\omega$ of unbounded degree rooted planar polygonal complexes is a dense G_δ subspace of $\mathbf{RC}$.*

The next property we advance is introduced to delineate special compact subsets of $\mathbf{RC}^n$. We say that a planar polygonal complex K satisfies a θ-**isoperimetric inequality** if every combinatorial disk D of K satisfies the θ-isoperimetric inequality, meaning that

$$\mathsf{size}(D) < \theta(\mathsf{size}(\partial D)).$$

[7] Note that $\mathbf{RC}^3 = \emptyset$ since there are no planar 3-gon complexes of degree ≤ 3. We include this case for the convenience of certain statements later.

Here, $\theta : \mathbb{N} \to \mathbb{R}$ is a positive function, $\mathsf{size}(D)$ is the number of faces of the combinatorial disk D, and $\mathsf{size}(\partial D)$ is the number of edges in the simple closed edge path ∂D forming the combinatorial boundary of D. For each integer $n \geq 3$ and positive function θ, let $\mathbf{RC}^{n,\theta}$ be the set that consists of those $\mathsf{k} \in \mathbf{RC}^n$ for which the complex k^* satisfies a θ-isoperimetric inequality.

Theorem 2.4. *For each $n \geq 3$ and positive function θ, the subset $\mathbf{RC}^{n,\theta}$ is a compact, nowhere dense subspace of $\mathbf{RC}$.*

Proof. We assume that $\mathbf{RC}^{n,\theta}$ is non-empty as otherwise there is nothing to prove. Since $\mathbf{RC}^{n,\theta} \subset \mathbf{RC}^n$ and $\mathbf{RC}^n$ is nowhere dense in $\mathbf{RC}$, so too is $\mathbf{RC}^{n,\theta}$. That $\mathbf{RC}^{n,\theta}$ is closed in $\mathbf{RC}$ is proved the same way that $\mathbf{RC}^n$ is shown to be closed. The point is that, if $\mathsf{k}_i \to \mathsf{k}$ where $\mathsf{k}_i \in \mathbf{RC}^{n,\theta}$ and $\mathsf{k} \in \mathbf{RC}$, then every combinatorial disk D in k^* appears isomorphically as a disk in some k_i^*, and this guarantees that the θ-isoperimetric inequality holds for D since it holds for the isomorphic copy of D in k_i^*.

To verify compactness, we show that $\mathbf{RC}^{n,\theta}$ is totally bounded in $\mathbf{RC}$, which for closed subsets of complete metric spaces is equivalent to compactness. Recall that a metric space is totally bounded if, for each $\varepsilon > 0$, there exists a finite ε-dense subset of the space. Let $\varepsilon > 0$ and choose a positive integer m so that $e^{-m} < \varepsilon$. Let $(H_1, f_1), \ldots, (H_J, f_J)$ be a list of all the finite, connected planar polygonal CW complexes of degree at most n that appear as the filled m-neighborhood of a face in some planar polygonal complex in $\mathbf{RC}^{n,\theta}$, up to isomorphism. This list is finite precisely because, for all complexes in $\mathbf{RC}^{n,\theta}$, the degree is bounded by n *and* the θ-isoperimetric inequality holds; indeed, the bounded degree condition guarantees that there are only finitely many isomorphism classes of combinatorial m-neighborhoods of the form $C_K(f, m)$ as K ranges over $\mathbf{RC}^{n,\theta}$ and f ranges over K, and for each such class, the θ-isoperimetric inequality guarantees that there are only finitely many ways to fill the holes of any combinatorial m-neighborhood to obtain a filled m-neighborhood. The upshot is that these two conditions together place a bound $M > 0$ on the number of faces in any such filled neighborhood, and there are then only finitely many ways to arrange a set of at most M combinatorial polygons of polygonal type bounded by n to form a filled neighborhood. For each $i = 1, \ldots, J$, let K_i be a planar polygonal complex that contains an isomorphic copy of H_i as the filled m-neighborhood $B_{K_i}(f_i, m)$. Letting $\mathsf{k}_i = (K_i, f_i)$, the set $\{\mathsf{k}_1, \ldots, \mathsf{k}_J\}$ is a finite ε-dense subset of $\mathbf{RC}^{n,\theta}$. $\square$

The feature used in this proof guaranteed by the combination of the two requirements of degree bounded by n and satisfying a θ-isoperimetric inequality is very important in the sequel. This feature is reminiscent of the condition of finite local complexity in the traditional setting of aperiodic tilings. The most straightforward translation of finite local complexity to the combinatorial setting would be the rather pedestrian condition of having bounded degree. This guarantees that there are only finitely many combinatorial types of polygonal complexes of a given combinatorial diameter, but because these complexes need not be simply connected, and in particular combinatorial neighborhoods of the form

$C_K(f,m)$ need not be simply connected, these complexes can have "holes" whose fillings may be arbitrarily complex. The addition of the isoperimetric condition remedies this. This isoperimetric condition is not needed as an explicit requirement in the traditional setting because the Euclidean geometric structure of the traditional tilings automatically imposes this condition. In our first paper [4], we defined a conformal tiling to have finite local complexity exactly in case it has finite degree. There we did not need the benifits that an isoperimetric inequality entails and so did not include that as part of the definition; here, though, we need the full effect that finite local complexity entails in the traditional theory, and in particular the compactness results it engenders. Therefore, we modify the notion of combinatorial finite local complexity introduced in [4] to include an isoperimetric condition. We say, then, that the planar polygonal complex K has combinatorial **finite local complexity**, abbreviated as FLC, if K has degree bounded by some natural number n and satisfies a θ-isoperimetric inequality for some positive function θ. To be rather clear on its intended use, we will separate out as a corollary that important feature of the preceding proof as applied to a single complex K.

Corollary 2.5. *Suppose the planar polygonal complex K has FLC. Then, up to isomorphism, there are only finitely many filled m-neighborhoods in K of a given radius m.*

2.2. **Subspaces of planar n-gon complexes.** There are several subspaces of planar n-gon complexes that are useful. First, for a fixed $n \geq 3$, let $\mathbf{RC}_n$ be the subspace of $\mathbf{RC}$ that consists of all those $\mathsf{k} \in \mathbf{RC}$ for which k^* is a planar n-gon complex, and let $\mathbf{C}_n$ be its image in $\mathbf{C}$ under $*$. Thus, $\mathbf{C}_n$ is the set of (isomorphism classes of) planar n-gon complexes. For fixed $m \geq 3$, let $\mathbf{RC}_{n,m}$ be the subspace of $\mathbf{RC}_n$ with the property that each vertex of k^* is incident to at most m faces of k^*, and let $\mathbf{RC}_{n,\omega} = \cup_{m=3}^{\infty} \mathbf{RC}_{n,m}$, with $\mathbf{C}_{n,m}$ and $\mathbf{C}_{n,\omega}$ their respective images under $*$. Notice that $\mathbf{RC}_{n,m} \subset \mathbf{RC}^{\max\{n,m\}}$ since any element of $\mathbf{RC}_{n,m}$ has degree at most $\max\{n,m\}$. Finally, let $\mathbf{RC}^{\theta}_{n,m}$ be the subset of $\mathbf{RC}_{n,m}$ that consists of those elements that satisfy the θ-isoperimetric inequality. The following theorems are immediate upon reflection.

Theorem 2.6. *For $m, n \geq 3$, the spaces $\mathbf{RC}_n$ and $\mathbf{RC}_{n,m}$ are closed subspaces of $\mathbf{RC}$ and, as such, are complete in the metric ρ.*

Theorem 2.7. *For $m, n \geq 3$ and positive function θ, the space $\mathbf{RC}^{\theta}_{n,m}$ is compact.*

2.2.1. *The preeminence of 4-gon tilings and the mapping $\mathbf{C} \to \mathbf{C}_4$.* We uncover here in what sense the quad tiling $T_{K^\diamond}$ associated to the planar polygonal complex K stands as *primus inter pares* among the four tilings T_K, $T_{K^\dagger}$, $T_{K^\diamond}$, and $T_{K^\beta} = T_{K^{\dagger\beta}}$. Define the **quad mapping**

$$\diamond : \mathbf{C} \to \mathbf{C}_4$$

from the set of isomorphism classes of planar polygonal complexes $\mathbf{C}$ to the set of isomorphism classes of planar 4-gon complexes $\mathbf{C}_4$ by $\diamond(K) = K^\diamond$. The mapping $\diamond$ is generally

2-to-1 with $K^\diamond = K^{\dagger\diamond}$ for every $K \in \mathbf{C}$. It fails to be 2-to-1 only when K is self-dual with K isomorphic with $K^\dagger$.

Theorem 2.8. *The mapping $\diamond$ is surjective.*

Proof. Let L be a planar 4-gon complex. All we need do is show that the vertex set $L^{(0)}$ is 2-colorable. Indeed, if $c : L^{(0)} \to \mathbb{Z}_2 = \{\pm 1\}$ is a coloring map with $c(e^-) \neq c(e^+)$ whenever the vertices $e^\pm$ are adjacent, then define planar polygonal complexes K and $K^\dagger$ that are dual to one another as follows. The vertex set of K is $K^{(0)} = c^{-1}(+1)$ and of $K^\dagger$ is $K^{\dagger(0)} = c^{-1}(-1)$, and the edge set of K is the set of diagonals of the 4-gon faces of L that span two vertices of $K^{(0)}$ and the edge set of $K^\dagger$ is the set of the other diagonals of the 4-gon faces of L that span two vertices of $K^{\dagger(0)}$. Each vertex of $c^{-1}(-1)$ of degree n is surrounded by a cycle of n diagonal edges from $K^{(1)}$ that determines a face of K of type n, and similarly for $K^\dagger$. The reader may check that K and $K^\dagger$ are planar polygonal complexes that are dual to one another and that $K^\diamond = L = K^{\dagger\diamond}$, confirming the surjectivity of $\diamond$.

It remains to demonstrate the 2-colorability of the vertex set $L^{(0)}$. This follows easily from the fact that each simple closed edge cycle in L has an even number of vertices, which may be proved as follows. Let C be a simple closed edge cycle in L and let J be the bounded subcomplex of L bordered by C that decomposes a closed disk into 4-gons that meet along boundary subcomplexes. Double J along its boundary to get $2J$, a 4-gon regular CW-decomposition of the 2-sphere. Let F be the number of 4-gon faces of J, E_∂ the number of boundary edges of J, and E_{int} the number of edges of J that are not boundary edges. Note that the number of 4-gon faces of $2J$ is $2F$ and the number of edges of $2J$ is $E_\partial + 2E_{\mathrm{int}}$. We count the number of angles of $2J$ in two differing ways. First, each face has 4 angles, so there are $4 \cdot 2F = 8F$ angles in the complex $2J$. Second, each angle may be paired with an oriented edge for which that angle sits at the initial vertex of that edge and lies in the left-hand 4-gon that the edge bounds. This describes a one-to-one correspondence between angles and oriented edges, and since each edge has two orientations, there are $2(E_\partial + 2E_{\mathrm{int}})$ angles. Setting $8F = 2(E_\partial + 2E_{\mathrm{int}})$ gives $E_\partial = 4F - 2E_{\mathrm{int}}$, so there are an even number of edges in the boundary curve C of J, and therefore an even number of vertices in C. $\square$

The interest in this result is precisely that it says that the study of general planar polygonal complexes may take place in the setting of planar 4-gon complexes, and since the β-equilateral conformal structures on the planar polygonal complex K, its dual $K^\dagger$, and its associated 4-gon complex $K^\diamond$ are compatible in the sense already articulated, this extends to the study of the conformally regular β-equilateral structures on the elements of $\mathbf{C}$ and their associated tilings. In particular, general conformally regular tilings in the β-equilateral structure are all obtained from reflective conformal square tilings of the plane $\mathbb{C}$ and the disk $\mathbb{D}$ by 2-coloring the vertices of the conformal square tilings and cutting out the dual pair T and $T^\dagger$ using the conformal diagonals of the conformal squares. In particular, the type problem may be restricted to the realm of planar 4-gon tilings, provided the β-equilateral structure is the conformal structure of choice.

2.3. **Local isomorphism classes of complexes.** In studying conformal tilings, we are not so much interested in a single tiling T_K associated to the planar polygonal complex K as much as the family of all tilings T_L that arise from planar polygonal complexes L that are locally isomorphic to the fixed complex K. This notion of local isomorphism arises naturally in the discipline of traditional aperiodic tiling, helping to clarify the details of the hull of a fixed tiling, already mentioned, on which the translation group acts to build the tiling dynamics. Its appearance there has a strong Euclidean-geometric flavor, but here takes on combinatorial overtones rather than the geometric.

For planar polygonal complexes K and L, we say that K **locally embeds** in L, written as $K \preceq L$, if every finite connected subcomplex of K isomorphically embeds in L, and K is **locally isomorphic** with L, written as $K \sim L$, if $K \preceq L$ and $L \preceq K$. Notice that we may just as well use combinatorial patches in place of finite connected subcomplexes in this definition, and sometimes we do so. The relation $\preceq$ is reflexive and transitive, and hence a pre-order on C, and defines a partial order on the set of local isomorphism classes of elements of C. We let (K) denote the local isomorphism class of the planar polygonal complex K so that $(K) = \{L \in \mathsf{C} : K \sim L\}$. If (K) is a singleton, we say that K is **singular**; otherwise, K is **plural**. We will see that either K is singular and (K) is a singleton, or K is plural and (K) is uncountably infinite. Let $\mathsf{LI}(K)$ denote the pre-image of (K) under the map $*$ defined on RC, so that $\mathsf{LI}(K)$ is the set of rooted polygonal complexes $\mathsf{l} \in \mathsf{RC}$ with $\mathsf{l}^* \sim K$.[8] Our interest in this section is in uncovering the structure and properties of the local isomorphism class (K) and of the **rooted local isomorphism class** $\mathsf{LI}(K)$. The next result is a working lemma that will be used to aid in this endeavor.

Lemma 2.9. *Let K and L be planar polygonal complexes and $L_1 \subset L_2 \subset \cdots$ a sequence of finite, connected subcomplexes of L that exhausts L, meaning that $L = \cup_{n=1}^{\infty} L_n$. For each positive integer n, let $h_n : L_n \to K$ be an isomorphic embedding of complexes. If there are faces $g = g_0$ of L_1 and $f = f_0$ of K such that $h_n(g) = f$ for all $n \geq 1$, then K is isomorphic to L.*

Proof. Since $h_n(g_0) = f_0$ for all $n \geq 1$, infinitely many of these mappings agree on the vertex set of the face f_0. Let $(h_{0,n})_n$ be a subsequence of the sequence $(h_n)_n$ such that all the mappings of this subsequence agree on the vertex set of f_0. Now list all the faces of L, say $g_0, g_1, g_2, \ldots$, and note that since the sequence of complexes L_n exhausts L, each g_k is a face of all but finitely many of the L_n. Since the face g_1 is a fixed combinatorial distance from the face g_0 in L, and since all the embeddings $h_{0,n}$ agree on g_0, there are infinitely many of the embeddings from the sequence $h_{0,n}$ whose domains contain the face g_1 and that agree on that face and on its vertices. Let $(h_{1,n})_n$ be a subsequence of $(h_{0,n})_n$ for which $h_{1,n}(g_1) = f_1$, a fixed face of K, for all $n \geq 1$, and that agree on the vertices of g_1. Inductively construct a sequence of sequences $((h_{k,n})_{n=1}^{\infty})_{k=0}^{\infty}$ such that $(h_{k,n})_n$ is a

[8] The set $\mathsf{LI}(K)$ is the combinatorial analogue of the canonical transversal $\Xi = \Xi_T$ for the action of the translation group on the continuous hull in the traditional theory of aperiodic tilings.

subsequence of $(h_{k-1,n})_n$ for all $k \geq 1$, $h_{k,n}(g_k) = f_k$, a fixed face of K, for all $n \geq 1$, and that all agree on the vertices of g_k. Note that for each fixed integer $k \geq 0$, $h_{n,n}(g_k) = f_k$ for all $n \geq k$ and the limit mapping $h = \lim h_{n,n}|_{L^{(0)}} : L^{(0)} \to K^{(0)}$ exists, since for each vertex w of L, the sequence $h_{n,n}(w)$ stabilizes. It follows that h extends to a cellular mapping from L to K, still called h, such that $h(g_k) = f_k$ for all $k \geq 0$. It is easy to see that h is an isomorphic embedding of CW complexes since all the mappings h_n are isomorphic embeddings. Being a CW-embedding, h is an open mapping, and is then surjective because it is cellular and both L and K are locally finite decompositions of the plane into combinatorial polygons. Therefore h is an isomorphism. □

The first application of the working lemma is to confirm that planar polygonal complexes that are highly symmetric globally are uninteresting in terms of local isomorphism.

Theorem 2.10. *If* $\mathrm{Aut}(K)$ *acts cocompactly on the planar polygonal complex* K *and* $L \preceq K$*, then* $L \cong K$*. It follows that* (K) *is a singleton and therefore* K *is singular.*

Proof. Let F be a finite subcomplex of K that serves as a fundamental region for the action of $\mathrm{Aut}(K)$ on K and write $L = \cup_{n=1}^{\infty} L_n$, where $L_1 \subset L_2 \subset \cdots$ is an increasing sequence of finite, connected subcomplexes of L that exhausts L. Let g be a face of L_1 and let $(L_n, g) \cong (K_n, f_n)$, where K_n is a subcomplex of K isomorphic to L_n. Since $\mathrm{Aut}(K)$ acts with fundamental region F, we may assume that for each n, f_n is a face of F. Since F is finite, infinitely many of the faces f_n must be the same face f, and by removing appropriate L_n from the list, we may assume without loss of generality that $f_n = f$ for all n. Apply the working Lemma 2.9. □

A finite CW complex H is **represented** in K if H is isomorphic to a subcomplex of K, is **finitely represented** in K if it is represented, but only by finitely many subcomplexes of K, and is **infinitely represented** in K if there are infinitely many isomorphic copies of H in K. In this latter case, there are infinitely many pairwise disjoint subcomplexes of K all isomorphic with H. Finally, H is **quasi-dense** in K if there exists a positive integer n such that every vertex of K is in an edge-path of length less than or equal to n that meets a subcomplex of K that is isomorphic to H. Note that when H is quasi-dense in K, then H is infinitely represented in K, but the converse fails. Quasi-denseness implies not only that there are infinitely many copies of H in K, but that the copies of H are uniformly distributed in K. Note that every finite connected[9] subcomplex of K is quasi-dense in K when K is globally highly symmetric, i.e., when $\mathrm{Aut}(K)$ acts cocompactly on K. When this condition prevails for K, that every finite connected subcomplex of K is quasi-dense in K, whether or not $\mathrm{Aut}(K)$ acts cocompactly, we say that K is **combinatorially repetitive**.

[9]We need not restrict to connected complexes, but that is how the authors always imagine them, and since there is no loss in generality in assuming that these finite complexes are connected, we do so.

Theorem 2.11. *If there is a finite connected complex H that is finitely represented in K, then K is singular. Alternately, if K is plural, then every finite connected subcomplex of K is infinitely represented in K.*

Proof. The proof is another application of the working lemma and is similar to that of Theorem 2.10. Assume that L is a planar polygonal complex such that $L \sim K$, and let L_1 be a finite, connected subcomplex of L that contains at least one face as well as a subcomplex that is isomorphic to H, which exists since $K \preceq L$. Let F be a finite, connected subcomplex of K that contains all the subcomplexes of K that are isomorphic to L_1, which exists since H, and therefore L_1, is finitely represented in K. Write $L = \cup_{n=1}^{\infty} L_n$, where $L_1 \subset L_2 \subset \cdots$ is an increasing sequence of finite, connected subcomplexes of L that exhausts L. Let g be a face of L_1 and let $(L_n, g) \cong (K_n, f_n)$, where K_n is a subcomplex of K isomorphic to L_n. By our choices of F and L_0, for each n, f_n is a face of F. Since F is finite, infinitely many of the faces f_n must be the same face f, and by removing appropriate L_n from the list, we may assume without loss of generality that $f_n = f$ for all n. Apply the working Lemma 2.9. □

Symmetries of a complex K usually are thought of in terms of the automorphism group $\mathrm{Aut}(K)$ with a homeomorphism of the plane that preserves the combinatorics of K providing a **global symmetry**, a self-isomorphism of K to itself. The notion of local isomorphism provides a refined notion of symmetry that the eye picks out of the familiar aperiodic tilings of Penrose and others. These tilings have no nontrivial global symmetries, yet the eye sees local symmetries abounding, where copies of large finite patches of the tiling appear in many places. These numerous **local symmetries** appear as isomorphisms between finite patches of tilings in differing regions of the tiling that do not extend to global symmetries. In the combinatorial setting of this paper, Theorems 2.10 and 2.11 position singular complexes at the two ends of the spectrum of symmetry for planar polygonal complexes, and plural ones somewhere in the middle. Specifically, Theorem 2.10 says that complexes that are highly symmetric globally are singular, and Theorem 2.11 says that ones that are rather asymmetric locally are singular; further, the plural complexes cannot be highly symmetric globally, but must be so locally in that they share with the globally highly symmetric complexes the fact that all finite connected subcomplexes, if not quasi-dense as in the globally highly symmetric case, are at least infinitely represented. The strictly stronger symmetry condition of combinatorial repetitiveness, that every finite connected subcomplex of K be quasi-dense in K, turns out to be important in identifying when the space $\mathsf{LI}(K)$ is compact. We close this section of the paper with a detailed examination of the topology of the rooted local isomorphism class $\mathsf{LI}(K)$ and an examination of combinatorially repetitive complexes.

Lemma 2.12. *The set $\mathsf{LI}(K)$ is a dense G_δ subspace of the complete metric space $\overline{\mathsf{LI}(K)}$, the closure of $\mathsf{LI}(K)$ in $\mathbf{RC}$, and as such is a completely metrizable Baire space.*

Proof. Obviously $\mathsf{LI}(K)$ is dense in the closure $\overline{\mathsf{LI}(K)}$. Using the notation $H \hookrightarrow K$ to mean that the CW complex H isomorphically embeds in the CW complex K, fix a face f of K and for each positive integer n let

$$U_n = \left\{ \mathsf{l} \in \overline{\mathsf{LI}(K)} : B_K(f,n) \hookrightarrow \mathsf{l}^* \right\}.$$

To verify the theorem, we observe that each U_n is open in $\overline{\mathsf{LI}(K)}$ and that $\mathsf{LI}(K) = \cap_{n=1}^{\infty} U_n$.

U_n is open in $\overline{\mathsf{LI}(K)}$: Let $\mathsf{l} = (L,g) \in U_n$ and choose $N \geq n$ such that $B_K(f,n) \hookrightarrow B_L(g,N)$, which is possible since $B_K(f,n)$ embeds isomorphically in L. Then the set $B_\rho(\mathsf{l}, e^{-N}) \cap \overline{\mathsf{LI}(K)}$ is an open neighborhood of l in $\overline{\mathsf{LI}(K)}$ that is contained in U_n, and U_n is open in $\overline{\mathsf{LI}(K)}$.

$\mathsf{LI}(K) = \cap_{n=1}^{\infty} U_n$: The "$\subset$" containment follows from the observation that $\mathsf{LI}(K) \subset U_n$ for each n. For the containment "$\supset$", suppose that $\mathsf{l} = (L,g) \in \overline{\mathsf{LI}(K)}$ is an element of U_n for all n. Then $B_K(f,n) \hookrightarrow \mathsf{l}^* = L$ for all n, implying that every finite connected subcomplex of K embeds isomorphically in L, so that $K \preceq L$. For any positive integer n, choose $\mathsf{e} = (E,h) \in \mathsf{LI}(K)$ such that $\rho(\mathsf{e}, \mathsf{l}) < e^{-n}$, which is possible since l is in the closure of $\mathsf{LI}(K)$. Then $B_L(g,n) \cong B_E(h,n) \hookrightarrow K$, the existence of the embedding following from $E \sim K$. Since n is arbitrary, this implies that every finite connected subcomplex of L embeds isomorphically in K, so that $L \preceq K$. We conclude that $L \sim K$, hence $\mathsf{l} \in \mathsf{LI}(K)$.

$\mathsf{LI}(K)$ is completely metrizable since it is a G_δ subspace of a complete metric space (by Mazurkiewicz's Theorem), and is a Baire space since it is a dense subspace of a complete metric space (or, by the Baire Category Theorem, because it is completely metrizable). □

The following "invariance of domain" observation will be used several times in the sequel.

Lemma 2.13. *Let K and L be planar polygonal complexes, f a face of K and g a face of L. Let H be a subcomplex of K containing f that is isomorphic to either (i) the filled n-neighborhood $B_L(g,n)$ or (ii) the combinatorial n-neighborhood $C_L(g,n)$, by an isomorphism taking g to f. Then $H = B_K(f,n)$ in case (i) and $H = C_K(f,n)$ in case (ii).*

Proof. We verify case (i). Let $h : B_L(g,n) \to H$ be a combinatorial isomorphism with $h(g) = f$ and let f' be any face of the combinatorial n-neighborhood $C_K(f,n)$. Let $f = f_0, \ldots, f_n = f'$ be a chain of faces of K of length n from f to f' and let k be the largest index such that $f_1, \ldots, f_k \subset H$. If $k < n$, then an edge common to both f_k and f_{k+1} must be on the boundary of the disk H and applying the inverse isomorphism h^{-1} would provide a chain of faces from g to the face $g'' = h^{-1}(f'')$ of $B_L(g,n)$ that meets the outer sphere boundary of $B_L(g,n)$ and has length less than n. This contradicts the observation of page 18 that any face of a combinatorial n-neighborhood that meets the boundary of the neighborhood must have combinatorial distance from the center at least as large as n. It follows that $k = n$, and from this we get $C_K(f,n) \subset H$. Since $B_K(f,n)$

is the smallest simply connected subcomplex of K that contains the combinatorial n-neighborhood $C_K(f,n)$, we have $B_K(f,n) \subset H$.

It is easy to see that the image of the combinatorial n-neighborhood $C_L(g,n)$ under h satisfies $h(C_L(g,n)) \subset C_K(f,n) \subset B_K(f,n)$. As $B_K(f,n) \subset H$, we have $C_L(g,n) \subset h^{-1}(B_K(f,n))$ and therefore $h^{-1}(B_K(f,n))$ is a combinatorial disk containing the combinatorial n-neighborhood $C_L(g,n)$. As $B_L(g,n)$ is the smallest simply connected subcomplex of L containing $C_L(g,n)$, we conclude that $B_L(g,n) \subset h^{-1}(B_K(f,n))$, or $H \subset B_K(f,n)$. $\square$

Theorem 2.14. *If K is plural, then the space $\mathsf{LI}(K)$ has no isolated points. It follows that $\mathsf{LI}(K)$ is a completely metrizable, uncountably infinite perfect Baire space.*

Proof. Let $\mathsf{l} = (L,g) \in \mathsf{LI}(K)$. Since K is plural, there exists a planar polygonal complex J that is locally isomorphic to, but not isomorphic to L. Let n be a positive integer. Since $L \sim J$, there is an isomorphic embedding $h : B_L(g,n) \hookrightarrow J$ with, say, $h(g) = f$. By the preceding lemma, the image of the n-neighborhood $B_L(g,n)$ under h is precisely the n-neighborhood $B_J(f,n)$, and this implies that $0 \neq \rho(\mathsf{j},\mathsf{l}) \leq e^{-n}$, where $\mathsf{j} = (J,f)$. It follows that l is not an isolated point of $\mathsf{LI}(K)$.

Lemma 2.12 guarantees that the metric space $\mathsf{LI}(K)$ is a completely metrizable Baire space and the argument of this proof thus far guarantees that it is perfect. This implies that $\mathsf{LI}(K)$ cannot be countable. $\square$

The next corollary confirms the interesting dichotomy that the local isomorphism class (K) is either a singleton or uncountably infinite.

Corollary 2.15. *For the planar polygonal complex K, the local isomorphism class (K) has either one element or uncountably many elements.*

Proof. The fact that (K) is uncountably infinite when K is plural is an immediate consequence of the fact that (K) is the image of the uncountable set $\mathsf{LI}(K)$ under the countable-to-one function $*$. $\square$

The set $\mathsf{LI}(K)$ generally fails to be closed in $\mathbf{RC}$. The next theorem demonstrates that its closure captures precisely those complexes that precede K under the pre-order $\preceq$.

Theorem 2.16. *For an arbitrary planar polygonal complex K, the closure of $\mathsf{LI}(K)$ in $\mathbf{RC}$ is*

$$\overline{\mathsf{LI}(K)} = \{\mathsf{l} \in \mathbf{RC} : \mathsf{l}^* \preceq K\}.$$

Proof. $(\subset)$: Let $(L,g) = \mathsf{l} \in \overline{\mathsf{LI}(K)}$ and H a combinatorial patch in L. Choose n so that $H \subset B_L(g,n)$ and let $(J,f) = \mathsf{j} \in \mathsf{LI}(K)$ such that $\rho(\mathsf{j},\mathsf{l}) < e^{-n}$. Then the neighborhoods $B_L(g,n)$ and $B_J(f,n)$ are isomorphic and therefore $H \hookrightarrow J$. Since $J \sim K$, we conclude that $H \hookrightarrow K$, and therefore $\mathsf{l}^* = L \preceq K$. $(\supset)$: Suppose that $(L,g) = \mathsf{l} \in \mathbf{RC}$ with $L = \mathsf{l}^* \preceq K$.

Then for any n, there is an isomorphic embedding $h_n : B_L(g, n) \hookrightarrow K$. By Lemma 2.13, the image $h_n(B_L(g, n)) = B_K(h_n(g), n)$. Hence $\rho(\mathsf{l}, \mathsf{k}_n) \le e^{-n}$, where $\mathsf{k}_n = (K, h_n(g)) \in \mathsf{LI}(K)$, and therefore $\mathsf{l} \in \overline{\mathsf{LI}(K)}$. $\square$

We now want to ask when the perfect, totally disconnected metric space $\mathsf{LI}(K)$ is compact, and therefore a Cantor set, whenever K is plural. First note that if K has FLC, say with degree bounded by the positive integer d and satisfying the θ-isoperimetric inequality for the positive function θ, then $\mathsf{LI}(K)$ is a subspace of the compact set $\mathbf{RC}^{d,\theta}$ (Theorem 2.4), and the question becomes when is the space $\mathsf{LI}(K)$ closed in $\mathbf{RC}^{d,\theta}$. The condition that guarantees that $\mathsf{LI}(K)$ is closed in $\mathbf{RC}^{d,\theta}$ is that K be combinatorially repetitive. Recall that when K is combinatorially repetitive, for each finite connected subcomplex H of K, there exists an integer $n > 0$ such that every face of K is n-close to an isomorphic copy of H, where n-close is measured in the combinatorial distance between faces. This is a sort of regularity condition that says every finite connected subcomplex appears in every large enough neighborhood of any face, where the size of the neighborhood depends only on the subcomplex. We will see later that any complex K that arises as the expansion complex of certain subdivision operators is combinatorially repetitive and satisfies FLC.

Theorem 2.17. *If K is combinatorially repetitive, then $\mathsf{LI}(K)$ is a closed subspace of $\mathbf{RC}$, and therefore complete in the metric ρ, and every complex in (K) is combinatorially repetitive. If in addition K is plural and has FLC then $\mathsf{LI}(K)$ is a Cantor set.*

Proof. By Theorem 2.16, to verify that $\mathsf{LI}(K)$ is closed all we need show is that $\mathsf{l} \in \mathsf{LI}(K)$ whenever $(L, g) = \mathsf{l} \in \mathbf{RC}$ satisfies $L \preceq K$. Let $L \preceq K$ and let H be a combinatorial patch in K. Since K is combinatorially repetitive, there exists an integer n such that an isomorphic copy of H appears in the combinatorial n-neighborhood of every face of K. Since $L \preceq K$, there is an isomorphic embedding $h : B_L(g, n) \hookrightarrow K$. By Lemma 2.13, $h(B_L(g, n)) = B_K(h(g), n)$ and our choice of n guarantees that $H \hookrightarrow B_K(h(g), n)$. It follows that $H \hookrightarrow B_L(g, n) \subset L$ and therefore $K \preceq L$. Therefore $K \sim L$ and $(L, g) = \mathsf{l} \in \mathsf{LI}(K)$.

That L is combinatorially repetitive whenever $K \sim L$ is left as an exercise. If K is of bounded degree d, satisfies the θ-perimetric inequality, and is combinatorially repetitive, then $\mathsf{LI}(K)$ is a closed subspace of the compact set $\mathbf{RC}^{d,\theta}$. If in addition K is plural, Theorem 2.14 implies that $\mathsf{LI}(K)$ has no isolated points and, therefore, is a Cantor set. $\square$

The next result is a consequence of the two preceding theorems.

Corollary 2.18. *If $\mathsf{LI}(K)$ is a closed subspace of $\mathbf{RC}$, then $L \preceq K$ if and only if $L \sim K$. In particular, if K is combinatorially repetitive, then $L \preceq K$ if and only if $L \sim K$.*

3. Hierarchy, Subdivision Operators, and Type

Traditional aperiodic substitution tilings exhibit a hierarchical structure in which aggregates of tiles coalesce to form patches of tiles that are homothetic images of patches of the original tiling, and this occurs at coarser and coarser scales. The conformal tilings that arise as expansion complexes that are studied in Section 4 fail to posses the sort of hierarchical structures found in the traditional substitution tilings based on homotheties, but the combinatorics underlying these conformal tilings do possess a hierarchical structure based on combinatorial isomorphisms. Though there seems to be no precise definition of the term *hierarchical structure* in the traditional setting, we do provide a precise definition for what we mean for a planar polygonal complex to possess a **combinatorial hierarchical structure**. This we do using a combinatorial subdivision operator and its inverse, a combinatorial aggregation operator. After introducing this combinatorial hierarchy, the next task is to understand the geometric effect, if any, of an existent combinatorial hierarchy for a complex K on its corresponding tiling T_K. We define what it means for a conformal, reflective tiling T to possess a **conformal hierarchical structure** and we give conditions on the subdivision operator τ of the combinatorial hierarchy of K that guarantee that the tiling T_K possesses a conformal hierarchy induced by τ. We then examine the type problem in the context of hierarchical conformal tilings, proving several results implying that type is parabolic either for the planar polygonal complex K, or across the whole of the local isomorphism class (K). Along the way, we are compelled to identify important properties of hierarchies and subdivision rules—**(strongly) expansive** hierarchies and **shrinking** subdivision rules—that aide in proving a complex to be parabolic and in using the parabolic type of a complex K to infer the parabolic type of one locally isomorphic with K. We end this section with an examination of a possible symmetry of a planar polygonal complex with a subdivision of that complex, called a **supersymmetry**, that may be used to generate a conformal hierarchy for the complex.

3.1. **Combinatorial hierarchy.** The idea behind combinatorial hierarchy is that the faces of a planar polygonal complex K may be aggregated to produce another planar polygonal complex that locally looks like K, that the faces of this may then be aggregated to produce yet another that locally looks like K, ad infinitum. In this way, the local structure evident in K appears at coarser and coarser scales as one blurs the edges of faces to aggregate several faces into one. Our primary interest will be in plural planar polygonal complexes that are combinatorially repetitive and of bounded degree, and in hierarchical structures generated by subdivision operators with a certain amount of regularity that will be described as we develop this section. At first, though, we will not enforce restrictions on the planar polygonal complex K nor on the subdivision operator τ. It is only as we turn our attention to the local isomorphism class (K) that we begin to enforce restrictions on K and τ that are designed to manifest the hierarchy across the whole of the local isomorphism class.

Recall that a combinatorial subdivision of the planar polygonal complex K is a planar polygonal complex L for which each open cell (vertex, edge, or face) of L is contained in an open cell of K. In particular, each closed face of K is the union of finitely many closed faces of L, and each closed edge of K is a union of finitely many closed edges of L. Recall that the subdivision L is **totally nontrivial** if it is nontrivial on every face of K, meaning that each closed face of K is the union of at least two closed faces of L. In this case, we say that K is obtained from L by **aggregation**, and we call K a **combinatorial aggregate** of L. We say that the planar polygonal complex K **exhibits a combinatorial hierarchy** if there is a bi-infinite sequence $\{K_n : n \in \mathbb{Z}\}$, called a **combinatorial hierarchy** for K, of planar polygonal complexes indexed by the integers such that the following three conditions hold:

(1) $K_0 = K$;

(2) K_{n+1} is a totally nontrivial combinatorial subdivision of K_n, for all $n \in \mathbb{Z}$;

(3) $K_n \sim K_{n+1}$, for all $n \in \mathbb{Z}$.

Our primary concern will be combinatorial hierarchies that are defined in terms of subdivision operators. Recall that the subdivision operator $\tau = \{\tau_n\}$ is a subdivision rule that partitions the faces of any planar polygonal complex K by polygonal type, then subdivides the faces of type n by the pattern of the CW-decomposition τ_n of the standard n-gon Δ_n. The rule is edge compatible and, because each seed τ_n is oriented and rotationally symmetric, can be applied to a k-gon face of K in a unique way up to isomorphism. It is immediate that, whenever $K \cong L$, we have $\tau K \cong \tau L$ by an isomorphism induced from an isomorphism of K to L. This of course implies that τ induces a function $\hat{\tau} : \mathsf{C} \to \mathsf{C}$ in the obvious way, by sending the isomorphism class of K to the isomorphism class of τK. The simple subdivision rules of Fig.'s 1 and 2 define **simple subdivision operators** that may be applied to an arbitrary planar polygonal complex. An example of a non-simple subdivision operator is one that applies hexagonal subdivision to each 3-gon face of K while applying quad subdivision to all other faces of K. Our notation is a bit lacking since we have chosen to use the complex K to name its isomorphism class in C. This is where the hat symbol $\hat{}$ is useful: τK means a specific combinatorial subdivision of K while $\hat{\tau} K$ means the isomorphism class of τK in C. So, for instance, if Z is the **integer lattice 4-gon complex**, the planar 4-gon complex whose 1-skeleton is $\mathcal{Z}^2$, the integer lattice graph whose vertices form the integer lattice $\mathbb{Z}^2$ and whose edges span vertices a unit distance apart, then the quad subdivision operator ν produces the combinatorial subdivision νZ that, though not equal to Z since it subdivides each 4-gon into four 4-gons, nonetheless is isomorphic with Z. While $\nu Z \neq Z$ as planar polygonal complexes, in our notation we have $\hat{\nu} Z = Z$ in C so that Z is a fixed point of the mapping $\hat{\nu}$. This notation should cause no difficulty in the sequel.

The subdivision operator τ is said to **manifest a combinatorial hierarchy** for the planar polygonal complex K if, in the definition, K exhibits a combinatorial hierarchy for which $\tau K_n = K_{n+1}$, for each $n \in \mathbb{Z}$. It is convenient in this case to call the inverse operator τ^{-1} that yields K_{n-1} when applied to K_n an **aggregation operator** for K, which, unlike τ, is, a priori, defined only on the sequence $\{K_n\}$. Notice that by reindexing, the operator

τ manifests a combinatorial hierarchy of every K_n. For a specific example, the subdivision operator ν manifests a combinatorial hierarchy for the complex Z of the preceding paragraph. In this case Z is singular and item (3) of the definition necessarily reduces to $Z_n \cong Z_{n+1}$. The regular pentagonal complex P that produces the regular pentagonal tiling of [2] exhibits a combinatorial hierarchy manifested by the simple pentagonal operator, and though item (3) again reduces to $P_n \cong P_{n+1}$, P is not singular, but rather plural. We will see that this implies that every complex K that is locally isomorphic with P also exhibits a combinatorial hierarchy manifested by the pentagonal subdivision operator, but for the typical hierarchy, $K_n \ncong K_{n+1}$. It might seem that it would be rather difficult to find examples of complexes that exhibit combinatorial hierarchies beyond the rather pedestrian ones like Z and P, but we will see in Section 4 that every expansion complex of certain appropriate subdivision rules exhibits a combinatorial hierarchy.

In addressing the type problem for tilings that exhibit a hierarchical structure, it will be important to aggregate the tiling T iteratively using the hierarchy so that a given compact subset is engulfed by a set of aggregated tiles with a common vertex. For this to occur, the combinatorial hierarchy exhibited by the underlying complex K_T needs to have the combinatorial analogue of this engulfing property. To explicate this in the setting of the planar polygonal complex K, we need to develop a bit of notation and introduce some ancillary concepts. The first goal is to define what we mean for the combinatorial hierarchy $\{K_n\}$ for $K = K_0$ to be **expansive**, and for a **core** of the complex K_n to **engulf** a finite subcomplex of K. First, a **core** of any planar polygonal complex K is a combinatorial patch that comes in one of three flavors—its faces consists either of a single face of K, a **face core**, two faces that meet along a common edge, an **edge core**, or all the faces that meet at a vertex, a **vertex core**. In the general case of a combinatorial hierarchy, let σ_m be the rule that subdivides the complex K_m to obtain the complex K_{m+1}, and given integers $m \leq n$, let σ_m^n be the rule for K_m that applies $\sigma_m, \ldots, \sigma_{n-1}$ successively to obtain the complex K_n.[10] The collection $\sigma = \{\sigma_m\}$ is called the **subdivision sequence** for the hierarchy $\{K_m\}$. Let F be a finite subcomplex of $K = K_0$ and let $n \leq 0$ be an integer. We say that the subcomplex E of K_n **engulfs** the complex F if F is a subcomplex of $\sigma_n^0 E$, itself a subcomplex of $K_0 = K$, and that the combinatorial hierarchy of K is **expansive** if, for every finite subcomplex F of K, there exists a integer $n \leq 0$ such that F is engulfed by a core of K_n. This formalizes the idea that a finite subcomplex of K may be thought of as a subset of a core of one of the complexes K_n, for $n \leq 0$, that is obtained from K by aggregating faces. Notice that if a core of K_n engulfs a finite complex F, then, for every integer $m \leq n$, a core of K_m also engulfs F. When a subdivision operator τ manifests the combinatorial hierarchy for K, then $\sigma_m = \tau$ for all $m \in \mathbb{Z}$, and $\sigma_m^n = \tau^{n-m}$ for all $m \leq n \in \mathbb{Z}$.[11] The combinatorial hierarchies for Z and P described in the preceding

[10]The rule σ_m^m is the identity and $\sigma_m^{m+1} = \sigma_m$.

[11]It might seem that hierarchies may arise only from subdivision operators so that always there exists a subdivision operator τ such that $\sigma_m = \tau$ for all m. This is not so, at least when there is no bound on the degree of the complex K. Hierarchies determined by a subdivision sequence that cannot be manifested by

paragraph are both expansive. It might be helpful to see an example of a non-expansive combinatorial hierarchy.

Example 3.1. A non-expansive combinatorial hierarchy. The combinatorial complex we describe in this example underlies the *Penrose hyperbolic tiling* familiar to the traditional tiling community and the *discrete hyperbolic plane* familiar to the geometric group theory and conformal geometry communities. There are various ways to describe the planar polygonal complex H of this example, including using a finite subdivision rule, or using a hyperbolic isometry to build a metric model of the complex in the upper-half-plane model of hyperbolic geometry, or using a standard presentation of a Baumslag-Solitar group to see copies of the 1-skeleton of H in the Cayley graph. We give a rather pedestrian description. The vertices of H lie along the horizontal lines in the complex plane $\mathbb{C}$ with integer imaginary parts. At level $m \in \mathbb{Z}$, the vertices are $\{v_{m,k} = 2^m k + m\mathbf{i} : k \in \mathbb{Z}\}$ and the edges are the horizontal segments incident to $v_{m,k}$ and $v_{m,k+1}$, for all $k \in \mathbb{Z}$. The remaining edges are vertical segments incident to the vertices $v_{m,k}$ and $v_{m-1,2k}$ for all $m, k \in \mathbb{Z}$. The faces of H are pentagonal with cyclically ordered vertices $v_{m,k}, v_{m-1,2k}, v_{m-1,2k+1}, v_{m-1,2k+2}, v_{m,k+1}$ for the face $f_{m,k}$, for $m, k \in \mathbb{Z}$. See Fig. 4. For any integer n, the nth complex in the combinatorial hierarchy $\{H_n\}$ is just a copy of H translated vertically n units; explicitly, $H_n = H + n\mathbf{i}$. The subdivision rule σ_0 subdivides each face $f_{m,k}$ by bisecting each of the three horizontal edges of $f_{m,k}$ by adding a midpoint vertex, then adding a single vertical edge incident to the midpoint of the edge $\langle v_{m,k}, v_{m,k+1}\rangle$ and the vertex $v_{m-1,2k+1}$. This subdivides each of the pentagonal faces of H into two pentagonal faces and, easily, $\sigma_0 H = H + \mathbf{i}$. The hierarchy $\{H_n\}$ is not expansive. Indeed, aggregation (and subdivision) takes place horizontally, so that iterated aggregation produces faces that remain sandwiched between two horizontal lines at two consecutive levels m and $m+1$. It follows that no amount of aggregation can produce a core that contains any subcomplex that spans across three or more horizontal levels. It turns out that the Riemann surfaces $\mathcal{S}_Z$ and $\mathcal{S}_P$ are parabolic while $\mathcal{S}_H$ is hyperbolic. Indeed, it is easy to see that $\mathcal{S}_Z$ is parabolic, and that $\mathcal{S}_P$ is parabolic was proved in [2] by showing the existence of a loxodromic automorphism of the surface. This is no accident as we will see later that expansive hierarchies with certain assumptions on subdivision operators, assumptions satisfied by the quad subdivision operator of Z and the pentagonal subdivision operator of P, always lead to parabolic tilings, not only for those determined by the complexes of the hierarchy, but also for any determined by a planar polygonal complex locally isomorphic to one of these. As for $\mathcal{S}_H$, one may identify in the 1-skeleton of the complex H an infinite binary tree, which implies by various standard results that the surface is hyperbolic. $\surd$

Recall that the combinatorial distance between two faces f and g of a planar polygonal complex K is defined by $d_K(f,g) \le n$, for an integer $n \ge 0$, if there exists a chain of faces $f = f_0, \dots, f_n = g$ such that f_{i-1} and f_i share an edge for $i = 1, \dots, n$; further, $d_K(f,g) = n$

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any subdivision operator exist for the $\preceq$-maximal planar polygonal complexes constructed in the Addendum to this section. We, however, will be concerned with hierarchies that are manifested by subdivision operators.

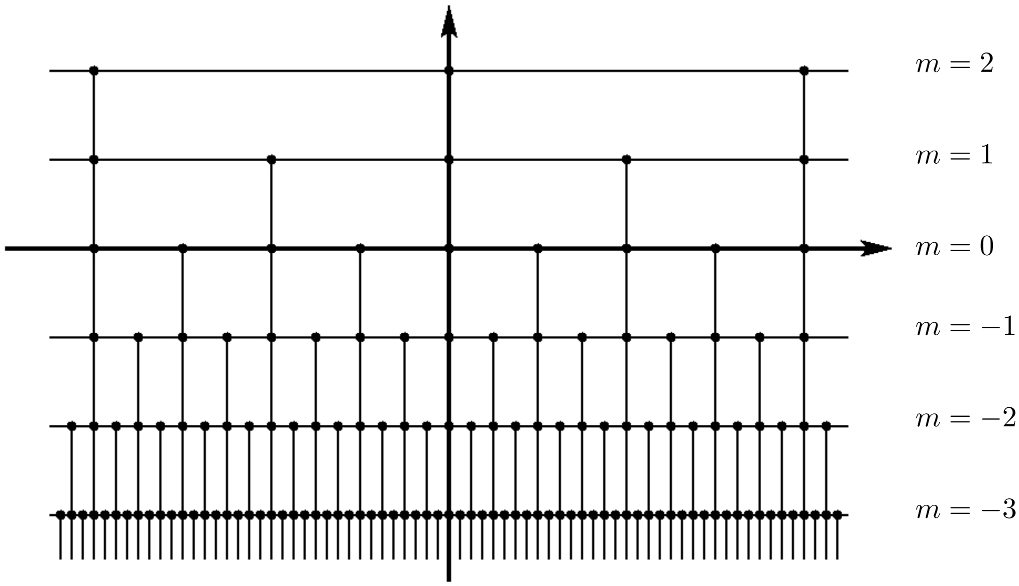

FIGURE 4. The planar pentagonal complex H.

if $d_K(f,g) \le n$ and $d_K(f,g) \not\le n-1$. For the purposes of analyzing expansive hierarchies, it is easier to work with a slightly modified notion of combinatorial distance where we measure how close two faces are by stepping from face to face, not only across edges, but across vertices also. Consequently, if there is a **simple chain** of faces $f = f_0, \ldots, f_n = g$ such that f_{i-1} and f_i meet[12] for $i = 1, \ldots, n$, we write $\delta_K(f,g) \le n$; further, $\delta_K(f,g) = n$ if $\delta_K(f,g) \le n$ and $\delta_K(f,g) \not\le n-1$. If F is a finite subcomplex of K, the **δ_K-diameter** of F is the largest value of $\delta_K(f,g)$ as f and g range over all the faces of K that meet F. Notice that if L is a subdivision of K and f' is a face of L contained in the face f of K and g' is a face of L contained in the face g of K, then $\delta_L(f',g') \ge \delta_K(f,g)$, and we write $\delta_L \ge \delta_K$. Symbolically, $L \le K$ implies $\delta_L \ge \delta_K$. In this sense we may say that the sequence of metrics δ_{K_n} for the combinatorial hierarchy $\{K_n\}$ increases distances as $n \to \infty$ and decreases distances as $n \to -\infty$. This observation has some important implications, as we will see later. Notice that this increase and decrease of distance among faces in a hierarchy is not necessarily strict, as Example 3.1 illustrates.

With the help of the distance function δ_K, stronger versions of expansive hierarchy may be defined. Let $\phi : \mathbb{N} \to \mathbb{R}$ be a non-decreasing function. We say that the combinatorial hierarchy $\{K_n\}$ is **ϕ-expansive** if there is a positive constant M such that, for any integer $m > M$, any finite subcomplex F of K of δ_K-diameter at most $\phi(m)$ is engulfed by a core

[12]The difference between the two notions of distance is, of course, that f_{i-1} and f_i may meet only at a set of vertices in the definition for δ_K, while for d_K they must share at least one edge. Obviously, $\delta_K \le d_K$.

of K_{-m}. This of course says that a finite subcomplex F of K is engulfed in a core after a number of aggregations that is dependent only on the diameter of F: m aggregations are sufficient to engulf into some core any finite subcomplex of δ_K-diameter at most $\phi(m)$. Here are a couple of important examples. If there exists positive constants M and r such that, for any positive integer $m > M$, any finite subcomplex F of K of δ_K-diameter at most rm is engulfed by a core of K_{-m}, then the hierarchy is **linearly expansive**. This of course says that a finite subcomplex F of K is engulfed after a number of aggregations that is linear with respect to the diameter of F. Most pertinent to our subsequent development is the case where ϕ is an increasing exponential function of the variable m, say of the form $\phi(m) = t^m$ for some $t > 1$. Being ϕ-expansive for this function ϕ says that, for each large enough positive integer m, m aggregations are sufficient to engulf all finite subcomplexes of δ_K-diameter at most t^m. We say in this case that the hierarchy is **exponentially expansive**. In case $\{K_n\}$ is ϕ-expansive for some non-decreasing function ϕ, we say that it is **strongly expansive**.

We now present a criterion, simple to check, that guarantees that a combinatorial hierarchy manifested by a subdivision operator is strongly expansive. Let τ be a subdivision operator. Recall that τ is determined by a collection $\{\tau_n\}$ of oriented regular CW complexes where τ_n is a nontrivial, rotationally symmetric CW-decomposition of an oriented n-gon Δ_n that decomposes each edge into N sub-edges, for a value of N independent of n. If v is a vertex of Δ_n, the set $\angle v = \{v\} \cup d^\circ \cup e^\circ \cup \Delta_n^\circ$, where d and e are the edges of Δ_n that are incident with the vertex v, is called the **open angle** of Δ_n at v. We say that τ is **strictly shrinking** if, for every n, every closed face of τ_n is contained in an open angle of Δ_n. In particular, no closed face of the subdivision τ_n can contain two vertices of Δ_n, nor meet more than two edges of Δ_n nontrivially, and if a face does meet two edges, those edges must be adjacent. Similarly, if f is a face of the planar polygonal complex K with vertex v incident to edges d and e of the face f, then the set $\angle_f v = \{v\} \cup d^\circ \cup e^\circ \cup f^\circ$ is the open angle of f at v and, if τ is strictly shrinking, then every closed face of τf is contained in an open angle of f. For each positive integer k, let τ^k be the subdivision operator obtained by iterating τ k times. We say that τ is **shrinking** if τ^k is strictly shrinking for some integer $k \geq 1$. Notice that if τ is (strictly) shrinking, then τ^k is (strictly) shrinking for all positive integers k. Among simple subdivision operators, the star and delta rules are not shrinking, while the remaining ones showcased in Fig.'s 1 and 2 are strictly shrinking. The hex rule is not strictly shrinking, but is shrinking since its second iterate is strictly shrinking. See Fig. 5.

Lemma 3.1. *Let τ be a shrinking subdivision operator with τ^k strictly shrinking for the positive integer k. Then $\delta_{\tau^k K} \geq 2\delta_K - 1$ for any planar polygonal complex K.*

Proof. First we show that if τ is strictly shrinking and f' and g' are faces of τK with $\delta_{\tau K}(f', g') = 2$, then $\delta_K(f, g) \leq 1$, where f and g are the faces of K containing the respective faces f' and g'. Let $f' = f'_0, f'_1, f'_2 = g'$ be a simple chain of faces of τK with $f'_{i-1} \cap f'_i \neq \emptyset$ for $i = 1, 2$. For each $i = 0, 1, 2$, let f_i be the face of K that contains f'_i and

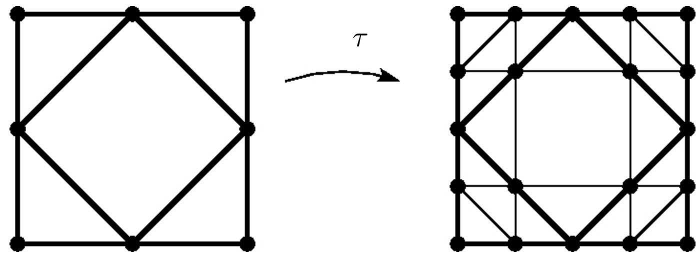

FIGURE 5. Hexagonal subdivision is shrinking, but not strictly shrinking. The 4-gon Δ_4 on the left has been hex subdivided and the "diamond" face in the center is not contained in any open angle. After a second hexagonal subdivision, on the right, each face is contained in an open angle.

observe that, $\emptyset \neq f'_{i-1} \cap f'_i \subset f_{i-1} \cap f_i$, for $i = 1, 2$. It follows that $f = f_0, f_1, f_2 = g$ is a simple chain of consecutively intersecting faces of K from f to g. Let u be a vertex of τK common to both f'_0 and f'_1 and w a vertex common to both f'_1 and f'_2. Since $\delta_{\tau K}(f', g') > 1$, $f'_0 \cap f'_2 = \emptyset$, so $u \neq w$. Since τ is strongly shrinking, the face f'_1 is contained in an open angle of f_1, say the open angle $\angle_{f_1} v = \{v\} \cup d^\circ \cup e^\circ \cup f_1^\circ$ at the vertex v incident with edges d and e of f_1. If the vertex u is in the open face f_1°, then f'_0 and f'_1 are contained in f_1 and we conclude that $f_0 = f_1$. It follows that $f = f_0, f_2 = g$ is a simple chain of consecutively intersecting faces of K from f to g of length 2, and therefore $\delta_K(f, g) \leq 1$. A similar argument shows that $\delta_K(f, g) \leq 1$ if w is in the open face f_1°. Assuming that neither u nor w is in the open face f_1°, we have $\{u, w\} \subset \{v\} \cup d^\circ \cup e^\circ$ and, since $u \neq w$, there are two further cases to consider: either $u = v$ and $w \in e^\circ$, or $u, w \in d^\circ \cup e^\circ$. In the first case, f_0 and f_1 have a common vertex $u = v$ and f_1 and f_2 have a common edge e that contains the vertex v, and so f_0 and f_2 have a common vertex v. It follows that f_1 may be removed from the chain $f = f_0, f_1, f_2 = g$ to get $\delta_K(f, g) \leq 1$. Similarly, in the second case, f_0 and f_1 have a common edge d or e and f_1 and f_2 have a common edge d or e, and as d and e have the common vertex v, again f_0 and f_2 have a common vertex v and $\delta_K(f, g) \leq 1$. In each case we conclude that $f = f_0, f_2 = g$ is a simple chain of consecutively intersecting faces of K from f to g of length 2, so that $\delta_K(f, g) \leq 1$.

Continuing under the assuption that τ is strictly shrinking, note that the lemma is true in case $\delta_{\tau K}(f', g') = 0$ or 1, so we assume that $\delta_{\tau K}(f', g') = n \geq 2$. The argument of the preceding paragraph may be applied to any of three consecutive faces f_{i-1}, f_i, f_{i+1} in a simple chain $f = f_0, \ldots, f_n = g$. In particular, it may be applied $\lfloor n/2 \rfloor$ times at $i = 1, 3, \ldots, 2\lfloor n/2 \rfloor - 1$, the odd integers in the list $1, \ldots, n-1$, to obtain a simple chain of consecutively intersecting faces of K from f to g of length $1 + (n/2)$ when n is even and $1 + (n+1)/2$ when n is odd. This implies that $\delta_{\tau K} \geq 2\delta_K - 1$, the minus one needed in case n is odd.

Now let τ be shrinking. Then there exists a positive integer k such that τ^k is strictly shrinking and for any such k, the arguments of the preceding paragraphs imply that $\delta_{\tau^k K} \geq 2\delta_K - 1$. $\square$

Corollary 3.2. *Let τ be a subdivision operator that manifests the combinatorial hierarchy $\{K_n\}$ for the planar polygonal complex K. If τ is shrinking, then there exists a positive integer k such that $\delta_{K_{n+k}} \geq 2\delta_{K_n} - 1$, for all integers n.*

Proof. Apply the lemma to K_n for any integer n to conclude that $\delta_{K_{n+k}} \geq 2\delta_{K_n} - 1$, where τ^k is strictly shrinking. □

Theorem 3.3. *Let τ be a subdivision operator that manifests a combinatorial hierarchy for the planar polygonal complex K. If τ is shrinking, then the hierarchy $\{K_n\}$ is exponentially expansive.*

Proof. Let k be the positive integer promised by the preceding corollary. Define $\phi : \mathbb{N} \to \mathbb{R}$ by

$$\phi(m) = 2^{m/2k} = t^m,$$

an increasing exponential function in the variable m with base $t = 2^{1/2k} > 1$. Our claim is that $\{K_n\}$ is ϕ-expansive, which would confirm that $\{K_n\}$ is exponentially expansive. Let F be a finite subcomplex of K of δ_K-diameter at most t^m, for some positive integer m, and let f be any face of K that meets F. We verify that F is engulfed by a core of K_{-m} as long as $m > 4k$.

First note that the condition that $\delta_{K_{n+k}} \geq 2\delta_{K_n} - 1$ for all integers n implies that, for all positive integers $r, \delta_K \geq 2^r\delta_{K_{-kr}} - 2^r + 1$. Choose r so that $2^{r-1} < t^m \leq 2^r$ and observe that this implies that $2k(r-1) < m \leq 2kr$. Since $\delta_K \geq 2^r\delta_{K_{-kr}} - 2^r + 1$, if g is a face of K that meets F with $\delta_K(f,g) > 1$, and a and b are faces of K_{-kr} that respectively contain f and g, then

$$t^m \geq \delta_K(f,g) \geq 2^r\delta_{K_{-kr}}(a,b) - 2^r + 1.$$

This implies, since $t^m \leq 2^r$, that when $\delta_K(f,g) > 1$, then $\delta_{K_{-kr}}(a,b) \leq 2 - 1/2^r < 2$, or $\delta_{K_{-kr}}(a,b) \leq 1$. This says that after kr aggregations, the subcomplex F is engulfed by the combinatorial patch $P_{K_{-kr}}(a,1)$ of radius 1, the patch whose faces c satisfy $\delta_{K_{-kr}}(a,c) \leq 1$. Our claim is that after k more aggregations, the patch $P_{K_{-kr}}(a,1)$, and therefore the subcomplex F, is engulfed by a core of $K_{-k(r+1)}$. Since $2k(r-1) < m$, it follows that

$$(\dagger) \qquad k(r+1) \leq 2k(r-1) < m \quad \text{as long as} \quad r \geq 3.$$

Let $m > 4k$. Then $t^m = 2^{m/2k} > 2^2$, which implies that $2^{r-1} \geq 2^2$, so $r \geq 3$ and hence Inequality ($\dagger$) holds. Since F is engulfed by a core of $K_{-k(r+1)}$ and $m > k(r+1)$, F

also is engulfed by a core of K_{-m}. This implies that the hierarchy $\{K_n\}$ is exponentially expansive, and the proof is complete modulo the claim.

As for the claim, we need only show that when τ is strictly shrinking, the combinatorial patch $P_{\tau K}(a,1)$ of any face a of τK is contained in a core of K. Let b be the face of K that contains the face a of the subdivision τK. Since τ is strictly shrinking, a is contained in an open angle $\angle_b v = \{v\} \cup d^\circ \cup e^\circ \cup b^\circ$ determined by the vertex v of b that is incident with the edges d and e of b. Immediately then, the vertices of a lie in this open angle and this implies that any face of τK that meets a is contained in the vertex core $c(v)$ of K determined by v, and so the core $c(v)$ engulfs $P_{\tau K}(a,1)$. This completes the proof; however, we comment that $P_{\tau K}(a,1)$ is engulfed by the face core determined by b if $a \subset b^\circ$ and that $P_{\tau K}(a,1)$ is engulfed by the edge core determined by d if a does not meet the edge e. $\square$

3.2. **Local isomorphism and combinatorial hierarchy.** It turns out that combinatorial hierarchy alone is not enough to guarantee the types of structures that one is trying to capture by the way the term *hierarchical tiling* is used in the traditional tiling community. Two other attributes appear in traditional hierarchical tilings, to wit, the tilings are repetitive and satisfy FLC. In this section, we enforce the combinatorial versions of repetitiveness as well as FLC on our combinatorial complexes. This allows us to prove that the existence of a combinatorial hierarchy is really a property not so much of a single complex K, but of its whole local isomorphism class (K).

Theorem 3.4. *Let K be a combinatorially repetitive, planar polygonal complex that has FLC. If the subdivision operator τ manifests a combinatorial hierarchy for K, then that same operator τ manifests a combinatorial hierarchy for any planar polygonal complex L that is locally isomorphic to K.*

Proof. Note that any planar polygonal complex locally isomorphic to K is combinatorially repetitive, has faces of the same polygonal type as the faces of K, has FLC, and, moreover, has degree bounded by d, where K has bounded degree d. Let $\tau = \{\tau_k\}$ be a subdivision operator that manifests the combinatorial hierarchy of K with $\tau K_n = K_{n+1}$ for each n, where $\{K_n\}$ is a combinatorial hierarchy for K. Since the degree of each complex K_n is bounded by d, only τ_k for $k \le d$ is used in subdividing, and we let λ be an upper bound on the number of faces into which τ subdivides any face of K. This means, for each $n \in \mathbb{Z}$, that every closed face of K_n is a union of at most λ closed faces of $\tau K_n = K_{n+1}$.

Let $K \sim L$ and define $L_0 = L$ and, for each positive integer n, $L_n = \tau^n L$ so that $L_n = \tau L_{n-1}$ when $n \ge 1$. Our first task is to define the complex L_{-1} so that $\tau L_{-1} = L_0 = L$. The data used to construct L_{-1} are the three complexes L_0, K_0 and K_{-1}. Write $L_0 = \cup_{\ell=1}^{\infty} B_\ell$, where $B_\ell = B_{L_0}(f, \lambda(\ell+1))$ is the filled $\lambda(\ell+1)$-neighborhood of the face f of L_0. Using that $K_0 \sim L_0$ and Lemma 2.13, there are isomorphic embeddings $h_\ell : B_\ell \hookrightarrow K_0$ where the image of h_ℓ is $B_{K_0}(f_\ell, \lambda(\ell+1))$, the filled $\lambda(\ell+1)$-neighborhood of $f_\ell = h_\ell(f)$ in K_0. There is a unique face g_ℓ of K_{-1} such that f_ℓ is a face of the subdivision τg_ℓ. The definition of λ may be used to verify the first containment in the observation that

$$\tau C_{K_{-1}}(g_\ell, \ell) \subset C_{K_0}(f_\ell, \lambda(\ell+1)) \subset B_{K_0}(f_\ell, \lambda(\ell+1)).$$

Since $B_{K_{-1}}(g_\ell, \ell)$ is obtained by merely "filling in the holes" of $C_{K_{-1}}(g_\ell, \ell)$, it follows from the containments above and the fact that $B_{K_0}(f_\ell, \lambda(\ell+1))$ is simply connected, that $\tau B_{K_{-1}}(g_\ell, \ell)$ is a subcomplex of $B_{K_0}(f_\ell, \lambda(\ell+1))$.

We now claim that we may extract a subsequence $(g_{j_i})_i$ of the sequence of faces $(g_\ell)_\ell$ with the following properties:

(i) the finite complex $D_i = B_{K_{-1}}(g_{j_i}, i)$ admits an isomorphic embedding e_i into the complex D_{i+1} with $e_i(g_{j_i}) = g_{j_{i+1}}$;

(ii) the direct limit $L_{-1} = \varinjlim(D_i, e_i)$ is a planar polygonal complex that is locally isomorphic to L_0;

(iii) $\tau L_{-1} \cong L_0$.

For item (i), we define the subsequence g_{j_i} inductively as follows. Since K_{-1} has FLC, Corollary 2.5 implies that there is a smallest subscript j_1 such that $D_1 = B_{K_{-1}}(g_{j_1}, 1)$ is isomorphic to $B_{K_{-1}}(g_\ell, 1)$ for infinitely many subscripts $\ell > j_1$. By Corollary 2.5 again, there is among these infinitely many subscripts a smallest subscript $j_2 > j_1$ such that $D_2 = B_{K_{-1}}(g_{j_2}, 2)$ is isomorphic to $B_{K_{-1}}(g_\ell, 2)$ for infinitely many subscripts $\ell > j_2$. Having chosen $g_{j_1}, \dots, g_{j_n}$ in this way so that $D_n = B_{K_{-1}}(g_{j_n}, n)$ is isomorphic to $B_{K_{-1}}(g_\ell, n)$ for infinitely many $\ell > j_n$, we again apply Corollary 2.5 to choose among these infinitely many subscripts the smallest subscript $j_{n+1} > j_n$ such that $D_{n+1} = B_{K_{-1}}(g_{j_{n+1}}, n+1)$ is isomorphic to $B_{K_{-1}}(g_\ell, n+1)$ for infinitely many subscripts $\ell > n+1$. This inductively defines the sequence g_{j_i} for $i = 1, 2 \dots$, and for each i, the choice of j_{i+1} implies that there is an isomorphic embedding

$$e_i : D_i = B_{K_{-1}}(g_{j_i}, i) \cong B_{K_{-1}}(g_{j_{i+1}}, i) \subset B_{K_{-1}}(g_{j_{i+1}}, i+1) = D_{i+1}$$

with, necessarily $e_i(g_{j_i}) = g_{j_{i+1}}$. This confirms item (i).

Since the complexes D_i are filled i-neighborhoods and hence combinatorial disks with D_i contained in the interior of D_{i+1}, the direct limit complex L_{-1} is a CW-decomposition of the whole plane, and hence a planar polygonal complex. By construction, $L_{-1} \preceq K_{-1}$ and as $K_{-1} \sim K_0 \sim L_0$, we have $L_{-1} \preceq L_0$. We have yet to use the hypothesis of combinatorial repetitiveness, but now it is invoked to prove that $L_0 \preceq L_{-1}$, implying that $L_{-1} \sim L_0$ and confirming item (ii). Let F be a connected subcomplex of L_0. Since $L_0 \sim K_{-1}$, there exists an isomorphic copy F' of F in K_{-1}. Since K_{-1} is combinatorially repetitive, there exists an integer $s > 0$ such that every combinatorial s-neighborhood of any face in K_{-1} contains an embedded copy of F', and therefore of F. Then $D_s = B_{K_{-1}}(g_{j_s}, s)$ contains an embedded copy of F, and therefore so does L_{-1}. We conclude that $L_0 \preceq L_{-1}$, and this finishes the verification of item (ii).

For item (iii), we will use the fact derived above that for any positive integer ℓ, $\tau B_{K_{-1}}(g_\ell, \ell)$ is a subcomplex of $B_{K_0}(f_\ell, \lambda(\ell+1))$. Write $L_{-1} = \cup_{i=1}^{\infty} D_i'$, where D_i' is the canonical isomorphic copy of D_i in L_{-1}, and let $g = (g_{j_i})$ be the face of the direct limit L_{-1} that corresponds to the faces g_{j_i} of the factors. Then $\tau L_{-1} = \cup_{i=1}^{\infty} \tau D_i'$ so that $\tau D_1' \subset \tau D_2' \subset \tau D_3' \subset \cdots$ is a sequence of finite subcomplexes of τL_{-1} that exhausts τL_{-1}, as in the hypothesis of the working Lemma 2.9. For each positive integer i, the mappings

$$\tau D_i' \cong \tau D_i = \tau B_{K_{-1}}(g_{j_i}, i) \subset \tau B_{K_{-1}}(g_{j_i}, j_i) \subset B_{K_0}(f_{j_i}, \lambda(j_i+1)) \xrightarrow[h_{j_i}^{-1}]{\cong} B_{j_i} \subset L_0$$

define an isomorphic embedding $\tau D_i' \hookrightarrow L_0$. Notice that the image of one of the faces f_i' of the subdivided face $\tau g \subset \tau D_i'$ under this embedding is equal to the face f of L_0. Since there are only finitely many faces in the subdivided face τg, by passing to a subsequence if necessary, we may assume without loss of generality that all the faces f_i' are the same face f' of τL_{-1}. An application of the working Lemma 2.9 now implies that $\tau L_{-1} \cong L_0$, and item (iii) is proved.

Having confirmed items (i)–(iii), we now may use the isomorphism $\tau L_{-1} \cong L_0$ to replace L_{-1} by an isomorphic copy that aggregates the faces of L_0 according to the pattern of L_{-1} and assume, without loss of generality, that $\tau L_{-1} = L_0$. Repeat the argument using the data L_{-1}, K_{-1}, and K_{-2} in place of L_0, K_0, and L_{-1} to construct a planar polygonal complex L_{-2} such that $\tau L_{-2} = L_{-1}$ with $L_{-2} \sim L_{-1}$. Iterating ad infinitum, this produces a sequence $\{L_n\}_{n<0}$ such that $\tau L_n = L_{n+1}$ and $L_n \sim L_{n+1}$ for $n \le -1$. Already we have defined L_n for $n \ge 0$ so that $\tau L_n = L_{n+1}$ and it follows from its definition that $L_n \sim K_n \sim K_{n+1} \sim L_{n+1}$, since $K \sim L$ implies that $\tau K \sim \tau L$.

All the ingredients now are in place as we have produced a bi-infinite sequence $\{L_n\}$, for $n \in \mathbb{Z}$, such that (1) $L_0 = L$, (2) $\tau L_n = L_{n+1}$, and (3) $L_n \sim L_{n+1}$, and this implies that subdivision operator τ manifests a combinatorial hierarchy for L. □

3.2.1. *A brief warning.* In the proof we have neither proved nor claimed that the backward sequence $\{L_n\}_{n<0}$ is uniquely determined by $L = L_0$. There indeed may be differing ways in which aggregation of the faces of a complex L can produce a complex locally isomorphic to L whose τ-subdivision is equal to L. The integer lattice 4-gon complex Z has differing aggregates, though in this case all the aggregates are isomorphic with Z.[13] Despite these cautions, the theorem does show that, whenever K is combinatorially repetitive and has FLC, a combinatorial hierarchy manifested by a subdivision operator τ is a property of the

[13] See page 56.

whole local isomorphism class (K) rather than of just the complexes K_n that manifest a combinatorial hierarchy for that single planar polygonal complex $K_0 = K$.

3.3. **Conformal hierarchy.** Just as there is an appropriate and useful conformal version for tilings of combinatorial subdivision rules, there is an appropriate and useful conformal version for tilings of combinatorial hierarchy. In fact, the conformal version of subdivision was defined with the conformal version of hierarchy in mind. A conformal tiling that exhibits an expansive conformal hierarchy in the next section will be seen to be, necessarily, of parabolic type.

Recall that the polygonal subdivision S of the conformal tiling T is a conformal subdivision if the tiling S shares with T the properties of conformal regularity of its tiles as well as conformal reflectivity of the tiling. The subdivision S is **totally nontrivial** if it is nontrivial on every face of T. When S is a totally nontrivial conformal subdivision of T, we call T a **conformal aggregate** of S. We say that the conformal tiling T **exhibits a conformal hierarchy** if there is a bi-infinite sequence $\{T_n : n \in \mathbb{Z}\}$, called a **conformal hierarchy** for T, of conformal tilings indexed by the integers such that the following three conditions hold:

(1) $T_0 = T$;

(2) T_{n+1} is a totally nontrivial conformal subdivision of T_n, for all $n \in \mathbb{Z}$;

(3) $K_{T_n} \sim K_{T_{n+1}}$, for all $n \in \mathbb{Z}$.

Let τ be a subdivision operator that manifests a combinatorial hierarchy for the planar polygonal complex K. Then $K_0 = K$, and $\tau K_n = K_{n+1}$ and $K_n \sim K_{n+1}$ for all $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, let $T_n = T_{K_n}$ be a conformal tiling associated to the complex K_n. Items (1) and (3) automatically are satisfied for the bi-infinite sequence $\{T_n\}$, and if the tilings T_n can be chosen so that item (2) also is satisfied, we say that τ is a **conformal subdivision operator for the sequence** $\{T_n\}$ that **manifests a conformal hierarchy** for T. Explicitly, the subdivision operator τ is a conformal subdivision operator for the sequence if $T_{n+1} = T_{\tau K_n}$ is a conformal subdivision of $T_n = T_{K_n}$, for each $n \in \mathbb{Z}$. We write that $T_{n+1} = \tau T_n$ so that $T_n = \tau^n T$. For example, Theorem 1.1 implies that whenever τ is a dihedrally symmetric simple subdivision rule that manifests a combinatorial hierarchy for K, then τ will be a conformal subdivision operator for a corresponding bi-infinite sequence of conformal tilings. Since this will be so useful to us, we formally separated it out as a theorem.

Theorem 3.5. *Let τ be a dihedrally symmetric simple subdivision operator that manifests a combinatorial hierarchy for the planar polygonal complex K. Then τ is a conformal subdivision operator for a corresponding conformal hierarchy $\{T_n\}$, where $K_{T_n} \cong K_n$ for all $n \in \mathbb{Z}$.*

When $\{T_n\}$ is a conformal hierarchy for the conformal tiling T of the plane $\mathbb{E} \in \{\mathbb{C}, \mathbb{D}\}$, the sequence $\{K_n = K_{T_n}\}$ is a combinatorial hierarchy for the planar polygonal complex

$K = K_T$. If the hierarchy $\{K_n\}$ is (strongly) expansive, then we say that the conformal hierarchy $\{T_n\}$ is **(strongly) expansive**. If c is a core of the complex K_T, then $|c|$, the union of the tiles of T that correspond to the faces of c, is a **core** of the tiling T. That a core $|c|$ of T **engulfs** a compact set D just means that $D \subset |c|$.

Lemma 3.6. *If the conformal hierarchy $\{T_n\}$ is expansive for the tiling T of the plane $\mathbb{E} \in \{\mathbb{C}, \mathbb{D}\}$, then every compact subset D of $\mathbb{E}$ is engulfed by a core of one of the tilings T_n, for some value of $n \leq 0$.*

Proof. Since T is a locally finite tiling of $\mathbb{E}$ and D is compact, only finitely many tiles of T meet D and there is a finite subcomplex F of K_T such that $|T_F| \supset D$. Since the hierarchy is expansive, there is an integer $n \leq 0$ such that F is engulfed by a core of K_{T_n}. This means that there is a core c of K_{T_n} such that the subcomplex $\sigma_n^0 c$ of K_T contains the complex F. Since F is a subcomplex of $\sigma_n^0 c$, $|T_F|$ is a subset of $|T_{\sigma_n^0 c}|$. By item (2) in the definition of conformal hierarchy, $T = T_0$ is a conformal subdivision of T_n[14] and, in particular, is a polygonal subdivision. From this it follows that $|c| = |T_{\sigma_n^0 c}|$. Thus, $D \subset |T_F| \subset |T_{\sigma_n^0 c}| = |c|$, and D is engulfed by $|c|$. □

3.4. **The conformal type of hierarchical tilings.** We now are in a position to examine the type problem for conformally hierarchical tilings. Our primary focus in this section is the proof that a conformal tiling that exhibits an expansive conformal hierarchy is parabolic when the degree is bounded. An important corollary is that when a combinatorial hierarchy of a complex K is manifested by a dihedrally symmetric and shrinking simple subdivision operator τ with bounded degree, the tiling T_K is parabolic and tiles the complex plane $\mathbb{C}$. In fact, type is constantly parabolic across the whole local isomorphism class (K). After verifying this result, we observe that when a power of a dihedrally symmetric, conformal subdivision operator for a bi-infinite sequence of tilings yields a tiling isomorphic to the base tiling $T_0 = T$, then the tiling T admits a *supersymmetry*. This also implies that T is parabolic and the supersymmetry is realized as the action of a nontrivial, orientation-preserving similarity transformation of the complex plane that contracts.

3.4.1. *Expansive conformal hierarchy and parabolic type.* We begin with a theorem that gives conditions that guarantee that a hierarchical conformal tiling is parabolic. Of course when we say that a conformal tiling T has **bounded degree**, we mean that the corresponding planar polygonal complex K_T has bounded degree.

Theorem 3.7. *Let T be a conformal tiling that exhibits an expansive conformal hierarchy. If T has bounded degree, then T is parabolic and tiles the complex plane $\mathbb{C}$.*

The tool we use to confirm that T is parabolic is one of the classical criteria for determining that a non-compact, simply connected Riemann surface is parabolic. Before the proof of

[14] Here we are using the fact that conformal subdivision of tilings is a transitive relation.

the theorem, we review this criterion, referring the reader to the classic references Lehto and Virtanen [8] and Ahlfors [1], as well as the modern treatment given in Fletcher and Markovic [7].

A **ring domain** is a doubly connected domain in the Riemann sphere $\mathbb{P}$, i.e., a domain of $\mathbb{P}$ whose complement consists of two connected components. A classical theorem of Koebe implies that a ring domain is conformally equivalent to a **circle domain**, a domain for which each complementary component is either a closed circular disk or a point. It follows that any ring domain is conformally equivalent to an **annulus**, a domain of the form

$$A(r, R) = \{z \in \mathbb{C} : 0 \leq r < |z| < R \leq \infty\}.$$

The **modulus** of the annulus $A = A(r, R)$ is defined as $\mathrm{Mod}(A) = \log(R/r)$, with the obvious interpretations that $\mathrm{Mod}(A) = \infty$ if $r = 0$ or $R = \infty$. Two annuli A and B with finite moduli are conformally equivalent if and only if $\mathrm{Mod}(A) = \mathrm{Mod}(B)$. The annuli with infinite moduli determine two conformal equivalence classes according to whether one or both of the complementary domains are points. We define the **modulus** of the ring domain C to be $\mathrm{Mod}(A)$, denoted of course as $\mathrm{Mod}(C)$, where A is any annulus conformally equivalent to C. This is well-defined by the preceding discussion.

With these facts under our belt, we can give a tool for determining the type of a non-compact, simply connected Riemann surface S. Let D be a closed disk in S. By the Uniformization Theorem, the set $S - D$ is conformally equivalent to a ring domain C in $\mathbb{C}$. Obviously, S is parabolic if and only if $\mathrm{Mod}(C) = \infty$ and is hyperbolic if and only if $\mathrm{Mod}(C) < \infty$. One of the useful results of quasiconformal mapping theory is the following characterization of parabolic surfaces.

Theorem 3.8 (Criterion for Parabolicity)**.** *A non-compact, simply connected Riemann surface S is parabolic if and only if there is a constant $\mu > 0$ such that, for every compact subset D of S, there is a ring domain C separating D from ∞ that has conformal modulus $\mathrm{Mod}(C) \geq \mu$.*

This is proved by applying the superadditivity of the modulus, which avers that if $C_1, C_2, \ldots$ is a sequence of pairwise disjoint ring domains that are subdomains of the ring domain C, and every C_n separates the boundary components of C from one another, then

$$\sum_{n=1}^{\infty} \mathrm{Mod}(C_n) \leq \mathrm{Mod}(C).$$

This implies also that the constant μ in the Criterion for Parabolicity may be taken to be unity. Armed with this Criterion for Parbolicity, we are equipped to prove the theorem

Proof of Theorem 3.7. Suppose the bounded degree conformal tiling T tiles the plane $\mathbb{E} \in \{\mathbb{C}, \mathbb{D}\}$. Let $\{T_n\}$ be an expansive conformal hierarchy for T.

We focus our attention first on the tiling $T = T_0$. Let c be a core of the planar polygonal complex $K = K_T$, and let $B(c)$ be the full subcomplex of K whose faces are all the faces of K that meet c and meet the unbounded complementary domain of c, but are not faces of c. Note that $B(c)$ is connected since it may be described alternately as the subcomplex whose faces are precisely the faces of the unbounded complementary domain of c that meet the connected boundary of that complementary domain. The complex $B(c)$ is called the **combinatorial collar** of c, and the patch $T_{B(c)}$ of the tiling T that consists of the set of tiles of T corresponding to the faces of $B(c)$ determines an open domain $U(c)$, the interior of $|T_{B(c)}|$ in $\mathbb{E}$, that separates the core $|c|$ from infinity. Since $B(c)$ is a finite subcomplex of K, the domain $U(c)$ is finitely connected and by the Koebe Uniformization Theorem, is conformally equivalent to a circle domain $V(c)$, the complement of a finite number of closed round disks in the plane. Let $\kappa : U(c) \to V(c)$ be a conformal isomorphism. By applying an appropriate Möbius transformation, we may assume two things: first, that $V(c)$ is the complement of a finite number of closed round disks in the unit disk $\mathbb{D}$ with the unit circle boundary of $\mathbb{D}$ corresponding to the boundary component of $U(c)$ that meets the unbounded complementary domain of $U(c)$; second, that one of the complementary domains of $V(c)$ is a disk centered at the origin whose boundary corresponds to the boundary of the complementary domain of $U(c)$ that contains the core $|c|$. Let $A(c)$ be the largest annulus centered at the origin that is contained in $V(c)$, and let $R(c) = \kappa^{-1}(A(c))$ be its image under the inverse isomorphism κ^{-1}. Then $R(c) \subset U(c)$ is a ring domain that separates the core $|c|$ from infinity and is called the **standard collar** of the core $|c|$. Our claim is that there are only finitely many different conformal isomorphism types of standard collars $R(c)$ as c ranges over all the cores of K. This follows from the fact that T has bounded degree and from Theorem 1.3. Indeed, since T, and therefore K, has bounded degree, there exists only finitely many combinatorial types of cores of K, and for each one of these combinatorial types of cores, only finitely many combinatorial types of collars. Hence, there exists finitely many combinatorial collars $B_1 = B(\mathsf{c}_1), \ldots, B_N = B(\mathsf{c}_N)$ such that the combinatorial collar $B(c)$ of any core c of K is combinatorially equivalent to one from the list. By Theorem 1.3, the domain $U(c)$ of every core $|c|$ of T is conformally equivalent to one of the domains $U(\mathsf{c}_1), \ldots, U(\mathsf{c}_N)$. But this implies that the standard collar $R(c)$ is conformally equivalent to one of the standard collars from the list $R_1 = R(\mathsf{c}_1), \ldots, R_N = R(\mathsf{c}_N)$.

If K' is any planar polygonal complex locally isomorphic to K, then the combinatorial collar $B(c')$ of any core c' of K' isomorphically embeds in K as a combinatorial collar of a core of K, and this implies that $B(c')$ is combinatorially equivalent to one from the list $B_1, \ldots, B_N$. Since the β-equilateral conformal structure is used for both T_K and $T_{K'}$, the domains $U(c)$ and $U(c')$ are conformally equivalent, and this implies that the standard collar $R(c')$ is conformally equivalent to one of the standard collars from the list $R_1, \ldots, R_N$. Since each of the complexes K_{T_n} from the conformal hierarchy for T is locally isomorphic to $K = K_T$, it follows that the standard collar $R(c)$ of every core $|c|$ of any tiling T_n, $n \in \mathbb{Z}$, is conformally equivalent to a ring domain from the list $R_1, \ldots, R_N$.

We are now in a position to verify that the Criterion for Parabolicity Theorem 3.8 is satisfied for $\mathbb{E}$, implying that $\mathbb{E} = \mathbb{C}$ and that the tiling T is parabolic. Let μ be the smallest modulus of a ring domain from the list $R_1, \ldots, R_N$. Of course $\mu > 0$. Let D be a compact subset of $\mathbb{E}$. By Lemma 3.6, there exists an integer $n \leq 0$ such that D is engulfed by a core $|c|$ of the tiling T_n. It follows that the ring domain $R(c)$ separates D from infinity. By the observation of the preceding paragraph, $\mathrm{Mod}(R(c)) \geq \mu$, and this verifies the Criterion for Parabolicity and completes the proof. $\square$

3.4.2. *Parabolic type across local isomorphism classes.* We now look at two results that guarantee that conformal type is constantly parabolic across local isomorphism classes of planar polygonal complexes. The first result says that, while Theorem 3.7 guarantees only that the conformal tiling T is parabolic and says nothing about the type of its locally isomorphic cousins, constancy of type across the whole local isomorphism class may be achieved by strengthening the expansive hypothesis to that of strongly expansive.

Corollary 3.9. *Let T be a conformal tiling of bounded degree that exhibits a strongly expansive conformal hierarchy manifested by a conformal subdivision operator τ. Then any conformal tiling S whose complex K_S is locally isomorphic to the complex K_T is parabolic and tiles the complex plane $\mathbb{C}$. In particular, conformal type is constantly parabolic across the local isomorphism class (K_T).*

Proof. Theorem 3.7 implies that T is parabolic and tiles the complex plane $\mathbb{C}$. Let $\mathbb{E}$ be either $\mathbb{C}$ or $\mathbb{D}$ with S a conformal tiling of $\mathbb{E}$, where K_S is locally isomorphic to the complex K_T, but not isomorphic to K_T. We will verify the Criterion for Parabolicity Theorem 3.8 for the surface $\mathbb{E}$. For this we will use the facts exposed in the proof of Theorem 3.7 concerning the existence of a finite number of standard collars in T up to conformal isomorphism.

Let D be a compact subset of $\mathbb{E}$ and let F be a finite, connected full subcomplex of K_S whose corresponding patch of tiles S_F covers D, so that $D \subset |S_F|$. Since $K_S \sim K_T$ while K_S is not isomorphic to K_T, K_T is plural and Theorem 2.11 implies that the finite complex F is infinitely represented in K_T. Let f be a face of F and $(F_1, f_1), (F_2, f_2), \ldots$ be a pairwise unequal listing of all the pairs where F_i is a finite subcomplex of K_T and f_i is a face of F_i such that (F_i, f_i) and (F, f) are isomorphic as pairs. Since F is finite and connected, there exists a positive integer m such that both the δ_{K_S}-diameter of F and the δ_{K_T}-diameter of F_i, for all positive integers i, are less than m. Let $\{T_n\}$ be a strongly expansive conformal hierarchy for $T = T_0$ and, for each integer n, let $K_n = K_{T_n}$ so that $\{K_n\}$ is a strongly expansive combinatorial hierarchy for $K_T = K_0$. Since the hierarchy is strongly expansive, there exists a positive integer p such that p aggregations engulf any finite subcomplex of K_T of δ_{K_T}-diameter at most m in some core of the aggregate. In particular, each of the subcomplexes F_i of K_T is engulfed by a core c_i of K_{-p}. Let $B(c_i)$ be the combinatorial collar of c_i in K_{-p} as constructed in the proof of Theorem 3.7 and recall from that proof that, since $K_{-p} \sim K_T$, each complex $B(c_i)$ is combinatorially equivalent to a complex from a finite list, $B_1, \ldots, B_N$, of combinatorial collars of cores of K_T. Moreover, from this, as the proof of Theorem 3.7 avers, each standard collar $R(c_i)$ is conformally equivalent to a standard collar from a finite list, $R_1, \ldots, R_N$, of collars determined by the combinatorial collars $B_1, \ldots, B_N$.

Since c_i engulfs F_i, f_i is a face of $\tau^p c_i$. Choose a positive integer M so large that $\tau^p(c_i \cup B(c_i))$ is a subcomplex of the combinatorial M-neighborhood $C_{K_T}(f_i, M)$, and F is a subcomplex of $C_{K_S}(f, M)$. Such an M exists since T has bounded degree, there are only finitely many combinatorial types of cores in the list $c_1, c_2, \ldots$ and of collars in the list $B(c_1), B(c_2), \ldots$, and there is an upper bound on the number of faces in the subdivision $\tau^p g$ of any face g of K_{-p}. Consider the combinatorial M-neighborhood $C_{K_S}(f, M)$ in the complex K_S. Since $K_S \sim K_T$, there is a subcomplex H of K_T that is isomorphic to $C_{K_S}(f, M)$ via an isomorphism $\lambda : C_{K_S}(f, M) \to H$. By Lemma 2.13, $H = C_{K_T}(\lambda(f), M)$. Since F is a subcomplex of $C_{K_S}(f, M)$, there exists a positive integer i such that $(F_i, f_i) = (\lambda(F), \lambda(f))$, a subcomplex of H. It follows that

$$F_i \subset \tau^p c_i \subset \tau^p(c_i \cup B(c_i)) \subset C_{K_T}(f_i, M) = C_{K_T}(\lambda(f), M) = H.$$

The first containment is just the statement that the core c_i of K_{-p} engulfs F_i, the second is trivial, and the third is by choice of M. By applying the inverse isomorphism λ^{-1}, there is a subcomplex J of $C_{K_S}(f, M)$ isomorphic to $\tau^p B(c_i)$ that contains F in one of its bounded complementary domains of K_S, implying that J separates F from infinity. By Theorem 1.3, the patch S_J of tiles in the tiling S is conformally equivalent to the patch $T_{\tau^p B(c_i)}$ in the tiling T, and so the open domain $U(J)$, the interior of $|S_J|$ in $\mathbb{E}$, is conformally equivalent to the open domain $U(\tau^p B(c_i))$, the interior of $|T_{\tau^p B(c_i)}|$ in $\mathbb{C}$. Since the hierarchy $\{T_n\}$ is conformal, $\tau^p T_{-p} = T$ is a conformal subdivision of the tiling T_{-p}, and therefore is a polygonal subdivision. It follows that the open domain $U(c_i)$, the interior of $|(T_{-p})_{B(c_i)}|$ in $\mathbb{C}$, is precisely *equal* to the open domain $U(\tau^p B(c_i))$. This implies that the standard collar $R(c_i)$ is contained in the open domain $U(\tau^p B(c_i))$ and separates $|T_{F_i}|$ from infinity. From this, we conclude that $R = \lambda^{-1}(R(c_i))$ is a ring domain in $\mathbb{E}$ that is conformally equivalent to $R(c_i)$ and separates $D \subset |T_F|$ from infinity. But $R(c_i)$, and therefore R, is conformally equivalent to one of the standard collars from the list $R_1, \ldots, R_N$, and therefore has modulus at least μ, the smallest of the positive moduli of the ring domains $R_1, \ldots, R_N$. We have determined a positive constant μ such that, starting with an arbitrary compact subset of $\mathbb{E}$, there is a ring domain R of modulus $\geq \mu$ that separates D from infinity, and the Criterion for Parabolicity Theorem 3.8 applies to conclude that $\mathbb{E} = \mathbb{C}$, so that S is parabolic. $\square$

The proof makes clear the importance of strong expansivity in that it guarantees that there is a single complex, K_{-p}, in the hierarchy whose cores engulf all the subcomplexes F_i. This ensures that an isomorphic copy of a large enough combinatorial neighborhood of some F_i

in K_T will contain the τ^p-subdivision of a combinatorial collar of a core of K_{-p}, and this can be transferred back to K_S to separate F from infinity. Without this strengthening of the expansive property, though arbitrarily large combinatorial neighborhoods of F embed isomorphically in K_T, it may take so many aggregates to reach a core that engulfs that copy of F in a given isomorphic image of a large neighborhood that the combinatorial collar determined by that core fails to live inside that image of a large neighborhood. The standard collar then would not pull back to the tiling S to separate D from infinity.

The next corollary gives combinatorial conditions that guarantee constancy of type across local isomorphism classes.

Corollary 3.10. *If τ is a shrinking, dihedrally symmetric simple subdivision operator that manifests a combinatorial hierarchy for the planar polygonal complex K of bounded degree, then that combinatorial hierarchy is strongly expansive and the conformal tiling T_K exhibits a conformal hierarchy manifested by τ, has parabolic type, and tiles the complex plane $\mathbb{C}$. Also, conformal type is constantly parabolic across the local isomorphism class (K).*

Proof. Theorem 3.5 implies that the conformal tiling $T_0 = T_K$ has a conformal hierarchy $\{T_n\}$ for which $K_{T_n} \cong K_n$ for all $n \in \mathbb{Z}$, where $\{K_n\}$ is a combinatorial hierarchy for K manifested by τ. Theorem 3.3 implies that the hierarchy $\{K_n\}$, and hence the conformal hierarchy $\{T_n\}$, is strongly expansive. Theorem 3.7 implies that T_K is parabolic and, finally, Corollary 3.9 implies that type is constantly parabolic across the local isomorphism class (K), finishing the proof. $\square$

In an addendum to this section and in contrast to the two preceding corollaries, we construct simple examples of locally isomorphic planar polygonal complexes of differing conformal type. In Section 4, we will discuss examples of single tile type complexes that fit in a fairly general theoretical framework in which this latter corollary applies to confirm that the ones with dihedral symmetry are parabolic. For now we are content to offer a glimpse of a rather pleasant looking example of a planar polygonal complex with three polygonal types—3-, 4-, and 5-gons—whose hierarchy is manifested by the diamond edge subdivision operator pictured in Fig. 1. As the diamond edge subdivision operator is shrinking, dihedrally symmetric, and simple, Corollary 3.10 applies to conclude that it and its locally isomorphic cousins are parabolic. Fig. 6 gives the first four stages in the construction of one of the uncountably many conformal tilings associated to this simple subdivision operator and hints at the infinite complex as well as the conformal hierarchy of the tiling, which is illustrated in Fig. 7.

3.4.3. *Conformal hierarchy and supersymmetry.* In the pentagonal tiling of [2], a loxodromic Möbius transformation that generates a conformal hierarchy for the tiling is clearly discernible. In fact it was the existence of such a transformation that allowed the authors to claim that the pentagonal tiling is parabolic. This transformation actually is constructed

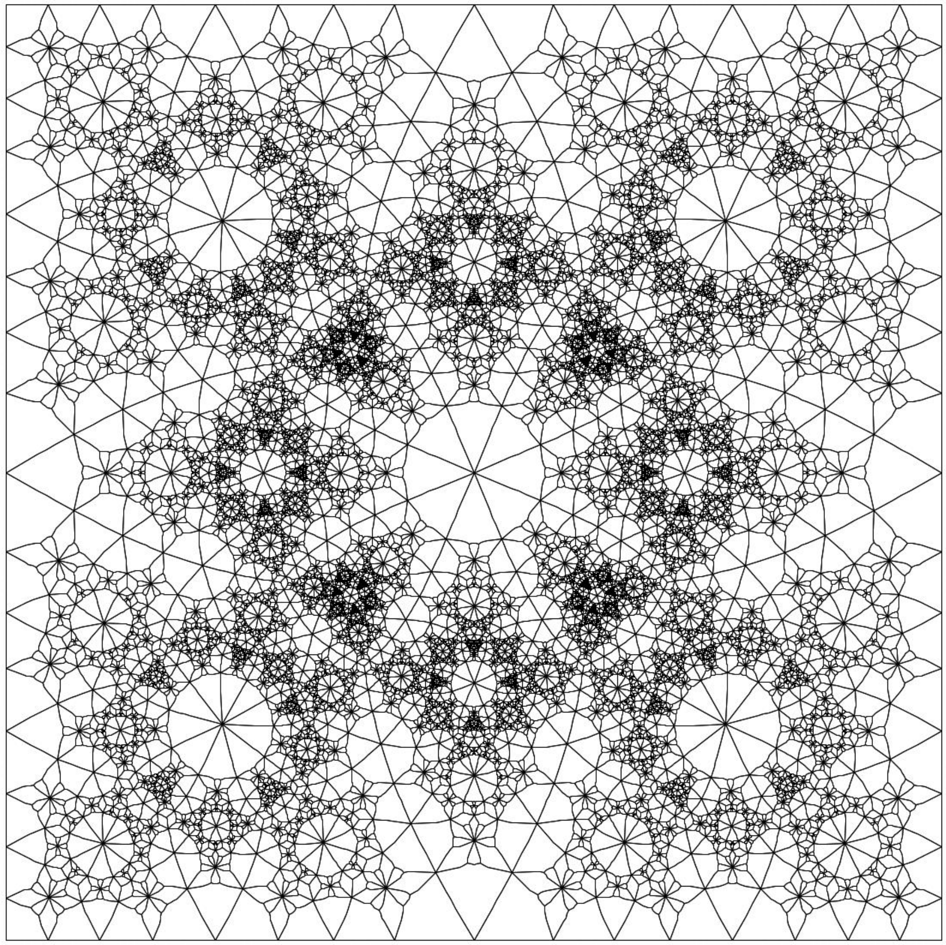

FIGURE 6. An expansion complex generated by the diamond edge subdivision operator: the first four stages of construction. This is an approximation to a patch of a conformally reflective tiling T by conformally regular 3-, 4-, and 5-gons. This sort of *diamond lace* tiling is parabolic and exhibits a conformal hierarchy manifested by the diamond edge subdivision operator. Four aggregate stages of the conformal hierarchy are color coded in the next figure.

first as a combinatorial symmetry of the pentagonal complex using a subdivision of the complex, which then is realized geometrically by a conformal automorphism. In the Cannon, Floyd, and Parry development of expansion complexes in [5, 6], these sorts of transformations are crucial in their verification that certain finite subdivision rules have desirable

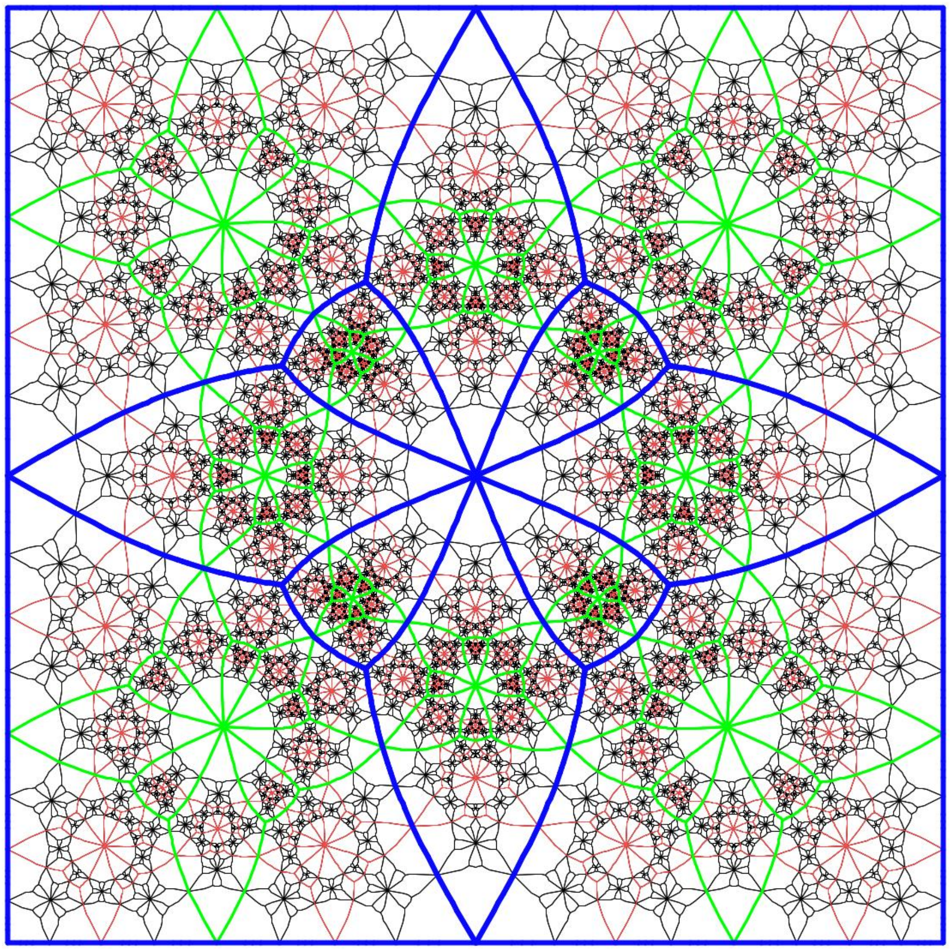

FIGURE 7. The small black-sided polygons approximate a patch in a diamond lace conformal tiling $T = T_0$, the red-sided ones approximate the first aggregate tiling T_{-1}, the green-sided ones approximate the second aggregate tiling T_{-2}, and the blue-sided ones approximate the third aggregate tiling T_{-3} of the conformal hierarchy. The large blue-sided square that borders the whole figure approximates a 4-gon in the fourth aggregate tiling T_{-4}.

conformal properties. We now examine conditions that guarantee their existence. The Möbius transformation μ promised by the next lemma is called a **supersymmetry**[15] and has the form $z \mapsto az+b$, where $a \neq 0$ and b are complex constants. That μ fixes ∞ and has

[15]The qualifier *super* is used since this is not a symmetry—a self-isomorphism—of the tiling; rather, μ is a conformal isomorphism of the tiling onto a subdivision of the tiling and generates a countable family

an attracting fixed point in $\mathbb{C}$ implies that $0 < |a| < 1$. Thus μ is an orientation-preserving similarity of the complex plane that contracts, and is **loxodromic**[16] whenever a is not a positive real number, i.e., whenever μ both contracts and rotates nontrivially.

Lemma 3.11. *Let τ be a shrinking subdivision operator for which τT is a conformal subdivision of the bounded degree conformal tiling T, with τT combinatorially isomorphic to T. Then T tiles the complex plane $\mathbb{C}$ and there exists a Möbius transformation μ that fixes ∞ and a single attracting fixed point of $\mathbb{C}$, and that realizes the combinatorial isomorphism $K_T \cong \tau K_T$; moreover, μ generates a conformal hierarchy for T manifested by τ. In particular, we have $\mu(T) \equiv \{\mu(t) : t \in T\} = \tau T$ and, moreover, type is constantly parabolic across the local isomorphism class (K_T).*

Proof. Since T and τT are combinatorially equivalent tilings by conformally regular polygons both of which are reflective, Theorem 1.2 applies and guarantees a conformal isomorphism μ of the tilings T and τT that realizes the combinatorial isomorphism $K_T \cong \tau K_T$. In particular $\mu : |T| \to |\tau T|$ is a conformal isomorphism such that $\mu(T) \equiv \{\mu(t) : t \in T\} = \tau T$, and either $|T| = |\tau T| = \mathbb{C}$ or $|T| = |\tau T| = \mathbb{D}$.

For each integer n, let T_n be the tiling defined by

$$T_n = \mu^n(T) \equiv \{\mu^n(t) : t \in T\}$$

and note that since μ is a conformal automorphism, each tiling T_n is conformal, a reflective tiling by conformally regular polygons. Since $\mu(T)$ conformally subdivides T, a moment's consideration should convince the reader that, for each integer n, the tiling T_{n+1} polygonally subdivides the tiling T_n. Since these are all conformal tilings, it follows that, for each integer n, T_{n+1} conformally subdivides T_n and, in fact, $T_{n+1} = \tau T_n$ since $\mu(T) = \tau T$. Since τ is shrinking, it is a totally nontrivial subdivision operator. All this shows that τ is a conformal subdivision operator for the sequence $\{T_n\}$ that manifests a conformal hierarchy for $T = T_0$. Since τ is shrinking, Theorem 3.3 implies that the hierarchy $\{T_n\}$ is strongly expansive. Theorem 3.7 and Corollary 3.9 imply that T is parabolic and that type is constantly parabolic across (K_T).

Now that T is seen to be parabolic, μ is recognized as a conformal automorphism of the complex plane $\mathbb{C}$. By the classification of conformal automorphisms of the plane, we conclude that μ is a Möbius transformation of the form $\mu(z) = az + b$ for some complex constants $a \neq 0$ and b. Since τT is a nontrivial subdivision, μ is not the identity. We now argue in turn that μ is not a translation ($a \neq 1$) and not a rotation (further, $|a| \neq 1$),

of pairwise isomorphic tilings by both forward and backward iteration that defines a conformal hierarchy, as the proof of the theorem shows. The supersymmetry is a symmetry of the hierarchy.

[16]The terminology comes from the classification of Möbius transformations on the Riemann sphere. It is derived from the navigational term *loxodrome* (or *rhumb line*), a line crossing all meridians of longitude at the same angle, which describes precisely the orbits of points under the flow associated to such Möbius transformations that fix ∞ and a point of $\mathbb{C}$.

but rather is contractive ($0 < |a| < 1$). If μ is a translation, then $\mu(z) = z + b$ for some non-zero complex constant b. Let d be a positive integer that is strictly larger than the number of faces that meet at any vertex of the complex K_T and, given an arbitrary tile t of $T = T_0$, let D be the union of the tiles $\mu^j(t) = t + jb$ for $j = 1, \dots, d$. Since the conformal hierarchy $\{T_n\}$ is expansive, there is, for some negative integer n, a core $|c|$ of the tiling T_n that contains D. Let p be a point interior to the tile t. Since the core $|c|$ is the union of fewer than d tiles, there is a tile s of T_n that contains at least two of the points from the list $p+b, \dots, p+db$. Let k the smallest integer in the list $1, \dots, d$ for which $p+kb \in s$, and let ℓ be the largest positive integer for which $p+\ell b \in s$. Note that $k < \ell$ so that $m = \ell - k > 0$ and note that it may well be that $\ell > d$. Consider the tile $t_k = \mu^k(t) = t + kb \in T_k$. Since p is interior to t, $p + kb$ is interior to t_k, and so the tile s in T_n meets the interior of the tile t_k of $T_k = \tau^{k-n}T_n$. Since T_k is a polygonal subdivision of T_n, this implies that $t_k \subset s$, and since $p + kb$ is interior to t_k, $p + kb$ is interior to s. The same argument applied to $t_\ell = t + \ell b$ implies that $t_\ell \subset s$ and that $p + \ell b$ is interior to s. Choose $\varepsilon > 0$ so that the disk neighborhoods $D(p + kb, \varepsilon)$ and $D(p + \ell b, \varepsilon)$[17] are contained in the tile s, and observe that $\mu^m(D(p + kb, \varepsilon)) = D(p + \ell b, \varepsilon)$. It follows that the tile $\mu^m(s)$ of T_{n+m} contains the open disk $D(p + \ell b, \varepsilon)$, as does s, and so the tile s and the tile $\mu^m(s)$ meet in an interior point of both. Since T_{n+m} polygonally subdivides T_n, we conclude that $\mu^m(s) \subset s$. This implies, since $p + \ell b \in s$, that $p + (\ell + m)b = \mu^m(p + \ell b) \in s$. Since the integer $m > 0$, this contradicts the choice of ℓ as the largest positive integer for which $p + \ell b \in s$. We conclude that μ cannot be a translation, so $a \neq 1$.

Since $a \neq 1$, μ has a unique fixed point in $\mathbb{C}$, namely the fixed point $z_0 = b/(1 - a)$. Let t be a tile of T that contains z_0 that has maximum area among all the tiles of T that contain z_0. Then $\mu(t)$ is a tile of the tiling τT that subdivides T, so $z_0 = \mu(z_0) \in \mu(t) \subset t^*$, for some tile t^* of T. Since t^* is a tile of T that contains $\mu(t)$, a tile of the totally nontrivial subdivision τT of T, we have $\text{area}(\mu(t)) < \text{area}(t^*)$. By our choice of t, since both t and t^* are tiles of T that contain z_0, $\text{area}(t^*) \leq \text{area}(t)$ and we conclude that $\text{area}(\mu(t)) < \text{area}(t)$. This means that μ is not an isometry and hence not a rotation, and so $|a| \neq 1$. Finally, the fact that $\text{area}(\mu(t)) < \text{area}(t)$ implies that μ must be contracting, so that $0 < |a| < 1$ and z_0 is an attracting fixed point of the similarity transformation μ. □

We emphasize that the lemma implies that no *hyperbolic* conformal tiling of bounded degree can have a conformal subdivision induced by a shrinking subdivision operator that is combinatorially equivalent to the tiling.

The next theorem showcases the types of examples of conformal hierarchies with supersymmetry that initiated the study of conformal tilings. The original pentagonal tiling of [2] as well as others constructed subsequently by the authors and by Cannon, Floyd, and Parry using finite subdivision rules fall under this setting of supersymmetric tilings. These turn out to be the exception rather than the rule, as one is more likely than not to encounter a conformal tiling that exhibits a conformal hierarchy that fails to be supersymmetric. We

[17] The disk neighborhood $D(z, \varepsilon)$ is defined to be the set $\{w \in \mathbb{C} : |w - z| < \varepsilon\}$.

will see in the section following how to construct examples with supersymmetry. An isomorphism of the planar polygonal complex K onto a subdivision $\tau^m K$ as in this theorem is called a **combinatorial supersymmetry**, and when one exists for some positive integer m, we say that K is a **τ-supersymmetric** complex. The least value of m for which $K \cong \tau^m K$ is called the **τ-period** of K, or just the **period** if τ is understood.

Theorem 3.12. *Suppose that τ is a shrinking subdivision operator that manifests a conformal hierarchy for the conformal tiling $T = T_K$, where K is a planar polygonal complex of bounded degree. If $K \cong \tau^m K$ for some positive integer m, then T exhibits a conformal hierarchy via a bi-infinite sequence $\{T_n\}$ of conformal tilings manifested by τ such that T_n is conformally equivalent to T_{m+n} for all integers $n \in \mathbb{Z}$. Moreover, T is parabolic and there is a supersymmetry μ that simultaneously realizes all the tiling equivalences $T_n \cong T_{m+n}$ for all $n \in \mathbb{Z}$. In particular,*

$$\mu(T_n) \equiv \{\mu(t) : t \in T_n\} = T_{m+n}$$

for all $n \in \mathbb{Z}$.

Proof. Apply the preceding lemma with τ replaced by τ^m to conclude that T is parabolic and to obtain the supersymmetry μ for which $\mu(T) = \tau^m T$. Letting $T_n = \tau^n T$ for each non-negative integer n, an exercise verifies that $\mu(T_n) = T_{m+n}$ for all positive integers n. Each negative integer n may be written uniquely as $n = -km + j$ for a positive integer k and an integer j between 0 and $m-1$ inclusive. Letting $T_n = \mu^{-k}(T_j)$ when n is negative, we have $T_n \cong \mu(T_n) = T_{m+n}$ for all integers n. Another exercise verifies that $\tau T_n = T_{n+1}$ for all integers n, so that τ is a conformal subdivision operator for the sequence $\{T_n\}$ that manifests a conformal hierarchy for T. $\square$

3.5. **Addendum. Examples of $\preceq$-maximal complexes.** Though the primary emphasis of this paper is on tilings that arise from complexes K that are combinatorially repetitive and of bounded degree, we pause to construct examples that are of more general interest. The first simple example will be modified to obtain two maximal tilings of differing type—one parabolic, the other hyperbolic—that are locally isomorphic to one another. This shows that we need not expect that conformal type is constant across local isomorphism classes, and it becomes interesting to find conditions on the complex K that would guarantee constancy of type across (K). Corollary 3.10 presents one such condition and is applied in the next section on specific concrete examples. It might be guessed that, perhaps, an ingredient sufficient for constancy of type across local isomorphism classes is repetitiveness, but Cannon, Floyd, and Parry close their paper [6] with examples of locally isomorphic repetitive complexes, the one parabolic and the other hyperbolic. Their examples, as well as others with exotic properties, will be studied anew in a later paper of this series, but for the present, we are content with the following illustrative examples.

Example 3.2. A maximal planar polygonal complex under $\preceq$. Let $D_1, D_2, \dots$ be a list, up to isomorphism, of all the finite CW decompositions of the closed disk $\mathbf{D}$ into combinatorial polygons. Then any planar polygonal complex M that contains pairwise disjoint finite subcomplexes $D'_1, D'_2, \dots$ with $D_n \cong D'_n$ is **universal** for finite, planar, connected CW decompositions into combinatorial polygons in the sense that each finite subcomplex of any planar polygonal complex embeds isomorphically in M, since any embeds so in at least one of the D_i's. As such, $K \preceq M$ for all $K \in \mathbf{C}$. Explicit constructions appear in the next example. $\surd$

Example 3.3. Maximal planar polygonal complexes of differing conformal type. As in Example 3.2, we begin with a list $D_1, D_2, \dots$ of all finite polygonal complexes obtained as finite polygonal CW decompositions of the closed disk $\mathbf{D}$. Note that any two planar polygonal complexes, both of which contain pairwise disjoint isomorphic copies of all the D_i's, are locally isomorphic. We construct two such examples, M_{par} and M_{hyp}, the first a parabolic and the second a hyperbolic planar polygonal complex.

For M_{par}, let K_6 be the constant 6-degree triangulation of the plane, which is a parabolic planar 3-gon complex since the equilateral metric space $|K_6^\beta|_{\text{eq}}$ is conformally equivalent to $|K_6|_{\text{eq}}$, which is isometric to the plane $\mathbb{C}$. Let $h : \mathcal{S}_{K_6} \to \mathbb{C}$ be a conformal isomorphism, and use the Criterion for Parabolicity Theorem 3.8 to choose a pairwise disjoint sequence of simple closed edge paths C_n in K_6 such that the following conditions hold:

(i) C_{n+1} separates C_n from ∞;

(ii) $\mathrm{Mod}(A_n) \geq \mu$ for each n, where A_n is the ring domain in $\mathbb{C}$ bounded by the simple closed curves $h(C_{2n-1})$ and $h(C_{2n})$ and $\mu > 0$ is a fixed constant;

(iii) between the curves C_{2n} and C_{2n+1} lies a combinatorial disk d_n in K_6 whose boundary path δ_n has combinatorial length equal to that of the combinatorial length of the boundary of D_n.

Let $\mathcal{A}_n$ be the combinatorial annulus in K_6 bounded by C_{2n-1} and C_{2n}, so that $|\mathcal{A}_n^\beta|_{\text{eq}}$ is the piecewise equilateral ring domain in the equilateral surface $|K_6^\beta|_{\text{eq}}$ bounded by C_{2n-1} and C_{2n} that h maps onto A_n. Note that the disks d_n are pairwise disjoint and do not meet any of the combinatorial annuli $\mathcal{A}_k$. Now for each positive integer n, remove the interior of the disk d_n and glue to δ_n the combinatorial disk D_n along its boundary. The result is the planar polygonal complex M_{par} that is maximal with respect to the pre-order $\preceq$.

Our claim is that the conformal type of the planar polygonal complex M_{par} is parabolic. We use the Criterion for Parabolicity Theorem 3.8 to verify this. Indeed, the equilateral surface $|M_{\text{par}}^\beta|_{\text{eq}}$ contains, for every n, the equilateral ring domain $|\mathcal{A}_n^\beta|_{\text{eq}}$, and the complex structure of $\mathcal{S}_{M_{\text{par}}}$ gives these ring domains the same respective moduli as does $\mathcal{S}_{K_6}$, since those atlases agree on $|\mathcal{A}_n^\beta|_{\text{eq}}$, for all n. Thus $|\mathcal{A}_n^\beta|_{\text{eq}}$ is a ring domain in $\mathcal{S}_{M_{\text{par}}}$ such that $\mathrm{Mod}(|\mathcal{A}_n^\beta|_{\text{eq}}) = \mathrm{Mod}(A_n) \geq \mu$. Now if D is a compact subset of $\mathcal{S}_{M_{\text{par}}}$, then $|\mathcal{A}_n^\beta|_{\text{eq}}$ separates D from ∞ if n is large enough, and this confirms the Criterion for Parabolicity. We conclude that $\mathcal{S}_{M_{\text{par}}}$ is parabolic, so that M_{par} is a parabolic planar polygonal complex that is maximal with respect to $\preceq$.

The construction of M_{hyp} is similar to that of M_{par}, except that we begin with a hyperbolic planar complex, say K_7, the constant 7-degree triangulation of the plane. Let $h : \mathcal{S}_{K_7} \to \mathbb{D}$ be a conformal isomorphism and $\mathcal{H}$ the upper half disk $\{z \in \mathbb{D} : \text{Im}(z) \geq 0\}$. Conformally, $\mathcal{H}$ is the complement of a nontrivial arc in the boundary of a closed disk and is to be distinguished, conformally, from its homeomorphic cousin, the complement of a point in the boundary of a closed disk. The fact we will use is that, though both $\mathbb{C}$ and $\mathbb{D}$ contain closed homeomorphic copies of $\mathcal{H}$, only $\mathbb{D}$ contains a closed *conformally equivalent* copy of $\mathcal{H}$, a closed subset that is the image of $\mathcal{H}$ under a homeomorphism that is conformal on the interior of $\mathcal{H}$. Choose simple closed edge paths δ_n in K_7 disjoint from $h^{-1}(\mathcal{H})$ whose complementary disks are pairwise disjoint and of respective combinatorial lengths agreeing with those of the respective boundaries of D_n. Let M_{hyp} be the complex obtained by replacing the complementary disks of the sequence δ_n by the combinatorial disks D_n. Then the $\preceq$-maximal planar polygonal complex M_{hyp} is of hyperbolic type since the Riemann surface $\mathcal{S}_{M_{\text{hyp}}}$ contains $h^{-1}(\mathcal{H})$, a closed conformal copy of $\mathcal{H}$. $\surd$

4. Subdivision Rules and Expansion Complexes

The first conformal hierarchy for a conformal tiling recognized as such was one constructed[18] from a loxodromic supersymmetry of the pentagonal tiling of [2], which is generated by a finite subdivision rule. Theorem 3.7 and its corollaries, 3.9 and 3.10, were uncovered in the attempt to understand why any conformal tiling combinatorially locally isomorphic with the pentagonal one is parabolic, and then to generalize this to a wider setting. Though the hierarchy of the pentagonal tiling may be generated by a loxodromic supersymmetry, this turns out to be a very special case in that the typical hierarchical conformal tiling admits no supersymmetry. In this section, we study the conformal tilings that arise as expansion complexes of rotationally invariant subdivision operators, and prove that these complexes exhibit a combinatorial hierarchy. This latter claim is proved after exploring some of the combinatorial properties possessed by these expansion complexes, including the fact that they are combinatorially repetitive. We then specialize to those with dihedral symmetry and apply the results of Section 3.4 to conclude that certain conformal tilings have a conformal hierarchy and are constantly parabolic across their local isomorphism class, including the original pentagonal one. Finally, we examine the subdivision map defined on the local isomorphism class (K) of the expansion complex K, relating the existence of finite orbits in (K) to the existence of supersymmetries on elements of (K). We prove that this map has at most finitely many k-orbits for any fixed positive integer k, and the proof points us to an elegant construction method for expansion complexes that

[18]Perhaps *constructed* is too strong a word, for the hierarchy appears automatically in constructing the tiling and is perceived immediately by the intuition when one first encounters the graphics of the tiling. The loxodromic supersymmetry just confirms precisely the intuition.

admit combinatorial supersymmetries, cellular isomorphisms to their τ^k-subdivisions. We close out this section, and indeed the paper, with a zoo of graphical examples.

In the following, we are going to use a definition of the *interior* of a CW complex that corresponds to usage of the term in manifold theory rather than in general topology. It will be used only in the case where F is a finite, 2-dimensional planar CW complex. In this case, the **interior** of F is defined as the largest 2-manifold (without boundary) contained in $|F|$, the underlying space of F. This is precisely the union of the open faces of F, the open edges that meet two closed faces of F, and the vertices of F that have a disk neighborhood contained in $|F|$, and is denoted as F°.

4.1. **Expansion complexes associated to subdivision operators.** Let τ_n be a nontrivial, regular oriented CW-decomposition of the oriented n-gon Δ that is rotationally symmetric and subdivides Δ into ℓ combinatorial n-gons. We can think of τ_n as the seed that defines the subdivision operator τ that is defined only on the set of planar n-gon complexes rather than on the set of all planar polygonal complexes, an (n,n)**-subdivision rule** in that it yields a planar n-gon complex upon its action on a planar n-gon complex. We call τ a **rotationally invariant** (n,n)**-subdivision operator** when emphasis is needed.[19] We will apply τ as a subdivision operator on the expansion complexes that are defined next. Under this convention that τ_n defines the subdivision operator τ, we may write $\tau_n = \tau\Delta$, and we define $\tau_n^k = \tau^k\Delta$ for each positive integer k. Obviously, τ_n^k is a subdivision of Δ into ℓ^k combinatorial n-gons. Let

$$(\ddagger) \qquad F_1 \hookrightarrow F_2 \hookrightarrow \cdots \hookrightarrow F_m \hookrightarrow F_{m+1} \hookrightarrow \cdots$$

be any sequence of isomorphic embeddings of CW complexes that satisfies the following properties.

(1) For each positive integer m, F_m is a connected subcomplex of $\tau_n^{i_m}$, for some integer $i_m \geq m$;

(2) for each m, $i_m < i_{m+1}$;

(3) each map $F_m \hookrightarrow F_{m+1}$ is a cellular, orientation-preserving isomorphic embedding of CW complexes;

(4) for each m, there exists a positive integer p for which the image of $|F_m|$ under the composition $F_m \hookrightarrow \cdots \hookrightarrow F_{m+p}$ is contained in the interior of F_{m+p};[20]

[19] Using Cannon, Floyd, and Parry's machinery of finite subdivision rules and model complexes, one may place this in the setting of their one-tile rotationally invariant finite subdivision rule, which is explored in their paper [6]. That machinery, though elegant, is a bit more sophisticated than our need demands and, as has been stated already, is avoided in this paper.

[20] We could have required that the image of $|F_m|$ under the embedding $F_m \hookrightarrow F_{m+1}$ be contained in the interior of F_{m+1} and arrived at the same collection of expansion complexes. The slightly more complicated condition (4) makes for easier proofs in what follows.

(5) for each m and each combinatorial simple closed edge path γ in F_m, there exists a positive integer q for which the image of γ under the composition $F_m \hookrightarrow \cdots \hookrightarrow F_{m+q}$ bounds a combinatorial disk in F_{m+q}.[21]

The CW complex $K = \varinjlim F_m$, the direct limit of the system (‡), is called an **expansion complex** associated to the subdivision operator τ.[22] In most cases, each F_m in the sequence defining the expansion complex will be a CW decomposition of a closed topological disk and the image of the closed disk $|F_m|$ under the single embedding $F_m \hookrightarrow F_{m+1}$ will be contained in the interior of the disk $|F_{m+1}|$. Properties (4) and (5) automatically adhere in this case. We can define a more general expansion complex by removing property (4) from the list. Property (4) guarantees that any expansion complex is a CW decomposition of a planar surface *without* boundary and, ultimately, is a planar n-gon complex. Removing the property (4) requirement would allow for expansion complexes that have nontrivial boundary components of infinite length, and removing property (5) would allow for holes, even an infinite number of these. We wish to avoid these pathologies in this study.

Theorem 4.1. *The expansion complex $K = \varinjlim F_m$ is a planar n-gon complex.*

Proof. Property (4) guarantees that the topological space $|K|$ underlying the CW complex K is a non-compact topological 2-manifold. Property (5) guarantees that $|K|$ is simply connected. It follows that $|K|$ is homeomorphic to the plane so that K is a CW-decomposition of the plane. Since each of the subdivided n-gons $\tau_n^{i_m}$ is a complex with only n-gon faces, each face of K is an n-gon, and since each subdivision $\tau_n^{i_m}$ is regular, so too is the CW decomposition K. It follows that K is a planar n-gon complex. □

The most useful property of an expansion complex is encased in the next theorem, Theorem 4.2. First we need a few definitions. A **τ-aggregate** of the planar n-gon complex K is a combinatorial aggregate L of K for which $\tau L = K$. We do not claim that τ-aggregates are unique. As an example, let Z be the integer lattice 4-gon complex and let τ be the subdivision of the regular 4-gon Δ_4 that subdivides each edge into three sub-edges and subdivides the single face into nine 4-gon sub-faces as in Fig. 8. Notice that there are two types of sub-faces in Δ_4, a single **center sub-face** surrounded by eight **edge sub-faces**. Let f be the unit square face of Z whose lower left-hand vertex is the origin $(0,0)$. Let L be the (unique) τ-aggregate of Z for which the face f of Z is a center sub-face of the

[21] We make no claim that these five properties are independent. Under mild restrictions on τ, property (5) is a consequence of properties (3) and (4).

[22] The expansion complexes defined by Cannon, Floyd, and Parry in [5] are more restricted than these in that they, like the original pentagonal example, come equipped with an **expansion map**, which is the inverse of a combinatorial supersymmetry of unit period. Theirs therefore do not include the expansion complexes defined herein that have no combinatorial supersymmetry, nor those with a combinatorial supersymmetry of period greater than 1. We will see subsequently that when K is plural, of the uncountably many pairwise distinct expansion complexes associated to τ, at most countably many have a combinatorial supersymmetry, and at most finitely many of those have period 1 and, therefore, an expansion map.

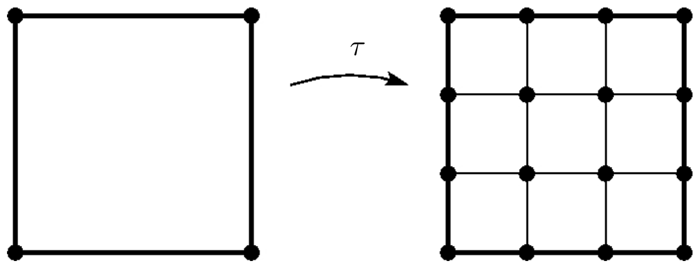

FIGURE 8. A 4-gon subdivision rule τ with non-unique aggregates.

unique face g of L containing f, and let L' be a τ-aggregate of Z for which the face f of Z is an edge sub-face of the unique face g' of L' containing f. Then $L \neq L'$ and so τ-aggregates fail to be unique. Of course, L is isomorphic to L', but no isomorphism of L to L' can take f set-wise to itself. The point is that the face f sits within the aggregate L differently than it sits within the aggregate L'. This causes a problem that requires the proof of Theorem 4.2 to be a bit more involved than one might at first suspect, and it will help in that proof to understand this problem. Suppose K and K' are locally isomorphic planar n-gon complexes with respective τ-aggregates L and L' and $h : F \to F'$ is an isomorphism of the subcomplex F of K onto a subcomplex F' of K'. Let H be the smallest subcomplex of L whose τ-subdivision contains F and H' the smallest subcomplex of L' whose τ-subdivision contains F'. It does not follow that H is isomorphic with H', even if K and K' are isomorphic, and even more importantly, when H and H' are isomorphic, it does not follow that the isomorphism h extends to an isomorphism of τH with $\tau H'$. In fact, even if $K = K'$ and H and H' are isomorphic, the CW complex pairs $(\tau H, F)$ and $(\tau H', F')$ may not be isomorphic as pairs. This is aptly illustrated in $Z = K = K'$ by setting $F = F'$, the smallest subcomplex of Z with the single face f, and letting H be the smallest subcomplex of L with the single face g and H' the smallest subcomplex of L' with the single face g'.

Theorem 4.2. *Any expansion complex $K = \varinjlim F_m$ associated to a rotationally invariant (n, n)-subdivision operator τ has a τ-aggregate that is itself an expansion complex associated to τ.*

Proof. For each $m > 1$, let G_m be the smallest subcomplex of $\tau_n^{i_m - 1}$ whose τ-subdivision τG_m, a subcomplex of $\tau_n^{i_m}$, contains F_m. Ideally, the complexes G_m would admit appropriate embeddings, $G_m \hookrightarrow G_{m+1}$ so that, first, the embedding $\tau G_m \hookrightarrow \tau G_{m+1}$ would extend the embedding $F_m \hookrightarrow F_{m+1}$, and, second, the direct limit $L = \varinjlim G_m$ would be an expansion complex associated to τ. It then would follow that $\tau L = K$, making the expansion complex L a τ-aggregate of K. Unfortunately this is not necessarily true and in trying to modify this idea to get a proof, the problem articulated in the paragraph above presents itself. The remainder of this proof merely adjusts this idea to make it work.

First, by identifying each F_m with its canonical embedded copy in the direct limit K, we may write K as the increasing union $\cup_{m=1}^{\infty} F_m$. By passing to a subsequence if necessary, property (4) allows us to assume, for each positive integer m, that F_m is contained in the interior of F_{m+1}. In fact the use of property (4) allows us to assume, without loss of generality, that the δ_K-distance from any face of F_m to any face in the complement of F_{m+1} is greater than $\ell + 1$, where we recall that ℓ is the number of faces of τ_n, the seed of the subdivision operator τ. A **partial τ-aggregate of F_m** is a finite connected n-gon complex H contained in $|K|$ whose τ-subdivision τH is a subcomplex of K that contains F_m as a subcomplex. The assumption that the δ_K-distance from F_m to the complement of F_{m+1} exceeds $\ell + 1$ implies that any partial τ-aggregate of F_m, each of whose faces meets F_m, is contained in F_{m+1}°, the interior of the complex F_{m+1}. We now describe a partial τ-aggregate of F_m. Let H_m be the smallest subcomplex of $\tau_n^{i_{m+1}-1}$ whose faces are precisely those faces f of $\tau_n^{i_{m+1}-1}$ whose τ-subdivision τf meets the image of F_m under the embedding $F_m \hookrightarrow F_{m+1}$. Since F_m is connected, so too is H_m. By our assumptions, we may consider that τH_m is a subcomplex of K contained in F_{m+1}, and so H_m is a partial τ-aggregate of F_m each of whose faces meets F_m. It follows that there are cellular containments

$$F_m \subset \tau H_m \subset F_{m+1}$$

with $|H_m| \subset F_{m+1}^{\circ}$. Now in this construction there appear other partial τ-aggregates of F_m. Indeed, for each integer $p \geq m$, let $H_{m,p}$ be the smallest subcomplex of H_p whose faces are precisely those faces of H_p that meet F_m. Then for every $p \geq m$, $H_{m,p}$ is a partial τ-aggregate of F_m contained in the interior F_{m+1}°. As explained in the paragraph preceding the theorem, the complexes $H_{m,p}$ for $p \geq m$ need not be isomorphic, and even if they are, the pairs $(\tau H_{m,p}, F_m)$ need not be isomorphic as pairs. Nonetheless, since for each $p \geq m$ the complex $\tau H_{m,p}$ is a subcomplex of the finite complex F_{m+1}, we may conclude that there are infinitely many indices among the integers p for which the complexes $\tau H_{m,p}$ are equal to one another. As there are at most finitely many ways to τ-aggregate any finite complex, we may conclude further that there are infinitely many indices among the integers p for which the complexes $H_{m,p}$ are equal to one another. This is the observation that drives the argument.

Applying the argument of the preceding paragraph with $m = 1$ provides a subsequence $\mathcal{H}_1' = (H_{1,p_i} : i \in \mathbb{N})$ of $\mathcal{H}_1 = (H_{1,p} : p \in \mathbb{N})$ for which $H_{1,p_i} = H_{1,p_{i+1}}$ for all positive integers i. Let $G_1 = H_{1,p_1}$ and observe that G_1 is a partial τ-aggregate of F_1 with cellular containments

$$F_1 \subset \tau G_1 \subset F_2$$

for which $|G_1| \subset F_2^{\circ}$. By construction, G_1 is a subcomplex of H_{2,p_i} for all positive integers $i \geq 2$. Moreover, we have $G_1 \subset H_{p_1} \subset \tau_n^{k_1}$ where $k_1 = i_{p_1+1} - 1$. We now apply the latter part of the argument of the preceding paragraph again, but this time to the sequence $\mathcal{H}_2 = (H_{2,p_i} : i \geq 2) = (H_{2,s} : H_{1,s} \in \mathcal{H}'_1$ and $s > p_1)$. The point is that for each integer $i \geq 2$, H_{2,p_i} is a partial τ-aggregate of F_2 contained in the interior F_3° and so $\tau H_{2,p_i}$ is a subcomplex of the finite complex F_3. We may conclude that there are infinitely many indices among the integers $i \geq 2$ for which the complexes $\tau H_{2,p_i}$ are equal to one another, and hence there are infinitely many such indices for which the complexes H_{2,p_i} are equal. This provides a subsequence $\mathcal{H}'_2$ of $\mathcal{H}_2$ that consists of pairwise equal terms. Let G_2 be the initial element of the subsequence $\mathcal{H}'_2$. Note that $G_2 = H_{2,p_j}$ for some $j \geq 2$, and therefore G_1 is a subcomplex of G_2 and $G_2 \subset H_{p_j} \subset \tau_n^{k_2}$ where $k_2 = i_{p_j+1} - 1$, so that $k_1 < k_2$ since $p_1 < p_j$. The cellular containments

$$F_1 \subset \tau G_1 \subset F_2 \subset \tau G_2 \subset F_3$$

hold, along with $|G_2| \subset F_3^\circ$. Since $|G_1| \subset F_2^\circ$, we have $|G_1| \subset G_2^\circ$. Of course, G_2 is a subcomplex of H for all complexes $H \in \mathcal{H}_3 = (H_{3,s} : H_{2,s} \in \mathcal{H}'_2$ and $s > p_j)$, and we may continue the construction. Continuing in this manner we may extract, via an inductive construction, a sequence $(G_m : m \in \mathbb{N})$ of finite n-gon complexes where, for each positive integer m,

(i) there are cellular containments $F_m \subset \tau G_m \subset F_{m+1}$.

(ii) G_m is a partial τ-aggregate of F_m contained in the interior of F_{m+1};

(iii) the finite n-gon complex G_m is a connected subcomplex of G_{m+1};

(iv) G_m is isomorphic to a subcomplex of $\tau_n^{k_m}$ where $k_m < k_{m+1}$;

(v) $|G_m| \subset G_{m+1}^\circ$;

The ingredients are in place now to complete the argument. Items (iii) through (v) show that the sequence of cellular containments

$$G_1 \subset G_2 \subset \cdots \subset G_m \subset G_{m+1} \subset \cdots$$

satisfies properties (1) through (4) of the definition of expansion complex, and item (i) along with the fact that the sequence (F_m) satisfies property (5) guarantee that property (5) holds for the sequence (G_m). We conclude that the direct limit, which in this case is the union $L = \varinjlim G_m = \cup_{m=1}^{\infty} G_m$, is an expansion complex associated to τ. Moreover, item (ii) guarantees that the τ-subdivision τL is equal to K so that L is a τ-aggregate of K. This completes the proof. □

From this theorem we are able to dissect the local isomorphism class of K, uncover some of the combinatorial structure of K, and prove that every expansion complex exhibits a combinatorial hierarchy. The important implications appear in the four corollaries that follow and culminate in Theorem 4.7. For each of these corollaries, τ is a rotationally invariant (n,n)-subdivision operator.

Corollary 4.3. *If $K = \varinjlim F_m$ and $L = \varinjlim G_m$ are expansion complexes associated to τ, then K is locally isomorphic with L.*

Proof. It suffices to verify that, for each positive integer k, the subdivided n-gon τ_n^k isomorphically embeds in K. Apply Theorem 4.2 iteratively k times, starting with K, to obtain a τ^k-aggregate of K, a planar n-gon complex K' such that $K = \tau^k K'$. Let f be any n-gon face of K' and observe that $\tau^k f$ is a subcomplex of K isomorphic to τ_n^k. □

The proof of this corollary shows more.

Corollary 4.4. *Every expansion complex associated to τ is combinatorially repetitive.*

Proof. Let H be any finite connected subcomplex of the expansion complex $K = \varinjlim F_m$ associated to τ. Then there exists a positive integer k for which H is contained in F_k, where we have identified F_k, a subcomplex of $\tau_n^{i_k}$, with its canonical copy in the direct limit K. Let K' be a τ^{i_k}-aggregate of K and recall that there are ℓ^{i_k} n-gon faces in the complex $\tau_n^{i_k}$. Since the subcomplex $\tau^{i_k} f$ contains an isomorphic copy of H for every face f of K', every face of K is ℓ^{i_k}-close to an isomorphic copy of H in the δ_K-metric, and this implies that each vertex of K is $n\ell^{i_k}/2$-close to an isomorphic copy of H. We conclude that H is quasi-dense in K, and this proves K to be combinatorially repetitive. □

Corollary 4.3 proves one half of the next corollary, that the local isomorphism class of an expansion complex associated to τ is nothing more than the set of all expansion complexes associated to τ.

Corollary 4.5. *The local isomorphism class (K) of the expansion complex $K = \varinjlim F_m$ associated to τ is precisely the set of all isomorphism classes of expansion complexes associated to τ.*

Proof. We need only show that if the planar n-gon complex L is locally isomorphic to the expansion complex K, then L is an expansion complex associated to τ. Again we identify each F_m with its canonical copy in K and write $K = \cup_{m=1}^{\infty} F_m$. Let C_m for $m \geq 1$ be a sequence of pairwise disjoint simple close edge-paths in L such that C_{m+1} separates C_m from infinity and let B_m be the combinatorial disk bounded by C_m. Then B_m is contained in the interior of B_{m+1}. Since K is locally isomorphic to L, B_m is isomorphic to a subcomplex B'_m of K and there exists an index $k(m)$ such that B'_m is contained in $F_{k(m)}$. We may assume by choosing $k(m)$ sequentially that $k(m) < k(m+1)$ so that $j_m = i_{k(m)} < i_{k(m+1)} = j_{m+1}$, where F_m is a subcomplex of $\tau_n^{i_m}$ as in property (1). We then have the isomorphic embedding $B_m \cong B'_m \subset F_{k(m)} \subset \tau_n^{j_m}$, and properties (1) through (5) in the definition of expansion complex are satisfied. We conclude that $L = \cup_{m=1}^{\infty} B_m \cong \varinjlim B'_m$ is an expansion complex associated to τ. □

Corollary 4.6. *If $K = \varinjlim F_m$ is an expansion complex associated to τ, then, for all positive integers k, the subdivision $\tau^k K$ is an expansion complex associated to τ and hence is locally isomorphic with K.*

Proof. First observe that τK is an expansion complex associated to the subdivision operator τ. Indeed, the sequence of embeddings defining an expansion complex for τK is merely $\tau F_m \hookrightarrow \tau F_{m+1}$, where τF_m is a subcomplex of $\tau \tau_n^{i_m} = \tau_n^{i_m+1}$. This and induction then imply that, for all positive integers k, $\tau^k K$ also is an expansion complex associated to τ, and an application of Corollary 4.3 finishes the proof. □

Finally, Dane Mayhook will prove in his doctoral thesis that any expansion complex associated to a shrinking, dihedrally symmetric (n, n)-subdivision operator of bounded degree satisfies a θ-isoperimetric inequality and hence has combinatorial FLC.

4.2. **Combinatorial hierarchy, conformal hierarchy, and expansion complexes.** The work of the preceding section pays off to verify rather easily that every expansion complex associated to τ exhibits a combinatorial hierarchy.

Theorem 4.7. *Every expansion complex associated to a rotationally invariant (n, n)-subdivision operator exhibits a combinatorial hierarchy.*

Proof. Let $K = K_0$ be an expansion complex associated to the rotationally invariant (n, n)-subdivision operator τ. For positive integers k, let $K_k = \tau^k K$ and define the planar n-gon complex K_{-k} inductively using Theorem 4.2 so that $\tau K_{-k} = K_{-k+1}$ for all k. Theorem 4.2 guarantees that each K_{-k} is an expansion complex associated to τ. Corollaries 4.3 and 4.6 imply that K_k is locally isomorphic to K_{k+1}, for all integers k. It follows that $\{K_k\}$ is a combinatorial hierarchy manifested by τ. □

In case the (n, n)-subdivision operator τ in Theorem 4.7 is additionally dihedrally symmetric, it serves as a conformal subdivision operator for a conformal hierarchy of the conformal tiling T_K determined by any expansion complex K associated to τ. This follows immediately by an application of Theorem 3.5. We state this formally as the next corollary.

Corollary 4.8. *The conformal tiling determined by any expansion complex associated to a dihedrally symmetric (n, n)-subdivision operator exhibits a conformal hierarchy.*

The (n, n)-subdivision operator τ has **bounded degree** if there is a positive constant β, called a **face bound** for τ, such that each vertex of τ_n^k meets at most β faces of τ_n^k, for all positive integers k. In particular, if K is a planar n-gon complex of bounded degree such that each vertex of K meets at most μ faces of K, then, for every positive integer k, the subdivision $\tau^k K$ has bounded degree with each vertex meeting at most $\beta\mu$ faces. We can say a bit more when K is an expansion complex associated to τ. In this case, K has bounded degree with at most β faces of K meeting at a vertex and, by an application of Theorem 4.2 and Corollary 4.6, the same holds for every τ^k-subdivision and τ^k-aggregate of K.

The following corollary uses the results of Section 3 to verify that type is constantly parabolic across the local isomorphism class of an expansion complex associated to an appropriate dihedrally symmetric subdivision operator. In particular, the results of this section and Corollary 3.10 imply the following result.

Corollary 4.9. *Let τ be a shrinking, dihedrally symmetric (n,n)-subdivision operator of bounded degree. Then τ manifests a combinatorial hierarchy for any expansion complex K associated to τ and the conformal tiling T_K exhibits a conformal hierarchy manifested by τ, has parabolic type, and tiles the complex plane $\mathbb{C}$. Moreover, conformal type is constantly parabolic across the local isomorphism class (K).*

This applies to the original pentagonal example P of [2] to answer Maria Ramirez-Solano's question that instigated this study of type among locally isomorphic conformal tilings. All conformal tilings locally isomorphic to the pentagonal tiling P are parabolic. In this pentagonal example, we don't need the full machinery of conformal hierarchies that has been developed in this paper. This is because a fairly straightforward proof of constancy of parabolic type across (P) exists using the fact that an aggregate may be defined uniquely by aggregating along "central" pentagons of the tiling, those whose vertices meet only three tiles. This is a very special property of the tiling P not shared by general conformal hierarchical tilings, and we will spare the reader the details.

4.3. **Supersymmetric expansion complexes and the action of $\hat{\tau}$ on (K).** If τ is a nontrivial subdivision operator, then τK is never equal to K for any planar polygonal complex K. Of course, it may be that τK is isomorphic to K. For example, if ν is the simple quad subdivision operator, then $\nu Z \cong Z$ for the integer lattice complex Z, though, of course, $\nu Z \neq Z$. As introduced on page 31, we use the symbol $\hat{\tau}$ to denote the function induced on appropriate isomorphism classes of planar polygonal complexes, so that τK is a specific planar polygonal complex that subdivides K while $\hat{\tau} K$ is the isomorphism class in C of τK. Our interest is in the action of $\hat{\tau}$ on a local isomorphism class (K). For example, since Z is singular, (Z) is a singleton and the map $\hat{\nu}$ is singularly uninteresting, so our interest resides in plural complexes K. In general, $\hat{\tau} : (K) \to (\tau K)$, but notice that Corollary 4.6 implies that the image of (K) under $\hat{\tau}$ is contained in (K) whenever K is an expansion complex associated to the rotationally invariant (n,n)-subdivision operator τ; moreover, Theorem 4.2 additionally implies that $\hat{\tau}$ takes (K) onto (K). We close this paper with a brief examination of the mapping $\hat{\tau}$ in this setting, that K is an expansion complex associated to the rotationally invariant (n,n)-subdivision operator τ, which is assumed for the remainder of the paper; additionally, to avoid the triviality expressed above, we make the assumption that K is plural. In the setting of this section then,

$$\hat{\tau} : (K) \to (K)$$

is surjective. As a first observation, note that if $L \in (K)$, then L is a point in a finite periodic orbit under the forward iteration of $\hat{\tau}$ if and only if L is τ-supersymmetric, and in this case, the number of elements in the periodic orbit is the τ-period of L. One of the goals of this section is to construct τ-supersymmetric expansion complexes with τ-period k, for arbitrary positive integers k, but first we prove that such complexes are the exception rather than the rule among shrinking subdivision operators of bounded degree. A **k-orbit** of the action of $\hat{\tau}$ on (K) is a forward periodic orbit with exactly k elements.

Theorem 4.10. *Let τ be a shrinking, rotationally invariant (n, n)-subdivision operator of bounded degree. Then up to isomorphism, at most countably many expansion complexes associated to τ are τ-supersymmetric. In fact, for each positive integer k, there are at most finitely many k-orbits of $\hat{\tau}$ in (K); equivalently, up to isomorphism, there are at most finitely many τ-supersymmetric expansion complexes associated to τ of τ-period k.*

Proof. The proof is divided into three parts.

Part 1: Classifying fixed points of the action of $\hat{\tau}$ on (K) by combinatorics. Let J be an expansion complex associated to τ that is a fixed point of the action of $\hat{\tau}$. Then J is a CW-decomposition of the plane $\mathbb{C}$ and τJ is a CW complex that subdivides J with $\hat{\tau} J = J$. This means that J is isomorphic to τJ and we let $\lambda_J : \mathbb{C} \to \mathbb{C}$ be a homeomorphism of the complex plane that is a cellular isomorphism of J onto τJ. In Part 1, we make two assumptions about the subdivision operator τ and the isomorphism λ_J. The first assumption is that τ is strictly shrinking and the second is that there is a closed cell b_J of J for which $\lambda_J(b_J) \subset b_J^\circ$. We will call b_J a **fixture** of the homeomorphism λ_J. The goals of this first part of the proof are to characterize combinatorially those fixed points J of $\hat{\tau}$ that satisfy the second assumption under the condition that τ is strictly shrinking and to use this characterization to see that there are at most finitely many fixed points of the action of $\hat{\tau}$ on (K) that satisfy the second assumption.

Let C_J be the core of J determined by the cell b_J and c_J be the core of τJ determined by the cell $\lambda_J(b_J)$. C_J and c_J are both vertex, edge, or face cores depending on whether the cell b_J is, respectively, a vertex, edge, or face of J, and, in fact, it is easy to see that $\lambda_J(C_J) = c_J$. We claim that the core c_J is contained in the interior $(\tau C_J)^\circ$ of the τ-subdivided core C_J. First we show that c_J is contained in the interior of the core C_J. When c_J is a face core, then automatically $c_J = \lambda_J(b_J)$ is contained in the open face $b_J^\circ = C_J^\circ$. When c_J is an edge core, the fact that the subdivision operator τ is both strictly shrinking and rotationally invariant and $\lambda_J(b_J) \subset b_J^\circ$ imply that a face f of τJ with boundary edge $\lambda_J(b_J)$ is contained in both open angles $\angle_g u$ and $\angle_g v$, where g is the face of J containing f and u and v are the vertices incident to b_J. This implies in turn that $c_J \subset C_J^\circ$. Finally, in case c_J is a vertex core with, say, $b_J = v$, a vertex of J, then the fact that τ is strictly shrinking implies that every face f of c_J is contained in the open angle $\angle_g v$, where g is the face of J that contains f. This implies that $c_J \subset C_J^\circ$. Since $C_J^\circ = (\tau C_J)^\circ$, we conclude also that c_J is contained in the interior of τC_J.

Let J and L be expansion complexes that serve as fixed points of the action of $\hat{\tau}$, both of which satisfy the second assumption. Define J and L to be **λ-equivalent** if there exists a homeomorphism $h : |C_J| \to |C_L|$[23] that is a cellular isomorphism of the cell complex τC_J onto the cell complex τC_L such that $h(c_J) = c_L$ and for which $h^{-1} \circ \lambda_L \circ h = \lambda_J|_{|C_J|}$. Our aim is to show that if J and L are λ-equivalent fixed points of $\hat{\tau}$, then $J \cong L$ so that $J = L$ in (K). To prove this, we first show how to recover J from the triple $(C_J, c_J, \lambda_J|_{C_J})$. Indeed, our claim is that

$$J = \cup_{i=1}^{\infty} \lambda_J^{-i}(\tau^i C_J),$$

where $\lambda_J^i : J \to \tau^i J$ is the isomorphism of complexes gotten from iterating the mapping λ_J i times. To verify the claim, note that, since $c_J = \lambda_J(C_J)$ is a subcomplex of τC_J, the core $C_J = \lambda_J^{-1}(c_J)$ is a subcomplex of $\lambda_J^{-i}(\tau^i C_J)$ for every positive integer i. It follows that the union $J' = \cup_{i=1}^{\infty} \lambda_J^{-i}(\tau^i C_J)$ is a connected subcomplex of J. To see that $J' = J$, it suffices to verify that, for each positive integer i, the complex $\lambda_J^{-i}(\tau^i C_J)$ is contained in the interior of $\lambda_J^{-i-1}(\tau^{i+1} C_J)$. But this follows from induction with the basis established from the fact that the complex c_J is contained in the interior of τC_J, implying after an application of λ_J^{-1} that C_J is a subcomplex of J contained in the interior of $\lambda_J^{-1}(\tau C_J)$. Now, assuming that the expansion complex L is λ-equivalent to J, as with J write $L = \cup_{i=1}^{\infty} \lambda_L^{-i}(\tau^i C_L)$ and let $h : |C_J| \to |C_L|$ be a homeomorphism with the properties described in the definition of λ-equivalence. This means that h is a homeomorphism of $|C_J|$ onto $|C_L|$ that is a cellular isomorphism of τC_J onto τC_L, and this implies by the obvious τ-aggregation that h is also a cellular isomorphism of the complex C_J onto the complex C_L. From $h^{-1} \circ \lambda_L \circ h = \lambda_J|_{|C_J|}$ we may infer that $h = \lambda_L^{-1} \circ h \circ \lambda_J|_{|C_J|}$. For each positive integer i, this allows us to extend the cellular isomorphism h of C_J onto C_L to a cellular isomorphism $h_i = \lambda_L^{-i} \circ h \circ \lambda_J^i|_{|J_i|}$ of $J_i = \lambda_J^{-i}(\tau^i C_J)$ onto $L_i = \lambda_L^{-i}(\tau^i C_L)$ and we obtain the diagram of commuting cellular containments and cellular isomorphisms h_i:

$$\begin{tikzcd}
C_J \arrow[r, hook] \arrow[d, "h"', "\cong"] & \lambda_J^{-1}(\tau C_J) \arrow[r, hook] \arrow[d, "h_1"', "\cong"] & \lambda_J^{-2}(\tau^2 C_J) \arrow[r, hook] \arrow[d, "h_2"', "\cong"] & \cdots \arrow[r, "\underrightarrow{\lim}"] & J \\
C_L \arrow[r, hook] & \lambda_L^{-1}(\tau C_L) \arrow[r, hook] & \lambda_L^{-2}(\tau^2 C_L) \arrow[r, hook] & \cdots \arrow[r, "\underrightarrow{\lim}"'] & L.
\end{tikzcd}$$

[23] Recall that if F is a subcomplex of the planar polygonal complex K, $|F|$ denotes the underlying space of F, the union of the cells of F, and is a subspace of the plane.

This induces an isomorphism of CW complexes $J \cong L$ and implies that $J = L$ in (K).

Armed with the observation of the preceding paragraph, we can verify that there are at most finitely many fixed points of the action of $\hat{\tau}$ on the local isomorphism class (K) that satisfy the second assumption. Indeed, we have shown that each fixed point J of the action of $\hat{\tau}$ that satisfies the second assumption identifies a CW pair $(\tau C_J, c_J)$, where C_J is a core of J and c_J is a core of τJ, and that $J \cong L$ whenever L is a fixed point of $\hat{\tau}$ satisfying the second assumption that is λ-equivalent to J. Since τ has bounded degree, the expansion complex K has bounded degree and, up to isomorphism, there are only finitely many pairs $(\tau C, c)$ where C is a core of K and c is a core of τK. Since the expansion complex J is locally isomorphic to K, the pair $(\tau C_J, c_J)$ is, up to isomorphism, one of these finitely many pairs $(\tau C, c)$ from K. Moreover, for any fixed pair (C, c), there are, up to cellular isotopy, only finitely many orientation preserving cellular isomorphisms λ from the cell complex C onto the cell complex c. This implies that each triple $(C_J, c_J, \lambda_J|_{C_J})$ is represented among the finitely many distinct triples (C, c, λ), and this implies that there are only finitely many λ-equivalence classes of expansion complexes in (K) that satisfy the second assumption, and therefore only finitely many fixed points of $\hat{\tau}$ that satisfy the second assumption. This completes the verification of Part 1.

Part 2: The existence of a fixture for a power of the homeomorphism λ_J. We are still under the assumption that τ is strictly shrinking. In this second part of the proof, we would like to show that any cellular isomorphism λ_J of any fixed point J of $\hat{\tau}$ onto its subdivision τJ has a fixture; unfortunately, though, this fails to be true. At the conclusion of the proof, we will give an example illustrating this. Our aim in Part 2, then, is to prove that the strictly shrinking subdivision operator τ determines a positive integer M such that, for every fixed point J of $\hat{\tau}$, there exists a positive integer $m \leq M$ such that the cellular isomorphism λ_J^m of J to $\tau^m J$ has a fixture. In fact, we will show that $M = 6\beta$ works, where β is a face bound for τ.

The verification of the existence of a fixture for a positive power of λ_J depends strongly on the fact that τ is rotationally invariant and shrinking so that any combinatorial hierarchy manifested by τ is expansive. As an illustrative aside, the hyperbolic complex H of Example 3.1 provides a fixture-free homeomorphism $z \mapsto z + \mathbf{i}$ of $\mathbb{C}$ that serves as a cellular isomorphism of H to $\sigma_0 H$, where σ_0 is the non-rotationally invariant and non-shrinking subdivision rule of the example that divides each pentagonal face into two pentagons and for which $\hat{\sigma}_0 H = H$.

Let J be any fixed point of the action of $\hat{\tau}$. We know from Theorem 4.7 that J exhibits a combinatorial hierarchy manifested by τ, but we can say more in this case. Indeed, forward and backward iteration of the mapping λ_J builds such a hierarchy $\{J_k\}$, where for each integer k, $J_k = \lambda_J^k(J)$. Since τ is shrinking, Theorem 3.3 implies that the hierarchy $\{J_k\}$ is exponentially expansive. Our first goal is to show that there exist positive integers p and q and a polygonal face g of J_{-p} such that $\lambda_J^q(g) \subset g$. Toward this goal, let f_0 be any face of J and, for any positive integer m, let f_m be that unique face of J that contains the face $\lambda_J^m(f_0)$ of J_m. Since the hierarchy is expansive, there is a positive integer p and a core c of J_{-p} that engulfs $f_0 \cup f_1 \cup \cdots \cup f_\beta$, where β is a face bound for τ, and therefore a bound on the number of combinatorial n-cell faces that meet at any vertex of any J_k of the hierarchy. Since the subcomplex c of J_{-p} is a core, it has at most β faces and it follows that, since each of the faces $f_0, f_1, \ldots, f_\beta$ is a face of the subcomplex $\tau^p c$, there is at least one face, say g, of c that contains two of the faces from the list $f_0, f_1, \ldots, f_\beta$. Assume that the two faces f_i and f_j satisfy $f_i \cup f_j \subset g$, where $0 \le i < j \le \beta$ are chosen so that $q = j - i$ is as small as possible. By the definition of the faces f_m, we have $\lambda_J^i(f_0) \cup \lambda_J^j(f_0) \subset g$ with $\lambda_J^q(\lambda_J^i(f_0)) = \lambda_J^j(f_0)$. It follows that the face $\lambda_J^q(g)$ of $\lambda_J^q(J_{-p}) = J_{-p+q}$ contains the face $\lambda_J^j(f_0)$ of its subdivision J_j, as does the face g of J_{-p}. This shows that the face $\lambda_J^q(g)$ of J_{-p+q} meets the open face g° since $\lambda_J^j(f_0)^\circ \subset \lambda_J^q(g) \cap g^\circ$. Since the face $\lambda_J^q(g)$ of the complex J_{-p+q} meets the open face g° of J_{-p}, and since J_{-p+q} subdivides J_{-p}, we conclude that $\lambda_J^q(g) \subset g$.

Since g is a face of J_{-p}, the combinatorial n-cell $a = \lambda_J^p(g)$ is a face of J with $\lambda_J^q(a) = \lambda_J^{p+q}(g) \subset \lambda_J^p(g) = a$. If λ_J^q fixes a vertex v of a, then $b_J = \{v\}$ is a fixture of λ_J^q, or if $\lambda_J^q(e) \subset e^\circ$ for an edge e of a, then $b_J = e$ is a fixture of λ_J^q. Assume that λ_J^q neither fixes a vertex of a nor maps an edge of a into the corresponding open edge. Since τ is strictly shrinking, so too is τ^q. This implies that $\lambda_J^q(a)$, a face of $\tau^q J$ and a subset of a, is a subset of an open angle $\angle_a v$ for a vertex v of a. If $\lambda_J^q(a) \subset a^\circ$, then $b_J = a$ is a fixture of λ_J^q; otherwise, $\lambda_J^q(a)$ meets one or both of the half-open edges $\{v\} \cup d^\circ$ and $\{v\} \cup e^\circ$, where d and e are the edges of a incident to v. The remainder of the argument rests on where v goes under the action of λ_J^q. There are four possibilities: $\lambda_J^q(v) \in \angle_a v = \{v\} \cup d^\circ \cup e^\circ \cup a^\circ$, and as this is a disjoint union, $\lambda_J^q(v)$ lies in exactly one of the open cells $\{v\}$, d°, e°, or a°. We have assumed though that λ_J^q fixes no vertex of a, so the first possibility is ruled out. The last possibility, that $\lambda_J^q(v) \in a^\circ$, implies that the image of the open angle $\angle_a v$ under λ_J^q is contained in the open cell a°, and from this we have $\lambda_J^{2q}(a) \subset a^\circ$, so that $b_J = a$ is a fixture of λ_J^{2q}. The remaining two possibilities are symmetric, so we assume that $\lambda_J^q(v) \in e^\circ$. This implies, since λ_J^q is a cellular isomorphism of J to a subdivision J_q, that $\lambda_J^q(e^\circ)$ is contained in either e° or a°. In the former case, since $\lambda_J^q(e)$ also is contained in the open angle $\angle_v a$, then $\lambda_J^q(e) \subset \{v\} \cup e^\circ$. But this implies that $\lambda_J^{2q}(e) \subset e^\circ$ and therefore $b_J = e$ is a fixture of λ_J^{2q}. In the latter case, $\lambda_J^{2q}(v) \in a^\circ$, implying that the image of the open angle $\angle_a v$ under λ_J^{2q} is contained in the open cell a°. From this we have $\lambda_J^{3q}(a) \subset a^\circ$, so that $b_J = a$ is a fixture of λ_J^{3q}. This paragraph's discussion verifies that at least one of the mappings λ_J^q, λ_J^{2q}, or λ_J^{3q} has a fixture. A quick inductive argument proves that, for any positive integer s, a fixture for λ_J^s is a fixture for λ_J^{st} for all positive integers t. We conclude that the mapping λ_J^{6q} has a fixture, and since $1 \le q \le \beta$, we may set $M = 6\beta$. This concludes the second part of the proof.

Part 3: The general case. We now assume that the subdivision operator τ is shrinking, with τ^s strictly shrinking for the positive integer s. Fix a positive integer k. Our aim is to count the number of k-orbits of the action of $\hat{\tau}$ on (K). Note first that an expansion complex for τ is automatically an expansion complex for any positive power of τ and vice-versa, so if $J \in (K)$, then J is an expansion complex associated to τ^{kr} for any positive integer r. In particular, the results of Parts 1 and 2 may be applied to any power of τ that satisfies the pertinent assumptions of those parts of the proof. In particular, Part 2 applied to the strictly shrinking (n, n)-subdivision operator τ^{ks} with face bound $\beta = \beta(k)$ implies that, for every fixed point J of $\hat{\tau}^{ks}$, there exists a positive integer $m \leq 6\beta$ such that λ_J^m has a fixture, where λ_J is a cellular isomorphism of J onto $\tau^{ks}J$.

We now count the k-orbits of $\hat{\tau}$. Let J be any point of (K) that lies in a k-orbit of $\hat{\tau}$. Now J is a fixed point of the action of $\hat{\tau}^k$ on (K) and hence a fixed point of $\hat{\tau}^{ks}$. Therefore, there is a positive integer $m \leq 6\beta$ such that λ_J^m has a fixture, where $\lambda_J : J \to \tau^{ks}J$ is a cellular isomorphism. Note that λ_J^m is a cellular isomorphism of J to the subdivision $\tau^{mks}J$. We see then that Part 1 applies to J since J is a fixed point of the action of $\hat{\tau}^{mks}$, τ^{mks} is a strictly shrinking, rotationally invariant (n, n)-subdivision operator of bounded degree, and the cellular isomorphism λ_J^m of J onto $\tau^{mks}J$ has a fixture. It follows that $J \in \mathcal{F}_m$, where $\mathcal{F}_m$ is the set of points of (K) that are fixed by $\hat{\tau}^{mks}$ and satisfy the second assumption of Part 1. But Part 1 implies that $\mathcal{F}_m$ is a finite subset of (K). We have shown that any expansion complex J in a k-orbit of $\hat{\tau}$ must lie in one of the finite sets $\mathcal{F}_m$, for some $1 \leq m \leq 6\beta$. It follows that there are at most finitely many k-orbits of τ. This finishes the proof of the theorem. $\square$

We already mentioned in Part 2 of the proof that the hyperbolic complex H of Example 3.1 provides an example of a cellular isomorphism of H onto a subdivided complex that fails to have a fixture. Of course the subdivision operator σ_0 of this example is neither shrinking nor rotationally invariant, and the hierarchy constructed in Example 3.1 is not expansive. In fact, it is not difficult to see that the mapping $\hat{\sigma}_0$ induced on the local isomorphism class (H) is the identity, so that every element of (H) is a fixed point even though H is plural and (H) is uncountably infinite.[24] The next example fulfills a promise of Part 2 of the proof to give an example of a cellular isomorphism of a planar n-gon complex onto its τ-subdivision that fails to have a fixture, even though τ is a strictly shrinking, rotationally invariant (n, n)-subdivision operator of bounded degree.

Example 4.1. A fixture-free cellular isomorphism. The integer lattice 4-gon complex Z described on page 31 admits many cellular isomorphisms onto its quad-subdivision νZ, some of which have a fixture, others of which do not. Quad-subdivision defines a rotationally invariant $(4, 4)$-subdivision operator that is strictly shrinking and of bounded degree. Let $R_{\pi/2}$ be the counterclockwise rotation of $\pi/2$ radians about $w = (1/2, 1/2)$, the center of the unit square face a of Z whose lower left-hand vertex is the origin $\mathcal{O} = (0, 0)$, and let $M_{1/2}$ be the map that multiplies each complex number by $1/2$, and so dilates the complex plane toward the origin $\mathcal{O}$. Then $\lambda_Z = M_{1/2} \circ R_{\pi/2}$ is a cellular isomorphism of

[24] Dane Mayhook's doctoral thesis will explore the local isomorphism class (H) of this hyperbolic complex and give a constructive description of all elements of (H).

Z onto νZ with no fixture. Though λ_Z^2 also fails to have a fixture, the face a serves as a fixture of λ_Z^3. $\surd$

4.3.1. *Identifying fixed points and k-orbits: A construction of supersymmetric expansion complexes from the proof of Theorem 4.10.* The proof of Part 1 of Theorem 4.10 may be backward engineered to build a procedure for constructing k-orbits of $\hat{\tau}$, for arbitrary positive integers k. For a given positive integer k, the procedure may be used to define an expansion complex F for which $\tau^k F$ is cellularly isomorphic to F, so that $\hat{\tau}^k F = F$. Moreover, an isomorphism λ_F of F onto $\tau^k F$ is apparent and the complex F may be encoded by a finite n-gon complex pair with distinguished vertices. Below we describe this construction method for expansion complexes and close the paper with several illustrative graphical examples.

For convenience, suppose that τ is strictly shrinking and of bounded degree with face bound β. Let k be a positive integer and choose any 4-tuple $\mathcal{C} = (C, v; c, w)$ that satisfies the following properties.

(1) C is a core of the complex $\tau_n^t = \tau^t \Delta$, for some positive integer t, that is contained in the interior of τ_n^t;

(2) c is a core of $\tau^k C$ that is contained in the interior of a combinatorial disk in C;

(3) v is a vertex of C that lies in its boundary;

(4) w is a vertex of c that lies in its boundary;

(5) there exists an orientation-preserving cellular isomorphism $\lambda_0 : C \to c$ with $\lambda_0(v) = w$.

Let $\mu_0 : c \hookrightarrow \tau^k C$ be the cellular inclusion and, for each positive integer m, let $\mu_m : \tau^{km} c \hookrightarrow \tau^{k(m+1)} C$ be the cellular inclusion induced from μ_0, and let $\lambda_m : \tau^{km} C \to \tau^{km} c$ be the cellular isomorphism induced from λ_0. Setting $F_m = \tau^{km} C$, for each non-negative integer m we have the isomorphic embedding $\Xi_m = \mu_m \circ \lambda_m : F_m \hookrightarrow F_{m+1}$ and a quick check verifies that this sequence of embeddings satisfies the five properties that define an expansion complex associated to τ. Let $F = \varinjlim F_m$ be the expansion complex defined by this sequence of embeddings. The idea here is that the isomorphic embedding Ξ_m shrinks C via λ to c and then includes c back into C via μ, and the subscript m merely tells one the level at which Ξ is cellular, namely, at the level of the τ^{km}-subdivision. Now set $f_m = \tau^{km} c$ and let $\xi_m = \lambda_{m+1} \circ \mu_m : f_m \hookrightarrow f_{m+1}$, an isomorphic embedding of f_m into the interior of f_{m+1}. Again, a quick check verifies that the sequence of embeddings $\xi_m : f_m \hookrightarrow f_{m+1}$ satisfies the five properties that define an expansion complex associated to τ, and we let $f = \varinjlim f_m$.

The following diagram commutes, and this has interesting implications for the expansion complexes F and f:

$$\begin{tikzcd}
F_0 \arrow[r, "\Xi_0"] \arrow[d, "\lambda_0"'] & F_1 \arrow[r, "\Xi_1"] \arrow[d, "\lambda_1"] & \cdots \arrow[r, "\Xi_{m-1}"] & F_m \arrow[r, "\Xi_m"] \arrow[d, "\lambda_m"'] & F_{m+1} \arrow[r, "\Xi_{m+1}"] \arrow[d, "\lambda_{m+1}"] & \cdots \arrow[r, "\lim\limits_{\longrightarrow}"] & F \\
f_0 \arrow[r, "\xi_0"'] \arrow[ur, "\mu_0"] & f_1 \arrow[r, "\xi_1"'] \arrow[ur, "\mu_1"] & \cdots \arrow[r, "\xi_{m-1}"'] \arrow[ur] & f_m \arrow[r, "\xi_m"'] \arrow[ur, "\mu_m"] & f_{m+1} \arrow[r, "\xi_{m+1}"'] & \cdots \arrow[r, "\lim\limits_{\longrightarrow}"'] & f.
\end{tikzcd}$$

Since the vertical arrows are all cellular isomorphisms, they induce a cellular isomorphism $\lambda : F \to f$ and, since the diagonal arrows are cellular inclusions of f_m into $F_{m+1} = \tau^k F_m$, they induce a cellular inclusion $\mu : f \to \tau^k F$. Since the image of each μ_m lies interior to F_{m+1}, the induced inclusion μ must be onto, and this implies that μ is a cellular isomorphism of f onto $\tau^k F$. That the diagram commutes further implies that the composition $\lambda_F = \mu \circ \lambda : F \to \tau^k F$ is a cellular isomorphism, and hence $\hat{\tau}^k F = F$ in (K). It follows that F generates an ℓ-orbit of the action of $\hat{\tau}$ on (K) for some positive integer $\ell \leq k$. The 4-tuple $\mathcal{C}$ that generates F, f, and λ_F is called a **combinatorial footprint** for F.

Before presenting examples, we make some observations about this construction. If C is a face core, then C is merely an oriented combinatorial n-gon and c is obtained by making a choice of one of the faces of the subdivided n-gon $\tau^k C$ that is interior to C. The only remaining choice to make to define $\mathcal{C}$—and therefore F, f, and the cellular isomorphism λ_F—is the choice of an orientation-preserving isomorphism λ_0 of the combinatorial n-gon C onto the combinatorial n-gon c. There are exactly n of these up to isotopy, and one may be distinguished uniquely by the choices of a vertex v of C and a vertex w of c and then requiring that $\lambda_0(v) = w$. The combinatorial n-gon C identified naturally as the base face of F is a fixture of the isomorphism λ_F since, under this identification, $\lambda_F(C) = c \subset C^\circ$. The situation for edge and vertex cores is a bit more complicated. In this case, the mapping λ_F may have no fixture, but of course the proof of Theorem 4.10 shows that a power $m \leq 6\beta$ of λ_F will have one.

4.4. **A zoo of graphical examples: Supersymmetric expansion complexes and their combinatorial footprints.** In this final section, we illustrate the ideas of the preceding one by exhibiting various footprints for supersymmetric complexes associated to the pentagonal subdivision operator τ. In each, we picture the combinatorial footprint $\mathcal{C}$ of F and a snapshot of a finite patch of the conformal tiling $T = T_F$ with a polygonal τ^k-subdivision. Since the pentagonal rule is dihedrally symmetric, τ is a conformal subdivision operator, and the combinatorial supersymmetry determined by a footprint may be realized by a conformal supersymmetry μ, a Möbius transformation of the plane whose application to the conformal tiling T determined by the footprint conformally subdivides T, à la Theorem 3.12 and Corollary 4.9.

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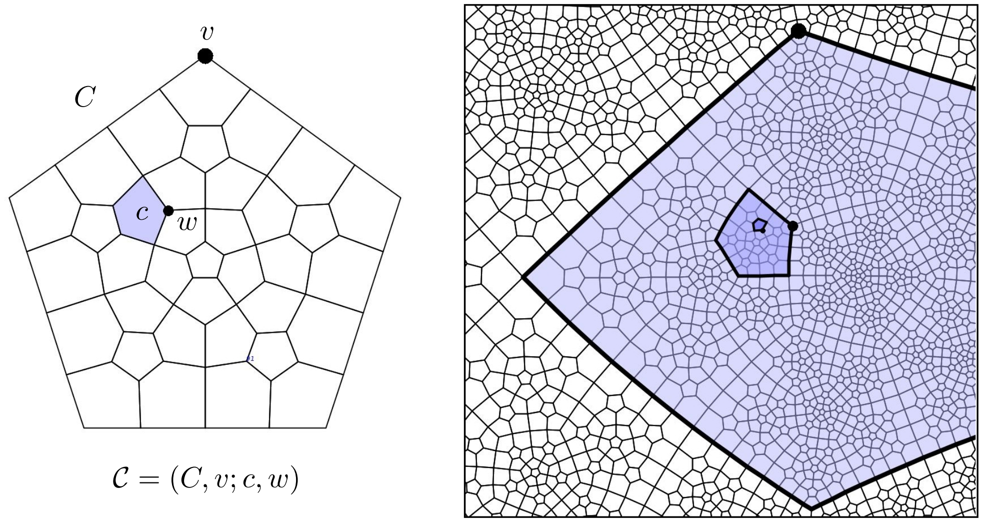

FIGURE 9. The conformal tiling T on the right is sampled in a neighborhood of the fixed point of its loxodromic supersymmetry μ, which is a conformal isomorphism of the tiling T with $\tau^2 T = \mu(T)$. The fixed point lies interior to the small bold-sided pentagonal tile near the center, and the image of the large bold-sided and shaded pentagon under the supersymmetry is the mid-sized bold-sided and shaded pentagon, whose image in turn is the small bold-sided and shaded tile. The tiling T fails to be isomorphic with the subdivision τT and so has τ-period 2. The tiling arises from the footprint $\mathcal{C} = (C, v; c, w)$ on the left for which C is a pentagonal face core and c is the shaded pentagon in the interior of the second subdivided complex $\tau^2 C$.

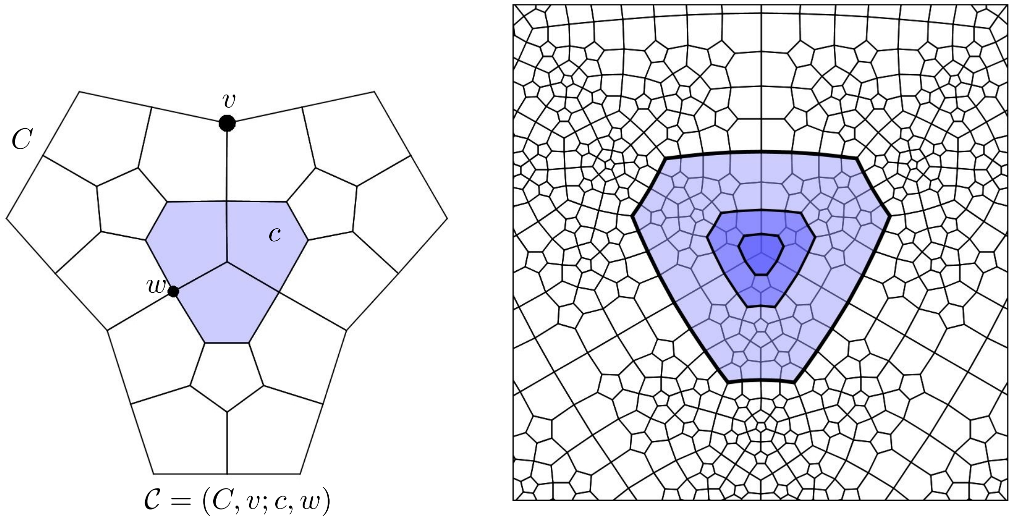

FIGURE 10. In this example, the conformal supersymmetry μ is a contraction followed by a counterclockwise rotation of angle $2\pi/3$. The fixed point is the central vertex in the shaded region. In the footprint on the left, the core C is a vertex core and c is the central vertex core of τC. The τ-period is 1 and this represents a fixed point of the action of $\hat{\tau}$ on (K).

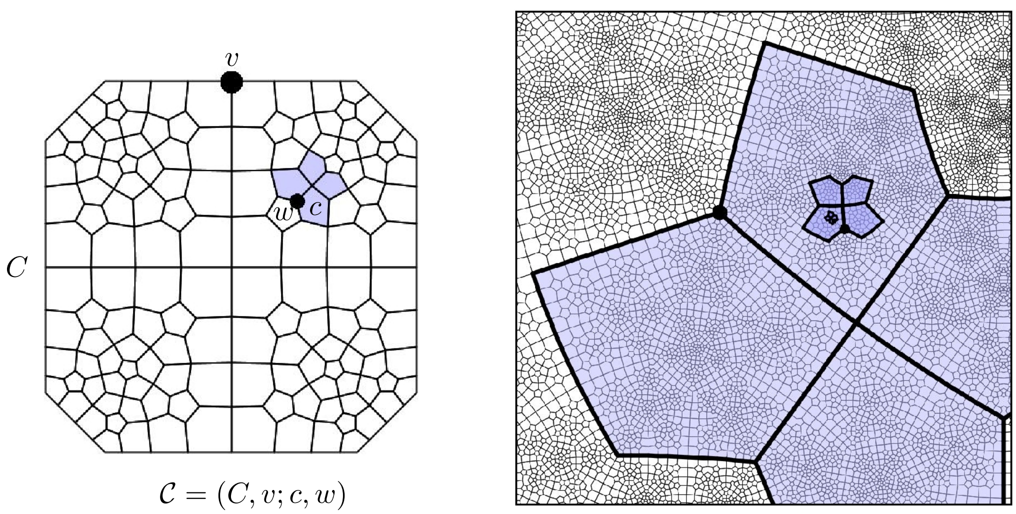

FIGURE 11. This example has a loxodromic supersymmetry of τ-period 2.

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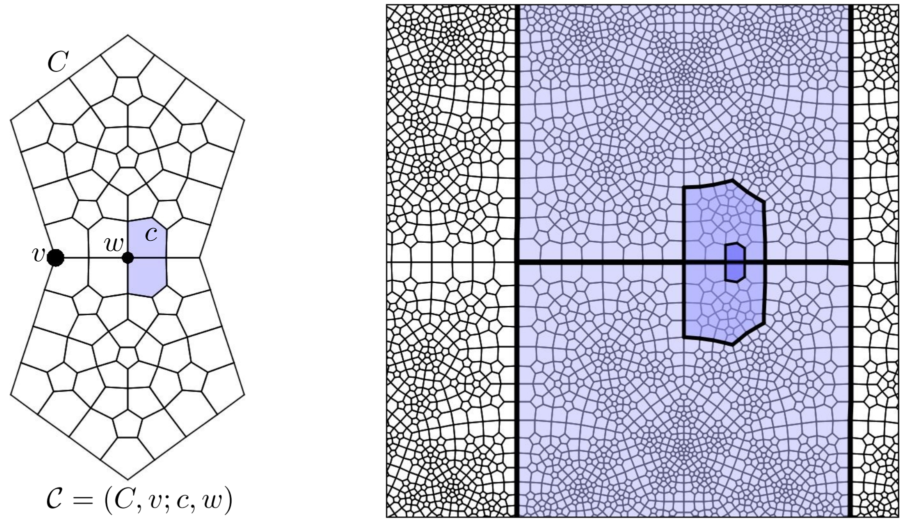

FIGURE 12. This example has τ-period 2 and the conformal supersymmetry is a pure contraction toward a fixed point that lies in the horizontal edge that determines the inner-most bold-sided edge core on the right. The horizontal line of edges the eye picks out lies along a Euclidean straight line that serves as a line of Euclidean-reflective symmetry of the tiling.

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